Integral Approach of Asymptotic Expansions for Low Reynolds Number Flow Past an Arbitrary Cylinder*

Teruhiko KIDA** and Takanori TAKE***

An incompressible stationary flow past an arbitrary two-dimensional cylindrical body at low Reynolds number is analyzed by an asymptotic approach. First, the integral formulations are derived from the vorticity equation and the continuity equation. Second, simple integral equations with only one variable are derived from them. From these integral equations, asymptotic integral equations on the Stokes and Oseen regions are finally obtained by using the concept of both the significant integral operator given by Kida and the generalized expansions given by Skinner. Furthermore, using analytical results given by Kida, it is shown that no other local region exists.

Key Words: Fluid Dynamics, Viscous Flow, Low Reynolds Number Flow, Asymptotic Analysis, Integral Equation

1. Introduction

The singular perturbation problem of finding a uniform valid asymptotic solution of the Navier-Stokes equation for incompressible flow past a two- or three-dimensional body at low Reynolds numbers is classical, and it is solved by considering two principal limits: Oseen limit and Stokes limit. For the two-dimensional problem, Proudman and Pearson⁽¹⁾ solved the flow around a circular cylinder, and Kaplun⁽²⁾ also solved an arbitrary body shape. Kaplun noted in his paper that the first terms of the Stokes and Oseen expansions cannot be matched (we note that the first outer (Oseen) expansion is only due to uniform flow), and introduced the concept of intermediate limits (we note that Proudman and Pearson⁽¹⁾ used the Oseen solution of the circular cylinder as the first outer expansion, so that this theoretical background is

based on Kaplun's intermediate limits). In both works, the outer expansion is obtained in powers of $1/\log Re$ where Re is the Reynolds number. Although the next high order approximation might be of order Re , as pointed out by Proudman and Pearson, they did not obtain it. Along lines of Kaplun's or Proudman and Pearson's analysis, the low Reynolds number flow problem past several cylindrical bodies has been treated by many investigators^{(3),(4)}. Lee and Leal⁽⁵⁾ solved it by an alternative approach. They expressed general solutions for both the Stokes and Oseen equations in terms of a boundary distribution of fundamental single- and double-layer singularities, and derived integral equations for unknown distributions by application of matching conditions between the Stokes and Oseen solutions. They obtained a uniformly valid numerical approximation to the solution for the whole domain by using the collocation method. However, we note that their integral formulations cannot bridge the $1/\log Re$ asymptotic series to the next-order Re . The first work aiming to bridge the first-order $1/\log Re$ series to the next-order approximation is Skinner's analysis⁽⁶⁾. He proposed a new concept, the generalized expansion, and solved the second-order approxi-

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** Department of Mechanical Engineering, University of Osaka Prefecture, 1-1 Gakuencho, Sakai, Osaka 593, Japan

*** Department of Mechanical Engineering, Shiga Junior College, 1900 Yasakacho, Hikone, Shiga 522, Japan

mation from the suggestion of Proudman and Pearson; that is, he obtained a few terms of the $1/\log Re$ power series by supposing that these expansions are expressed as a $1/\log Re$ series. As noted by Fraenkel⁽⁷⁾ and Eckhaus⁽⁸⁾, careful treatment of the asymptotic matching principle is necessary in the purely logarithmic case, such as the present problem. They noted that the correct truncation of expansions can be essential in application of the asymptotic matching principle. Hence, in Theorem 3.7.1 of the matching principle given by Eckhaus, the purely logarithmic expansions are excluded. According to these papers, it is very important to obtain the asymptotic expansions with the order function $\delta_m(Re) = Re^m$ (m ; integer), but this has not been established yet. Since Skinner's analysis uses purely logarithmic expansions, it is an open problem as to whether or not Skinner's series expansion of the second-order approximation is correct.

In the proof of the validity of the asymptotic matching principle and the existence of a composite expansion, the overlap hypothesis is used. However, this hypothesis is only used as a sufficient condition. For a purely logarithmic case, such as the present problem, it is necessary to examine this hypothesis. To do this, we will also need to obtain the asymptotic expansions with the order function $\delta_m(Re) = Re^m$.

From this background, the main purpose of the present paper is to derive an asymptotic expression with the order functions $\delta_m(Re) = Re^m$, which are called 'the generalized expansions' in the present paper. A new alternative approach is here proposed; we start from the vorticity equation, derive integral equations which satisfy the boundary conditions both on the surface of a cylindrical body and at infinity, give further alternative expressions with only one variable to make the discussion of asymptotic expansion easy, and obtain a set of integral expressions of the generalized Stokes and Oseen limits through application of the mathematical analysis of linear integral equations. We hence note that we can bridge the $1/\log Re$ series of first expansion to the next-order approximation with Re , and we can also check Skinner's $1/\log Re$ series expansions of the second order (the relationship between the present results and the $1/\log Re$ series expansions will be submitted in the series paper by the present authors). We here assume that the stream function and the vorticity function are sufficiently differentiable in the entire flow region and that there exist a regular and local expansions in its subdomain. We finally show that (a) Stokes and Oseen limits are significant, (b) there is no possibility of another significant local limit even if the contour of the body is not smooth enough, and (c) the generalized Oseen

and Stokes limits can be matched by the matching principle. We note here that the terminologies which are used or will be used in the present paper, regular, local, significant, and so on, are after Eckhaus⁽⁸⁾ and Kida⁽⁹⁾.

2. Integral Representations

Let Ox and Oy be rectangular coordinate axes having the origin O at some point in a two-dimensional body which is at rest in a uniform free-stream flow of an incompressible viscous fluid whose velocity is unity. The x -axis is taken as the uniform flow direction. The governing equations for this problem are well known as the Navier-Stokes equation and the continuity equation. From these equations, we easily obtain the following relations for the vorticity ζ and the stream function ϕ , normalized by the typical length of the body L and the uniform flow velocity U :

$$\nabla^2 \zeta - Re \partial \zeta / \partial x = Re f. \quad (2.1)$$

$$\zeta(x, y; Re) = -\nabla^2 \hat{\phi}. \quad (2.2)$$

$$f(x, y; Re) = \hat{u} \partial \zeta / \partial x + \hat{v} \partial \zeta / \partial y, \quad (2.3)$$

where $\hat{u}$ and $\hat{v}$ are the perturbation velocities, $\hat{u} = \partial \hat{\phi} / \partial y$ and $\hat{v} = -\partial \hat{\phi} / \partial x$. Here $\hat{\phi}$ is the perturbation stream function given by

$$\phi = y + \hat{\phi}. \quad (2.4)$$

The Reynolds number is defined by $Re = LU/\nu$, where ν is the kinematic viscosity. We have the following boundary conditions from no-slip and upstream conditions:

$$\hat{u} = 0, \quad \hat{v} = 0 \quad \text{at infinity.} \quad (2.5)$$

$$\partial \hat{\phi} / \partial x = 0, \quad \partial \hat{\phi} / \partial y = -1 \quad \text{on } S, \quad (2.6)$$

where S refers to the contour of the body. From Green's theorem, $\hat{\zeta}$ and $\hat{\phi}$ are obtained as

$$\begin{aligned} \hat{\zeta} = & -\frac{Re}{2\pi} \int_D \hat{f}(x_1, y_1; Re) G(x - x_1, y - y_1) dx_1 dy_1 \\ & + \frac{1}{2\pi} \int_S [\mu G(x - x_s, y - y_s) \\ & - \sigma \frac{\partial}{\partial n} G(x - x_s, y - y_s)] ds. \end{aligned} \quad (2.7)$$

$$\begin{aligned} \hat{\phi} = & \frac{1}{2\pi} \int_D \exp(Re x_1 / 2) \hat{\zeta}(x_1, y_1) \log \frac{1}{\rho} dx_1 dy_1 \\ & + \frac{1}{2\pi} \int_S \left[\frac{\partial \hat{\phi}}{\partial n} \log \frac{1}{\rho_s} - \hat{\phi} \frac{\partial}{\partial n} \log \frac{1}{\rho_s} \right] ds, \end{aligned} \quad (2.8)$$

where

$$G(x - x_1, y - y_1) = K_0(Re\rho/2). \quad (2.9)$$

$$\hat{f}(x, y; Re) = \exp(-Re x / 2) f(x, y; Re). \quad (2.10)$$

$$\hat{\zeta}(x, y) = \exp(-Re x / 2) \zeta(x, y). \quad (2.11)$$

$$\rho = \{(x - x_1)^2 + (y - y_1)^2\}^{1/2},$$

$$\rho_s = \{(x - x_s)^2 + (y - y_s)^2\}^{1/2}$$

$$\mu = \partial \hat{\zeta} / \partial n \quad \text{and} \quad \sigma = \hat{\zeta} \quad \text{on } S. \quad (2.12)$$

In these equations, (x_1, y_1) is a point on the integral area D which is the entire region of the outside of the body, (x_s, y_s) a point on S , and ds the differential change in length along S . The function $K_0(x)$ is the

modified Bessel function. $\partial/\partial n$ denotes the interior normal derivative on the contour S . Since $\partial\hat{\psi}/\partial n = -\partial y/\partial n$ and $\hat{\psi} = -y$ on S from Eq.(2.6), we are led to the fact that the last term on the right-hand side of Eq.(2.8) may be equal to zero if S is a piecewise-smooth surface. Since the Reynolds number is small, we define a small parameter ε by

$$\varepsilon = Re/2.$$

In order to apply the mathematical analysis given by Kida⁽⁹⁾, we consider the following relations which are a system of integral equations with single variable r :

$$2\tilde{F}(r, \theta : \phi : \varepsilon) - (F(r : \phi : \varepsilon) - \tilde{\xi}(r : \phi : \varepsilon))/\varepsilon = f^s(r, \theta : \phi : \varepsilon). \quad (2.13)$$

$$\Omega(r, \theta : \phi : \varepsilon) - \hat{\psi}(r : \phi : \varepsilon) = g^s(r, \theta : \phi : \varepsilon) + H(r : \phi). \quad (2.14)$$

$$\tilde{F}(r, \theta : \phi : \varepsilon) \triangleq \int_{r_s(\theta)}^{\infty} \tilde{f}(x_1, y_1 : \varepsilon) \times G(x - x_1, y - y_1) r_1 dr_1. \quad (2.15)$$

$$\Omega(r, \theta : \phi : \varepsilon) \triangleq \int_{r_s(\theta)}^{\infty} \exp(\varepsilon r_1 \cos \theta) \times \tilde{\xi}(x_1, y_1) \log \frac{1}{\rho} r_1 dr_1. \quad (2.16)$$

$$F(r : \phi : \varepsilon) \triangleq \int_s \left[\mu G(x - x_s, y - y_s) - \sigma \frac{\partial}{\partial n} G(x - x_s, y - y_s) \right] ds. \quad (2.17)$$

$$H(r : \phi) \triangleq \int_s \left[\frac{\partial y_s}{\partial n} \log \frac{1}{\rho_s} - y_s \frac{\partial}{\partial n} \log \frac{1}{\rho_s} \right] ds, \quad (2.18)$$

where

$$\begin{aligned} (x_1, y_1) &= (r_1 \cos \theta, r_1 \sin \theta) \\ (x, y) &= (r \cos \phi, r \sin \phi) \\ (x_s, y_s) &= (r_s \cos \theta, r_s \sin \theta). \end{aligned}$$

We assume that functions f^s and g^s , which are some analytic functions of r and θ , can be expressed as a Fourier series with respect to θ and that they become zero by integration with respect to θ from 0 to 2π . Then we easily see that the basic relations (2.7) and (2.8) are obtained by integrating Eqs.(2.13) and (2.14) from 0 to 2π with respect to θ , respectively. In the following section we will discuss these equations.

3. Significant Local Regions

The above simultaneous integral Eqs. (2.13) and (2.14) of $\hat{\psi}$ and $\tilde{\xi}$ with boundary conditions (2.6) are nonlinear, but unknown functions do not appear in their kernels. We may hence apply the singular perturbation theories of a linear integral equation obtained by Kida⁽⁹⁾ to the present equations. Let us assume that the unknown functions are sufficiently differentiable in the entire flow region D , except the boundary of the body S , and that their regular and local expansions exist on some subdomain of D (the definitions of "regular expansion" and "local expansion" are given

by Eckhaus⁽⁸⁾).

We first consider the functions F and H . We note that the function G defined by Eq.(2.9) is asymptotically expressed by

$$G \sim (\pi/2\varepsilon\rho)^{1/2} \exp(-\varepsilon\rho) \quad \text{for } \varepsilon\rho \gg 1 \quad (3.1)$$

$$G \sim -\log(\varepsilon\rho/2) - \gamma \quad \text{for } \varepsilon\rho \ll 1, \quad (3.2)$$

where γ is the Euler constant. Then the line integral of Eq.(2.17) is restricted to $r_s = O(1)$ for $\varepsilon \rightarrow 0$, so F is not uniformly convergent to a function $F_0(r : \phi)$ for $\varepsilon \rightarrow 0$ and $r \gg r_s$. Since H , f^s and g^s are assumed to be uniformly convergent to $H_0(r : \phi)$, $f_0^s(r : \phi)$ and $g_0^s(r : \phi)$ for $\varepsilon \rightarrow 0$ in the entire region D , respectively, we may conclude from these functions that a domain $r = O(1/\varepsilon)$ is significant, and a significant local variable is given by $R = \varepsilon r$.

We next consider the asymptotic behavior of the integral operators of Eqs.(2.13) and (2.14):

$$K_\varepsilon \varphi \triangleq \int_{r_s(\theta)}^{\infty} G(x - x_1, y - y_1) \varphi(r_1) dr_1 \quad (3.3)$$

$$T_\varepsilon \varphi \triangleq \int_{r_s(\theta)}^{\infty} \exp(\varepsilon r_1 \cos \theta) \log \frac{1}{\rho} \varphi(r_1) dr_1, \quad (3.4)$$

where φ is an arbitrary test function belonging to the class C_0^∞ in $[r_s, \infty)$ such that these integrals exist. Taking into account Eq.(3.2), we may obtain the degeneration of K_ε (the rigorous definition of "degeneration" is given by Kida⁽⁹⁾):

$$K_\varepsilon = \delta_0 K_0 + \delta_1 K_1 + O(\varepsilon^2 \log \varepsilon). \quad (3.5)$$

$$K_0 \varphi \triangleq \int_{r_s}^{\infty} \varphi(r_1) dr_1$$

$$K_1 \varphi \triangleq \int_{r_s}^{\infty} \varphi(r_1) \log \frac{1}{\rho} dr_1,$$

where $\delta_0 = -\gamma - \log(\varepsilon/2)$, $\delta_1 = 1$, and φ is a test function such that $K_0 \varphi$ and $K_1 \varphi$ exist. For operator T_ε , we can also obtain its degeneration:

$$T_\varepsilon = T_0 + \varepsilon T_1 + O(\varepsilon^2). \quad (3.6)$$

$$T_0 \varphi \triangleq \int_{r_s}^{\infty} \varphi(r_1) \log \frac{1}{\rho} dr_1$$

$$T_1 \varphi \triangleq \int_{r_s}^{\infty} \varphi(r_1) \cos \theta \log \frac{1}{\rho} r_1 dr_1,$$

where φ is also a test function such that the above integrals exist. Equation (3.5) implies that it is rational to define the first degeneration of K_ε by $\bar{K}_0$ such that

$$\bar{K}_0 \varphi \triangleq \int_{r_s}^{\infty} \left[-\gamma - \log \frac{\varepsilon}{2} - \log \rho \right] \varphi(r_1) dr_1. \quad (3.7)$$

Thus, we see from the asymptotic behavior of the integral operator that the generalized expansion introduced by Skinner⁽⁶⁾ is reasonable for the purpose of the present paper.

Let us consider the significant local operators of K_ε and T_ε (the rigorous definitions of "significant operator", "local operator", and "significant", are given by Kida⁽⁹⁾). We define the local integral operator K_ε^* of K_ε by setting the local variable as $R = \varepsilon^m r$ ($0 < m \leq 1$). Then we determine, from the definition of

significant operator, that the significant local operator is given by putting m as 1 and that its degeneration is given by

$$K_\varepsilon^* = K_0^*/\varepsilon + O(1) \quad \text{in } (0, \infty). \quad (3.8)$$

$$K_0^* \varphi^* \triangleq \int_0^\infty \varphi^*(R_1) K_0(\rho^*) dR_1 \quad \text{in } (0, \infty), \quad (3.9)$$

where $\varphi^*(R)$ is a test function such that $K_0^* \varphi^*$ exists, that

$$\rho^* = \{R^2 + R_1^2 - 2RR_1 \cos(\theta - \phi)\}^{1/2}.$$

The reason that the above operator is significant is that the transformed operator $T_R \bar{K}_0$ of $\bar{K}_0$ by $R = \varepsilon r$ is given by (the rigorous definition of "transformed operator" is given by Kida⁽⁹⁾):

$$T_R \bar{K}_0 \varphi^* = \int_0^\infty [-\gamma + \log 2 - \log \rho^*] \varphi^*(R_1) dR_1,$$

and $T_R \bar{K}_0$ is contained in K_0^* (the rigorous definition of "contained" is given by Kida⁽⁹⁾). Similarly, we see that the significant local operator T_ε^* of operator T_ε is obtained by the same transformation $R = \varepsilon r$ and the degeneration of the significant operator, T_0^* , is given by

$$T_0^* \varphi^* \triangleq \int_0^\infty \varphi^*(R_1) \exp(R_1 \cos \theta) \log \frac{1}{\rho^*} dR_1, \quad (3.10)$$

because we easily see that $T_R T_0$ is also contained in T_0^* . Thus, the integral operators of Eqs. (2.13) and (2.14) show us that $R (= \varepsilon r)$ is a significant variable and no other significant variable is yielded.

Finally, we will discuss from these considerations that R is certainly the significant local variable if it indeed exists. The first regular and significant solutions of the integral Eqs. (2.13) and (2.14) are denoted by $\hat{\psi}_0$ and $\hat{\xi}_0$, and $\hat{\psi}_0^*$ and $\hat{\xi}_0^*$, respectively (the definition of "regular solution of integral equations" is given by Kida⁽⁹⁾). Theorem 2.1 of Kida⁽⁹⁾ shows that $\hat{\psi}(r, \theta; \varepsilon) \rightarrow \hat{\psi}_0(r, \theta)$ uniformly in $D_0 = [r_s, 1/d]$ with fixed r where d is an arbitrary ε -independent small value, and $\hat{\psi}(r, \theta; \varepsilon) \rightarrow \delta_0 \hat{\psi}_0^*(R, \theta)$ uniformly in $D_0^* = [\bar{d}, \infty)$ with fixed R for $\varepsilon \rightarrow 0$ where $\hat{\psi}^* = \hat{\psi}(R/\varepsilon, \theta; \varepsilon)$, δ_0 is some order function dependent on ε , and $\bar{d}$ is also an arbitrary ε -independent small value. Let us define as $H \rightarrow H_0(r, \theta)$ and $g^s \rightarrow g_0^s(r, \theta)$ in D_0 and $H \rightarrow H_0^*(R, \theta)$ and $g^s \rightarrow \delta_0 g_0^*(R, \theta)$ in D_0^* uniformly for $\varepsilon \rightarrow 0$, respectively. We have, from Theorem 2.1,

$$\hat{\xi} \rightarrow \hat{\xi}_0(\hat{\psi}_0, g_0^s, H_0) \quad \text{uniformly in } D_0$$

$$\hat{\xi}^* \rightarrow \hat{\delta}_1 \hat{\xi}_0^*(\hat{\psi}_0^*, g_0^{s*}, H_0^*) \quad \text{uniformly in } D_0^*,$$

where $\hat{\xi}^* = \hat{\xi}(R/\varepsilon, \theta; \varepsilon)$ and $\hat{\delta}_1$ is some order function dependent on ε . In spite of whether $\hat{\psi}_0 \rightarrow \delta_0 \hat{\psi}_0^*$ or $\hat{\psi}_0 \not\rightarrow \delta_0 \hat{\psi}_0^*$ in D_0^* with fixed R for $\varepsilon \rightarrow 0$, Theorem 2.2 of Kida⁽⁹⁾ shows the following possibility: $\hat{\xi}_0 \not\rightarrow \delta_1 \hat{\xi}_0^*$ uniformly in D_0^* with fixed R for $\varepsilon \rightarrow 0$ because $T_R T_0$ is contained in T_0^* . If $\hat{\xi}_0 \not\rightarrow \delta_1 \hat{\xi}_0^*$ with fixed R for $\varepsilon \rightarrow 0$, then $\hat{\psi}_0 \not\rightarrow \delta_0 \hat{\psi}_0^*$ in D_0^* with fixed R for $\varepsilon \rightarrow 0$. Therefore, we may infer that D_0^* is significant. Since $T_R \bar{K}_0$ is

contained in K_0^* , we also have the following relations from Eq. (2.13) for $\varepsilon \rightarrow 0$:

$$\begin{aligned} \hat{f} &\rightarrow \hat{f}_0(\hat{\xi}_0, f_0^s, F_0) && \text{uniformly in } D_0 \\ \hat{f}^* &\rightarrow \delta_2 \hat{f}_0^*(\hat{\xi}_0^*, f_0^{s*}, F_0^*) && \text{uniformly in } D_0^* \\ \hat{f}_0 &\not\rightarrow \delta_2 \hat{f}_0^* && \text{uniformly in } D_0^* \\ &&& \text{with fixed } R, \end{aligned} \quad (3.11)$$

where $\hat{f}^* = \hat{f}(R/\varepsilon, \theta; \varepsilon)$, $f^s \rightarrow f_0^s$ and $f^{s*} \rightarrow \delta_1 f_0^{s*}$ uniformly in D_0 and D_0^* for $\varepsilon \rightarrow 0$, respectively, $F \rightarrow F_0$ and $F \rightarrow \delta_4 F_0^*$ uniformly in D_0 and D_0^* respectively, where $\delta_i (i=1, 2, \dots)$ are some order functions dependent on ε . On the other hand, $\hat{f}$ is related to $\hat{\psi}$ by

$$\hat{f}(r, \theta; \varepsilon) \triangleq L_\varepsilon \hat{\psi} \triangleq \frac{\partial \hat{\psi}}{\partial y} \left(\varepsilon \hat{\xi} + \frac{\partial \hat{\xi}}{\partial x} \right) - \frac{\partial \hat{\psi}}{\partial x} \frac{\partial \hat{\xi}}{\partial y}. \quad (3.12)$$

We hence have, from Eq. (3.11),

$$L_0 \hat{\psi}_0 \not\rightarrow \delta_2 L_0^* \hat{\psi}_0^* \quad \text{uniformly in } D_0^*,$$

where L_0 is the degeneration of the differential operator L_ε and L_0^* is also the degeneration of L_ε^* , which is the transformed operator of L_ε due to R . We again obtain the relation $\hat{\psi}_0 \not\rightarrow \delta_0 \hat{\psi}_0^*$ in D_0^* even if L_0 is equivalent to L_0^* (we note that the term "equivalent" is from Eckhaus⁽⁸⁾). Hence, D_0^* is certainly significant if the significant local region exists, but we cannot at this stage show that there is no any other significant local region. From the boundary condition that $\hat{\psi}_0^* \rightarrow 0$ uniformly for $R \rightarrow \infty$, we have $\hat{\psi}^* \rightarrow \delta_0 \hat{\psi}_0^*$ in D_0^* , that is, there does not exist any other significant local region in D_0^* . We hence assume that there exists a subdomain $D_0^s \subset D_0$ where D_0^s is significant. Then we have $\hat{\psi} \rightarrow \delta_s \hat{\psi}_0^s$ uniformly with the fixed certain variable $\hat{r}_s$ in D_0^s . We hence have, from Theorems 2.1 and 2.2 of Kida⁽⁹⁾ for $\varepsilon \rightarrow 0$,

$$\hat{\xi} \rightarrow \hat{\xi}_0(\hat{\psi}_0, g_0^s) \quad \text{uniformly in } D_0 - D_0^s$$

$$\hat{\xi} \rightarrow \delta_6 \hat{\xi}_0^s(\hat{\psi}_0^s, g_0^s) \quad \text{uniformly in } D_0^s$$

$$\hat{\psi}_0 \rightarrow \delta_5 \hat{\psi}_0^s \quad \text{uniformly in } D_0^s$$

$$\hat{\xi} \not\rightarrow \delta_6 \hat{\xi}_0^s(\hat{\psi}_0^s, g_0^s) \quad \text{uniformly in } D_0^s.$$

Therefore, we have for $\varepsilon \rightarrow 0$,

$\hat{f}_0(\hat{\xi}_0(\hat{\psi}_0, g_0^s), f_0^s, F_0) \not\rightarrow \delta_7 \hat{f}_0^s(\hat{\xi}_0^s(\hat{\psi}_0^s, g_0^s), f_0^s, F_0)$ uniformly in D_0^s . On the other hand, we have, by taking into account the assumption $\hat{\psi}_0 \rightarrow \delta_s \hat{\psi}_0^s$;

$$L_0 \hat{\psi}_0 \not\rightarrow \delta_6 L_0^s \hat{\psi}_0^s \quad \text{in } D_0^s,$$

where L_0^s is the degeneration of L^s which is the local differential operator of L_ε for $D_0 \rightarrow D_0^s$. Thus, we have the possibility of the existence of another significant local region. However, the above discussion implies that $\hat{\psi}_0$ or its derivatives are singular at some point for $\varepsilon \rightarrow 0$ in D_0 . If the contour of the body is sufficiently smooth, functions F and H defined by Eqs. (2.17) and (2.18) are not singular for $\varepsilon \rightarrow 0$ in D_0 . Therefore, the local expansions are sufficiently differentiable in D , except S , for any small ε . If there exists a point at which the contour of the body is not smooth, there is a possibility that the neighborhood of this point is the

significant local region. Let us define (x, y) as $(r_s \cos \theta_0 + \delta \bar{r}_s \cos \delta(\theta - \theta_0), r_s \sin \theta_0 + \delta \bar{r}_s \sin \delta(\theta - \theta_0))$, $\bar{r}_s = (r - r_s)/\delta$, where δ and $\bar{\delta}$ are some order functions with $\delta, \bar{\delta} \rightarrow 0$ for $\varepsilon \rightarrow 0$ and $(r_s \cos \theta_0, r_s \sin \theta_0)$ is the above-mentioned not-smooth point. Then, we have, by taking into account $\rho \triangleq \delta \bar{\rho} = \delta \{ \bar{r}_s^2 + \bar{r}_1^2 - 2 \bar{r}_s \bar{r}_1 \cos \delta(\theta - \theta_1) \}^{1/2}$,

$$\mathbf{K}_\varepsilon \rightarrow \delta \mathbf{K}_0, \quad \mathbf{T}_\varepsilon \rightarrow \delta \mathbf{T}_0 \quad \text{for } \varepsilon \rightarrow 0,$$

where

$$\bar{\mathbf{K}}_0 \varphi \triangleq \int_0^\infty \left[-\gamma - \log \frac{\varepsilon \delta}{2} - \log \rho \right] \varphi(r_1) dr_1,$$

$$\bar{\mathbf{T}}_0 \varphi \triangleq \int_0^\infty \left[-\log \delta + \log \frac{1}{\rho} \right] \varphi(r_1) dr_1.$$

These operators $\bar{\mathbf{K}}_0$ and $\bar{\mathbf{T}}_0$ are contained in $\bar{\mathbf{K}}_0$ and $\bar{\mathbf{T}}_0$, respectively, so we note that even in the region of the not-smooth point, the integral operators $\bar{\mathbf{K}}_0$ and $\bar{\mathbf{T}}_0$ are available. The differential local operator $\bar{\mathbf{L}}_0 (= \partial \bar{\psi} / \partial \bar{y} \cdot \partial \bar{\xi} / \partial \bar{x} - \partial \bar{\psi} / \partial \bar{x} \cdot \partial \bar{\xi} / \partial \bar{y})$ is equivalent to or contained in $\bar{\mathbf{L}}_0$, so that there exists no significant local region in D_0 . Thus, the simultaneous integral Eqs. (2.13) and (2.14) have only one significant local variable $R (= \varepsilon r)$ if their solutions indeed exist.

4. Generalized Stokes Expansion

We consider the generalized Stokes flow. From Eqs. (2.13) and (3.7), Theorem 2.3 of Kida⁽⁹⁾ may lead to the governing integral equation:

$$\begin{aligned} 2 \oint_{r_s}^\infty \bar{f}_0(r_1) \left[-\gamma - \log \frac{\varepsilon}{2} - \log \rho \right] r_1 dr_1 + \frac{\bar{\xi}_0}{\varepsilon} \\ = f_0^s + \frac{1}{\varepsilon} \int_s \left[\mu_0 \left(-\gamma - \log \frac{\varepsilon}{2} - \log \rho \right) \right. \\ \left. + \sigma_0 \frac{\partial}{\partial n} \log \rho \right] ds - \sum_{m=0}^\infty C_m r^m \cos \{m(\theta - \phi)\}, \end{aligned} \quad (4.1)$$

where C_m is an indeterminate regular function of θ , and $\oint_{r_s}^\infty [\] dr_1$ means the finite part of $\lim_{N \rightarrow \infty} \int_{r_s}^N [\] dr_1$.

We may also have, from Eq. (2.14),

$$\begin{aligned} \oint_{r_s}^\infty \bar{\xi}_0 \log \frac{1}{\rho} r_1 dr_1 - \bar{\psi}_0 \\ = g_0^s + \sum_{n=0}^\infty E_n r^n \cos \{n(\theta - \phi)\}, \end{aligned} \quad (4.2)$$

where E_n is also an indeterminate regular function of θ . We note that $\bar{\xi}_0 = O_s(1)$, so $\bar{f}_0$ is, at most, of order 1 from definition (2.3) (the definition of " O_s " is from Eckhaus⁽⁸⁾). Integrating the above equations from 0 to 2π , we arrive at

$$\begin{aligned} \bar{\xi}_0 = \frac{1}{2\pi} \int_s \left[\sigma_0 \frac{\partial}{\partial n} \log \rho - \mu_0 \left(\gamma + \log \frac{\varepsilon}{2} + \log \rho \right) \right] ds \\ + \frac{\varepsilon}{\pi} \int_0^{2\pi} \oint_{r_s}^\infty \bar{f}_0(r_1) \left(\gamma + \log \frac{\varepsilon}{2} + \log \rho \right) r_1 dr_1 d\theta \\ - \frac{\varepsilon}{2} \sum_{m=0}^\infty [C_m^s \sin m\phi + C_m^c \cos m\phi] r^m. \end{aligned} \quad (4.3)$$

$$\bar{\psi}_0 = \frac{1}{2\pi} \int_0^{2\pi} \oint_{r_s}^\infty \bar{\xi}_0 \log \frac{1}{\rho} r_1 dr_1 d\theta$$

$$\begin{aligned} + \frac{\varepsilon}{2\pi} \int_0^{2\pi} \oint_{r_s}^\infty \bar{\xi}_0 \log \frac{1}{\rho} r_1^2 \cos \theta dr_1 d\theta \\ - \frac{1}{2} \sum_{m=0}^\infty [E_m^s \sin m\phi + E_m^c \cos m\phi] r^m, \end{aligned} \quad (4.4)$$

where C_m^s, C_m^c, E_m^s , and E_m^c are unknown constants.

5. Generalized Oseen Expansion

We consider the generalized Oseen expansion. Since the velocity in this region may be considered to be of $O(1)$, the stream function $\bar{\psi}_0^*$ and the vorticity $\bar{\xi}_0^*$ are expressed by $\bar{\psi}_0^* = \varepsilon \bar{\psi}$ and $\bar{\xi}_0^* = \bar{\xi}/\varepsilon$, and functions $\bar{f}^*$ and F^* by $\bar{f}^* = \bar{f}/\varepsilon^2$ and $F^* = F$, respectively, where $\bar{\psi}^*, \bar{\xi}^*, \bar{f}^*$ and F^* are of $O(1)$ with respect to ε . Applying Theorem 2.3 of Kida⁽⁹⁾ and 11.4 of Watson⁽¹⁰⁾ to Eqs. (2.13) and (2.14), we finally have, from Eqs. (3.9) and (3.10),

$$\begin{aligned} \bar{\xi}_0^* = -\frac{1}{\pi} \int_0^{2\pi} \oint_0^\infty \bar{f}_0^* K_0(\rho^*) R_1 dR_1 d\theta \\ + \frac{1}{\pi} K_0(R) \int_s \mu_0 ds + \frac{1}{2\pi} K_1(R) \int_s \left[\mu_0 r_s \sin \theta \right. \\ \left. - \sigma_0 \frac{\partial}{\partial n} (r_s \sin \theta) \right] ds \sin \phi \\ + \frac{1}{2\pi} K_1(R) \int_s \left[\mu_0 r_s \cos \theta \right. \\ \left. - \sigma_0 \frac{\partial}{\partial n} (r_s \cos \theta) \right] ds \cos \phi \\ + \frac{1}{2} \sum_{m=0}^\infty K_m(R) [D_m^s \sin m\phi + D_m^c \cos m\phi]. \end{aligned} \quad (5.1)$$

$$\begin{aligned} \bar{\psi}_0^* = \frac{1}{2\pi} \int_0^{2\pi} \oint_0^\infty \exp(R_1 \cos \theta) \bar{\xi}_0^* \log \frac{1}{\rho^*} R_1 dR_1 d\theta \\ - H_0 \log R - \frac{1}{2} \sum_{m=0}^\infty [H_m^s \sin m\phi + H_m^c \cos m\phi] R^{-m}, \end{aligned} \quad (5.2)$$

where K_n is the modified Bessel function of n -th order, and D_m^s, D_m^c, H_m^s , and H_m^c are indeterminate constants.

6. Matching of the Stokes and Oseen Expansions

We use the principle of limit matching:

$$\mathbf{T}_r \bar{\mathbf{E}}_k^{(1)} \bar{\xi}_0 \sim \varepsilon E_r^{(1)} \bar{\xi}_0^* \quad \text{for } \varepsilon \rightarrow 0$$

$$\mathbf{T}_r \bar{\mathbf{E}}_k^{(1)} \bar{f}_0 \sim \varepsilon^2 E_r^{(1)} \bar{f}_0^* \quad \text{for } \varepsilon \rightarrow 0$$

$$E_r^{(1)} \bar{\psi}_0^* \sim \varepsilon \mathbf{T}_r \bar{\mathbf{E}}_k^{(1)} \bar{\psi}_0 \quad \text{for } \varepsilon \rightarrow 0,$$

where $\mathbf{T}_r$ is the transformed operator from R to r , and $E_r^{(1)} \bar{\xi}_0^*$ or $\bar{\mathbf{E}}_r^{(1)} \bar{\xi}_0^*$ is the regular expansion operator defined by Eckhaus⁽⁸⁾. Taking into account that $E_k^{(1)} \bar{\xi}_0$ and $E_k^{(1)} \bar{f}_0$ are, at most, of $O(\varepsilon)$ together with log powers of ε , we assume that $\bar{\xi}_0$ and $\bar{f}_0$ are expressed by

$$\bar{\xi}_0 = \sum_{n=1}^\infty \bar{\Theta}_n(r, \phi; \varepsilon) r^{-n} + \varepsilon \bar{\Theta}_0(\varepsilon r, \phi; \varepsilon). \quad (6.1)$$

$$\bar{f}_0 = \sum_{n=2}^\infty \bar{F}_n(r, \phi; \varepsilon) r^{-n} + \varepsilon \bar{F}_1(r, \phi; \varepsilon)/r. \quad (6.2)$$

For $\bar{\xi}_0^*$ and $\bar{f}_0^*$, we also suppose

$$\begin{aligned} \bar{\xi}_0^* = \sum_{n=1}^\infty \bar{\Theta}_n(R, \phi; \varepsilon) R^n + \bar{\Theta}_0(R, \phi; \varepsilon) \\ + \bar{\Theta}_1(R/\varepsilon, \phi; \varepsilon)/R. \end{aligned} \quad (6.3)$$

$$\begin{aligned}\hat{f}_0^* &= \sum_{n=1}^{\infty} \hat{F}_n(R, \phi : \epsilon) R^n + \hat{F}_0(R/\epsilon, \phi : \epsilon) \\ &+ \hat{F}_1(R/\epsilon, \phi : \epsilon)/R + \hat{F}_2(R/\epsilon, \phi : \epsilon)/R^2, \quad (6.4)\end{aligned}$$

where $\hat{\Theta}_n$, $\hat{\Theta}_n$, $\hat{F}_n$, and $\hat{F}_n$ are, at most, of $O(1/\epsilon)$ with respect to ϵ , and they are functions only of $\log r$ or $\log R$. By using these relations, we may determine unknown coefficients by the matching principle. Final expressions of the generalized Oseen and Stokes expansions are given by

$$\begin{aligned}\hat{\xi}_0^* &= (\alpha_s \sin \phi + \alpha_c \cos \phi) K_1(R) \\ &+ \frac{1}{2\pi\epsilon} K_0(R) \int_s \mu_0 ds \\ &- \frac{1}{\pi} \int_0^{2\pi} \int_0^\infty \hat{f}_0^* K_0(\rho^*) R_1 dR_1 d\theta \\ &- \frac{1}{\pi} \int_0^{2\pi} \int_{r_s}^\infty \hat{f}_0 r_1 dr_1 d\theta K_0(R) + O(\epsilon). \quad (6.5)\end{aligned}$$

$$\begin{aligned}\hat{\psi}^* &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \exp(R_1 \cos \theta) \hat{\xi}_0^* \log \frac{1}{\rho^*} R_1 dR_1 d\theta \\ &+ \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \exp(R_1 \cos \theta) \hat{\xi}_0^* \\ &\times \log R_1 R_1 dR_1 d\theta + O(\epsilon). \quad (6.6)\end{aligned}$$

$$\begin{aligned}\hat{\xi} &= \frac{1}{2\pi} \int_s \left[\sigma_0 \frac{\partial}{\partial n} \log \rho \right. \\ &\left. - \mu_0 \left(\gamma + \log \frac{\epsilon}{2} + \log \rho \right) \right] ds + O(\epsilon). \quad (6.7)\end{aligned}$$

$$\begin{aligned}\hat{\psi} &= -r \left[\frac{1}{2} (\alpha_s \sin \phi + \alpha_c \cos \phi) \log \epsilon \right. \\ &- \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \exp(R_1 \cos \theta) \hat{\xi}_0^* \\ &\times \cos(\theta - \phi) dR_1 d\theta \left. \right] \\ &+ \frac{1}{2\pi} \int_0^{2\pi} \int_{r_s}^\infty \hat{\xi}_0 \log \frac{1}{\rho} r_1 dr_1 d\theta + O(\epsilon), \quad (6.8)\end{aligned}$$

where

$$\begin{pmatrix} \alpha_s \\ \alpha_c \end{pmatrix} = \frac{1}{2\pi} \int_s \left[\mu_0 \begin{pmatrix} r_s \sin \theta \\ r_s \cos \theta \end{pmatrix} - \sigma_0 \frac{\partial}{\partial n} \begin{pmatrix} r_s \sin \theta \\ r_s \cos \theta \end{pmatrix} \right] ds.$$

The constants μ_0 and σ_0 are determined from the boundary conditions (2.6). Thus, we can derive a system of integral equations for the generalized Stokes and Oseen expansions for the first time. We must note that these generalized expansions are governed by nonlinear equations. It may be very difficult to solve these equations analytically. Thus, in another series paper by the present authors, asymptotic solutions expanded in powers of $1/\log Re$ will be shown.

7. Conclusions

The present paper treats a well-known classical problem, the low Reynolds flow problem, in which we have some unsolved problems, as pointed out in sec-

tion 1. In the present paper, these problems are discussed from the point of view of asymptotic analysis, by applying the analytical results given by Kida⁽⁹⁾. First, the governing integral expressions are obtained from the vorticity equation, and second, they are rewritten as simpler forms using only one variable. Finally, the integral expressions for the asymptotic expansions with the order function $\delta_m(Re) = Re^m$ (m : integer), that is, the generalized Stokes and Oseen expansions, are derived from them, and it is shown that there is no other significant local region by the application of the asymptotic analysis of the integral operator.

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