Direct Numerical Simulations of a Spatially Developing Plane Wake*

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In the present paper, direct numerical methods by which to simulate the spatially developing free shear flows in the transitional region are described and the numerical results of a spatially developing plane wake are presented. The incompressible time-dependent Navier-Stokes equations were solved using Padé finite difference approximations in the streamwise direction, a mapped pseudospectral Fourier method in the cross-stream direction, and a third-order compact Runge-Kutta scheme for time advancement. The unstable modes of the Orr-Sommerfeld equations were used to perturb the inlet of the wake. Statistical analyses were performed and some numerical results were compared with experimental measurements. When only the fundamental mode is forced, the energy spectra show amplification of the fundamental and its higher harmonics. In this case, unperturbed alternate vortices develop in the saturation region of the wake. The phase jitter around the fundamental frequency plays a critical role in generating vortices of random shape and spacing. Large- and small-scale distortions of the fundamental structure are observed. Pairing of vortices of the same sign is observed, as well as vortex coupling of vortices of the opposite sign.

Key Words: Computational Fluid Dynamics, Wake, Transition, Finite Difference Method, Turbulence Control, Pseudospectral Fourier Method

1. Introduction

Both developing higher-speed supercomputers and extensive numerical algorithm studies have allowed the simulation of the turbulent flows of simple boundary conditions at higher Reynolds numbers by using direct numerical methods of the Navier-Stokes equations with fine resolution. Direct numerical methods based on a pseudospectral orthogonal function such as the Fourier method have provided a means for studying the turbulent viscous flows. Since the numerical solutions are very precise, the fine structures of the turbulent flows are inspected by means of the direct numerical simulations⁽¹⁾. Furthermore, studies of the turbulence statistics using a database compiled from direct numerical simulations have assessed the proposed turbulence models quantitatively^{(2),(3)}.

The recent aeronautical engineering of developing more efficient propulsion systems has prompted a need to increase our understanding of the transition process from laminar to turbulent flows of the free shear flows such as jets and wakes. Investigations⁽⁴⁾ of free shear flows applying direct pseudospectral numerical methods have been important in the better understanding of the transition process.

In the past two decades, the direct numerical method for studying the free shear flows has been primarily applied to two-dimensional time-developing simulations⁽⁵⁾. Since computer resources required for use in the time-developing simulation are fewer than those required for the spatially-developing simulation, and also because inflow/outflow conditions are more favorable, most numerical simulations of mixing layers, jets and wakes have been restricted to addressing the time-developing cases. However, as a result of extensive studies, the well-established numerical algorithm based on the pseudospectral Fourier method to represent the time-developing shear layers, the mapped pseudospectral method⁽⁶⁾, has been produced.

As computer resources available for simulations have increased, three-dimensional time-develop-

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ing⁽⁴⁾⁽⁷⁾, and also two-dimensional spatially developing simulations⁽⁸⁾ have been developed.

The spacially developing simulation more closely mimics the experiments and is considered to be useful for engineering applications. The spacially developing simulations require more computer resources, therefore it is only recently that simulation of free shear flows so as to adequately resolve structures at high Reynolds numbers has been made possible⁽⁹⁾.

In the present study, the mathematical formulation of the spacially developing free shear flows is presented along with the numerical algorithm using the direct numerical method based on the pseudospectral Fourier and finite difference methods. The linear stability theory of the spatial problem is described briefly in this paper, because the free shear flow is forced by unstable modes.

Finally, the simulation results of the transition in a plane wake are reported. The vortical structures of the wake and the enhancement of the transition to the turbulent wake are presented in this paper.

2. Mathematical Formulation of the Problem

Before we detail the main part of the simulation, the formulation of the unstable modes for disturbances should be briefly introduced. The critical Reynolds numbers of the shear flow, such as wakes and mixing layers, are very small, as shown in the linear stability theory. Though Reynolds numbers of the shear flow in engineering are usually much higher than critical Reynolds numbers, the unstable modes solved by the linear theory are adapted in the simulations at high Reynolds numbers. Ho and Huang⁽¹⁰⁾ reported that the growth of subharmonic modes of the linear theory were important to enhance the pairing of vortices in the mixing layers.

2.1 Two linear stability theories

Temporal and spatial linear stability theory equations are solved to simulate the time-developing and spatially developing shear flows respectively. For a truly parallel mean flow, the normal-mode solutions are obtained as follows:

$$[u, v, w, p]^T = [\hat{u}(y), \hat{v}(y), \hat{w}(y), \hat{p}(y)]^T \exp[i(\alpha x + \beta z - \omega t)], \quad (1)$$

where α and β are the x and z components of the wave number vector k (see Fig. 1) and ω is the frequency. Notations $\hat{u}(y)$, $\hat{v}(y)$, $\hat{w}(y)$ and $\hat{p}(y)$ are eigenfunctions, which give the mode structure through the free shear layer. When α and β are real and ω is complex, the disturbance grows in time. If α and β are complex and ω is real, the disturbance grows spatially. The former case is referred to as the "time-amplification theory" and the latter as the "spacial-amplification theory"⁽¹¹⁾.

Since the shear flow at high Reynolds numbers appears in engineering applications, the solution of the inviscid Rayleigh equations shows good approximation as well⁽¹²⁾. In the present study, the solutions of Orr-Sommerfeld equations are used for the disturbances. The eigenvalues of the Orr-Sommerfeld equations for the reference free shear flow at a given Reynolds number show the growth rate. We use the fundamental mode of the maximum growth rate as well as the subharmonic modes.

2.2 Governing equations

The time-dependent incompressible Navier-Stokes equations are solved, which are described in cartesian tensor notation:

$$\frac{\partial u_i}{\partial t} = \epsilon_{ijk} u_j \omega_k - \frac{\partial P}{\partial x_i} + Re^{-1} \nabla^2 u_i, \quad (2)$$

where P is the dynamic pressure, $Re = Ub_{1/2}/\nu$, $b_{1/2}$ represents the half-width of the inlet Gaussian wake profile, and U is the free stream velocity. Conservation of mass for the fluid is expressed by the continuity equation,

$$\frac{\partial u_i}{\partial x_i} = 0. \quad (3)$$

The computational domain starts downstream of the trailing edge of a splitter plate. Figure 1 illustrates the computational setup and the coordinate system used. The x -axis represents the streamwise direction. The y -axis represents the cross-stream direction perpendicular to x .

All quantities are nondimensionalized by the appropriate characteristic scales, U and $b_{1/2}$. Passive scalar transportation is governed by the equation,

$$\frac{\partial \phi}{\partial t} + u_i \frac{\partial \phi}{\partial x_i} = \frac{1}{ReSc} \nabla^2 \phi, \quad (4)$$

where $Sc (= \nu/D)$ represents the Schmidt number. D is the diffusion coefficient for the passive scalar.

A Gaussian profile was chosen for an inlet flow condition, namely, the mean x -component of velocity at the inlet plane is represented by

$$U = 1 - 0.692 \exp(1 - 0.69315 y^2). \quad (5)$$

The coefficient 0.692 in Eq. (5) is used so that the center-line deficit will correspond to that in the exper-

Fig. 1 Coordinate system

iment by Sato and Kuriki (1961)⁽¹³⁾. The other coefficient, 0.69315, is set so that the half-width of the profile is 1 at $y=1$.

A time-dependent advection condition of the form,

$$\frac{\partial u_i}{\partial t} + U_a \frac{\partial u_i}{\partial x} = 0, \quad (6)$$

is invoked for each of the velocity components at the exit plane, where U_a represents the advection speed of the large-scale structures in the layer. This condition⁽¹²⁾ is "soft" at the exit plane. The choice of $U_a = U$ is appropriate for simulations of small-deficit wakes. We use a large computational domain in the x direction, so that we have a small-deficit wake at the exit plane. We observe that the vortical structures pass through the exit plane without deformation, as shown in Fig. 3.

A similar advection condition of the form,

$$\frac{\partial \phi}{\partial t} + U_a \frac{\partial \phi}{\partial x} = 0, \quad (7)$$

is invoked for the passive scalar field at the exit plane.

The velocity components are split into computational variables and "reference" profiles. Specifically,

$$u = u_c + U(y), v = v_c, w = w_c, P = P_c + U^2(y)/2, \\ \phi = \phi_c + \Phi(y), \quad (8)$$

where the computational variables are denoted by the subscript c . This decomposition implies that the boundary conditions at infinity on the computational variables are homogeneous Dirichlet conditions. Mapping in the y -direction is employed to move the points at infinity into the computable finite range.

The Gaussian profile prescribed for the mean u component at the inlet plane is distributed uniformly at all x locations in the domain at $t=0$. These initial conditions must be allowed to wash out before any statistical analysis can be performed on the layer.

2.3 Numerical algorithms

Time-developing simulations employ the pseudospectral Fourier method in the x , y and z directions. Since the pseudospectral method in the x direction is not applicable in spatially developing simulations, we use a finite difference method in this direction. Mapping in the y direction is employed to minimize the effect of the boundary in the y (major gradient) direction. Therefore, the Navier-Stokes equations are solved on a domain that is infinite in the y direction and finite in the x and z directions.

We can consider two numerical algorithms to solve the Navier-Stokes equations. The first is used to solve the equations obtained by eliminating the pressure term, and the second is used to solve the equations with the defining equation for the pressure field.

We describe the first algorithm⁽⁸⁾ of the equations. Pressure is eliminated by computing the curl of the two-dimensional Navier-Stokes equations twice. This yields a fourth order equation for the streamwise velocity u as follows:

$$\frac{\partial \nabla^2 u}{\partial t} = -\frac{\partial}{\partial y} \left(\frac{\partial H_1}{\partial y} - \frac{\partial H_2}{\partial x} \right) + Re^{-1} \nabla^4 u \quad (9a)$$

$$H_1 = \omega v \quad (9b)$$

$$H_2 = -\omega u. \quad (9c)$$

Equation (9) is advanced in time explicitly using a compact third-order Runge-Kutta scheme⁽¹⁴⁾. A fully explicit time-advancement scheme is appropriate to simulate the free shear flow of advection-dominated structures. Since the Laplacian is contained in the time-derivative term, a Poisson equation must be solved at each substep. We use the "compact" explicit scheme which requires only two storage locations per variable per degree derivative. The cross-stream velocity v is directly obtained from the continuity equation. The algorithm is based on a Fourier method with a cotangent mapping in the y direction of the form,

$$y = -a \cot(2\pi\zeta), \quad (10)$$

where a is a stretching parameter for the mapping. Fourier representation of the dependent arguments such as the vorticity in Eq. (9) is the form,

$$\psi = \sum_{k_y = -N_y/2}^{N_y/2-1} \hat{\psi} e^{i2\pi k_y \zeta}. \quad (11)$$

Therefore, with application of the chain rule, the representations of the first and second derivatives with respect to y are approximated in ζ -space via the following operators:

$$D_y[\psi] = \frac{-1}{4a} \sum_{k_y = -N_y/2}^{N_y/2-1} [(k_y-2)\hat{\psi}_{k_y-2} - 2k_y\hat{\psi}_{k_y} \\ + (k_y+2)\hat{\psi}_{k_y+2}] e^{i2\pi k_y \zeta}, \quad (12)$$

$$D_{yy}[\psi] = \frac{-1}{16a^2} \sum_{k_y = -N_y/2}^{N_y/2-1} [(k_y-4)(k_y-2)\hat{\psi}_{k_y-4} \\ - 4(k_y-2)(k_y-1)\hat{\psi}_{k_y-2} + 6k_y^2\hat{\psi}_{k_y} \\ - 4(k_y+1)(k_y+2)\hat{\psi}_{k_y+2} \\ + (k_y+2)(k_y+4)\hat{\psi}_{k_y+4}] e^{i2\pi k_y \zeta}. \quad (13)$$

The second-derivative operator is represented by block pentadiagonal matrix.

High-order accurate Padé approximations are used in the x direction. This finite difference method has 6th-order accurate approximations for the wave components up to the median wave number. The algorithm contains no numerical diffusion, which we believe is important for problems in the transitional and turbulent flow simulations. Furthermore, without numerical diffusion, marginal resolution will usually appear as high wave number oscillations, and is thus easily detected.

On the other hand, we show the second algorithm⁽¹⁵⁾ of the equations with a quasi-time-dependent

Poisson equation for the pressure field. The compact explicit scheme is also used for time advancement. A mass balance is performed at each "cell" of the computational domain. The location of the "cell" center is coincident with the location of v , w and P collocation points. The u and ϕ locate at the boundary of the so-called "staggered cell". Representing the convection term and the viscous term in the momentum equation by E_i , the following equations should be solved at each substep of the Runge-Kutta scheme.

$$\frac{u_i^{n+b_m} - u_i^{n+b_m-1}}{b_m \delta t} = E_i^{n+b_m-1} - D_i[P^{n+b_m}], \quad (14.a)$$

$$E_i = \varepsilon_{ijk} u_j \omega_k + Re^{-1} \nabla^2 u_i, \quad (14.b)$$

where the $(n+b_m)$ subscript reflects the equations at the $(n+b_m)$ th substep of the time-advancement scheme. D_i is a discrete operator ($D_i = d/dx_i$). Multiplying Eq. (14) by D_i , we obtain

$$\frac{D_i u_i^{n+b_m} - D_i u_i^{n+b_m-1}}{b_m \delta t} = D_i E_i^{n+b_m-1} - D_i D_i [P^{n+b_m}]. \quad (15)$$

The pressure is solved by the solenoidal condition of u at each time step, so that the first term of Eq. (15) must be zero. The final discrete form of the Poisson equation for the pressure field is then

$$D_i D_i [P^{n+b_m}] = D_i E_i^{n+b_m-1} + \frac{D_i u_i^{n+b_m-1}}{b_m \delta t}. \quad (16)$$

Since the finite difference scheme is used in the x direction, $D_i D_i$ of the left-hand term is represented by the following tridiagonal structure for the x dependence denoted here by matrix S :

$$S = \begin{bmatrix} [A-I] & I & & & \\ & I & [A-2I] & I & \\ & \ddots & \ddots & \ddots & \\ & & I & [A-2I] & I \\ & & & I & [A-I] \end{bmatrix}, \quad (17)$$

where A represents a block pentadiagonal matrix whose nonzero components are of the form

$$\left. \begin{aligned} a_{k,k-4} &= -(k-4)(k-2)/16a^2, \\ a_{k,k-2} &= (k-2)(k-1)/4a^2, \\ a_{k,k} &= -3k^2/8a^2, \\ a_{k,k+2} &= (k+1)(k+2)/4a^2, \\ a_{k,k+4} &= -(k+2)(k+4)/16a^2, \end{aligned} \right\} \quad (18)$$

where k in Eq. (18) is k_y . I is an identity matrix.

In the three-dimensional algorithm where spanwise dependence is included, the discrete representation of $\partial^2/\partial z^2$ resides solely in the diagonal elements of the A matrix. As a result, we have to solve as many S matrices as there are N_z modes and, due to the Fourier decomposition in the z direction, each S matrix is effectively decoupled from the others.

A flowchart of two cases is shown in Fig. 2.

3. Numerical Results and Discussions

3.1 Growth of the fundamental mode and the wake structure

We point out that for the Gaussian profile and Reynolds numbers considered in this study, a steady laminar wake solution is obtained with no forcing. A small-deficit wake profile approximates a Gaussian profile of the analytical laminar solution with high accuracy. Consequently, it is verified that the disturbances generated in the computational domain are due to the forcing, not the boundary conditions, the algorithm or the coding.

At $Re=600$, 768 grid points were used in the x direction over $0 \leq x \leq 200$, and 128 Fourier modes in the y direction.

Figure 3 shows the wake structure forced by the fundamental mode (the amplitude is 1% of the free stream) with isovorticity contour plots. The line spectrum at the fundamental frequency and its higher harmonics can be seen in Fig. 4, which is consistent with the experiments⁽¹⁶⁾. Figure 4 shows the energy spectra at $x=50$ and $y=2.1$. We observe that the energy at the fundamental frequency is higher than

Fig. 2 Flowchart of numerical algorithm

those of other modes. The growth of the mode is important to create the wake structures. The fundamental mode grows rapidly before the appearance of the alternate vortices, then saturates and decreases downstream⁽¹⁷⁾. These results show that the spatial evolution of the fundamental frequency is responsible for the generation of a Kármán vortex street.

On the other hand, the mean velocity profiles upon appearance of the vortex street indicate small-deficits wake, as shown in Fig. 5. We also observe in the same velocity profiles that the mean velocity at the outer edge exceeds the free stream velocity. These two characteristics of the velocity profile are consistent with the measurements⁽¹³⁾. This simulation shows good agreement with the experiment. Furthermore, the numerical results reveal that the distribution of uv changes sign⁽¹⁸⁾ upon the appearance of the vortex street. The experiments⁽¹³⁾ of the transitional wake show that the fluctuations of u and v are much larger

Fig. 3 Vorticity contours of the flow forced by the fundamental mode

Fig. 4 Energy spectrum of v of the flow forced by the fundamental mode ($x=50$, $y=2.1$); fundamental frequency is $\omega=0.608$

than the w fluctuation in the near wake. The w fluctuation grows gradually after the saturation of the u and v fluctuations. Since the two-dimensional fluctuations are dominant in the near wake and the three-dimensionalization is gradual, we conclude that the structure of the near wake is similar to the Kármán vortex structure shown by the simulation.

3.2 Effect of phase jittering

In this section, we will discuss how to enhance the wake to develop the turbulent flow. When the amplitude of the disturbances is 1% of the free stream and the phases between the disturbances are random, the structures are more complicated downstream than those of the wake forced by the fundamental mode. Figure 6 illustrates this case. Vortical structures of random shape and spacing are seen in Fig. 6, where the complicated vortex street rather than the Kármán vortex street can be observed downstream. We can observe the pairing of vortices of the same sign and the coupling of vortices of the different signs. The energy spectra show that low-frequency modes are more dominant than the fundamental mode.

Since the real flows of the complicated structures downstream are three-dimensional, the pairing and coupling processes shown in the simulation are computational physics: however, we think that the phase jittering effectively deforms the vortex street.

Fig. 5 Distribution of the mean velocity profiles and uv fluctuation of the flow forced by the fundamental mode:

Fig. 6 Vorticity contours for case of phase jittering at $Re = 600$

Fig. 7 Energy spectra of u for case of phase jittering at $Re = 600$ ($x = 175$, $y = -1.7$)

For more details on the numerical results, the reader is referred to Ref. (17).

4. Conclusions

We presented a numerical algorithm of incompressible Navier-Stokes equations to represent spatially developing shear flows and the numerical results of the wake. The method employs the finite difference method in the streamwise direction and the mapped pseudospectral method in the transversal direction. The numerical results show good agreement with the experiments, while some statistics are consistent with the experimental measurements of the transitional wake flow. The plane wake has alternate vortical structures. We found that phase jittering is effective in enhancing the wake to develop to the turbulent flow. We anticipate that the three-dimensional process of

the wake structures will be shown by direct numerical simulations in the future.

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References

- (1) Kim, J., Moin, P. and Moser, R., *J. Fluid Mech.*, Vol. 177(1987), p. 133.
- (2) Mansour, N., Kim, J. and Moin, P., *J. Fluid Mech.*, Vol. 194(1988), p. 15.
- (3) Mansour, N., Kim, J. and Moin, P., *AIAA J.*, Vol. 27, No. 8 (1989), p. 1068.
- (4) Chen, J., Cantwell, J. & Mansour, N., *Phys. Fluids*, A 2-6(1990), p. 984.
- (5) Patnaik, P. C., Corcos, G. M. and Sherman, F. S., *J. Fluid Mech.*, Vol. 73 (1976), p. 215.
- (6) Cain, A., Ferziger, J. and Reynolds, W. C., *J. Comput. Phys.*, Vol. 56 (1984), p. 272.
- (7) Metcalfe, R. W., Orszag, S. A., Brachet, M. C., Menon, S. and Riley, J. J., *J. Fluid Mech.* Vol. 184 (1987), p. 225.
- (8) Maekawa, H., Mansour, N. N. and Buell, N. J., *Nume. Methods Fluid Dyn.*, Vol. 2 (1990), p. 1000.
- (9) Buell, J. N. and Mansour, N. N., *Bull. Am. Phys. Soc.*, Vol. 34, No. 10, (1989), p. 2281.
- (10) Ho, C. and Huang, L., *J. Fluid Mech.*, Vol. 119 (1982), p. 443.
- (11) Mack, L., AGARD-R, 709(1984), p. 3-1.
- (12) Davis, W. and Moore, E. F., *Phys. Fluids* Vol. 28, No. 6. (1985), p. 1626.
- (13) Sato, H. and Kuriki, K., *J. Fluid Mech.* Vol. 11, (1961), p. 321.
- (14) Wary, A. A., *J. Comput. Phys.*, (1991) (accepted for publication).
- (15) Maekawa, H. and Mansour, N. N., *Proc. 4th Asian Cong. Fluid Mech.* Vol. 2 (1989), p. A 241.
- (16) Sato, H. and Saito, H., *J. Fluid Mech.* Vol. 67 (1975), p. 539.
- (17) Maekawa, H., Mansour, N. N. and Buell, J. C., *J. Fluid Mech.* Vol. 235 (1992), p. 223.
- (18) Sato, H. and Onda, Y., *Inst. Space Aero. Sci., Univ. Tokyo, Rep. No. 453(1970)*, p. 317.