An Unsteady Implicit SMAC Scheme for Two-Dimensional Incompressible Navier-Stokes Equations*

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A finite-difference method based on the SMAC (Simplified Marker and Cell) scheme for analyzing two-dimensional unsteady incompressible viscous flows is developed. The fundamental equations are the incompressible Navier-Stokes equations of contravariant velocities and the elliptic pressure equation in general curvilinear coordinates. With application of the Crank-Nicholson scheme, unsteady flow is calculated iteratively by the Newton method at each time step, and the elliptic pressure equation is solved by the Tschebyscheff SLOR method with alternating the computational directions. Therefore, the elliptic character of incompressible flow is well described. The present implicit scheme is stable under the proper boundary conditions, since spurious error and numerical instabilities can be suppressed by employing the staggered grid and upstream differences such as the modified QUICK scheme. Numerical results for two-dimensional flows through a decelerating cascade at high Reynolds numbers are shown. Some computed results of the surface pressure coefficient are in satisfactory agreement with the experimental data.

Key Words: Numerical Analysis, CFD, Incompressible Navier-Stokes Equations, Unsteady Flow, Finite-Difference Method, Implicit Scheme, Curvilinear Coordinates, Cascade Flow

1. Introduction

Much interest in computational fluid dynamics is focused on the simulation of practical flows in complex geometry such as turbomachinery flow. These flows may have unsteady flow phenomena including flow separation, recirculation, vortex shedding, three-dimensionality and turbulence. It is a formidable task to simulate these practical flows accurately.

A number of numerical methods for simulating unsteady incompressible viscous flow have been developed. Ghia et al.⁽¹⁾ developed an unsteady direct solution method which is formulated using the vorticity and the streamfunction in generalized curvilinear orthogonal coordinates. The alternating direction

implicit method and a block Gaussian elimination technique are used to solve the vorticity transport equation and the streamfunction equation, respectively. This method was applied successfully to the unsteady incompressible flow over a backward facing step. Also, Goodrich⁽²⁾ presented an algorithm for unsteady two-dimensional (2-D) incompressible Navier-Stokes calculations of high Reynolds numbers using the streamfunction as the only dependent variable. In this algorithm a multigrid solver is applied at each time step to reduce the computing time. On the other hand, a primitive variable (u, p) solution method using cartesian velocity components was presented by Rao et al.⁽³⁾ In this method, the third-order upwind scheme⁽⁴⁾ for the convection term is adopted to overcome numerical instability in high-Reynolds number computations, and a moving grid system is used to simulate vortex flows behind the 2-D oscillating circular cylinder. Recently, Rosenfeld et al.⁽⁵⁾ developed a fractional step solution method for determining the three-dimensional viscous flow in a generalized coordinate system. The governing equations were formulated using the pressure and the volume fluxes instead of cartesian components of the velocity as the

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dependent variables, and discretized by finite volumes using a staggered grid system. Methods for artificial compressibility have also been developed as a sophisticated approach using the primitive variable system. These methods have been used only in the case of steady-state flow, since they are based on the numerical scheme for a hyperbolic system of equations, so that the disturbance at one point does not spread throughout the flow field instantaneously. These methods, however, have been extended by Rogers and Kwak⁽⁶⁾ to unsteady flow by the introduction of the second-order backward-difference scheme and the Newton iteration with dual-time stepping to restore the time accuracy of the solution. To obtain accurate results for large fluctuating flow, it seems to be necessary to use a high artificial speed of sound and a small time step.

Among these methods, that with the velocity and pressure as the unknown variables is of great benefit to the practical calculation of the turbulent flow, because the conventional turbulence models can be used directly. The most widely used schemes belonging to this method are the MAC (Marker and Cell)-type schemes. In the MAC scheme⁽⁷⁾, in order to satisfy the continuity condition and to remove spurious error, an additional term is introduced in the difference equation of the Poisson equation of pressure, and a staggered grid is used. Improvements have been made in the MAC scheme, yielding the fractional step MAC scheme⁽⁸⁾, SMAC scheme⁽⁹⁾, SIMPLE (Semi-Implicit Method for Pressure Linked Equations) scheme⁽¹⁰⁾, etc., in which the scheme has been simplified and computing time has been reduced. Recently, the authors proposed an implicit SMAC scheme using a curvilinear coordinate grid for steady high-Reynolds number flow⁽¹¹⁾.

The purpose of this paper are to extend this implicit SMAC scheme to unsteady flow, and to apply it to unsteady 2-D cascade flow problems. The extension to the unsteady scheme is attained here using the second-order Crank-Nicholson method and calculation with the Newton iteration at each time step. For the flow which does not change rapidly with time, this implicit scheme can reduce the total computer time compared to the explicit one, even though 3 to 5 time subiterations are necessary, because the time increment Δt can be taken to be sufficiently large.

In this paper we first describe the incompressible Navier-Stokes equations expressed in terms of contravariant velocities, and the SMAC scheme generalized to these equations. Next, an implicit method based on the Crank-Nicholson method is explained for a simple one-dimensional scalar equation. Then, with application of this implicit method to the

generalized SMAC scheme, an implicit scheme for the unsteady 2-D flow is proposed. Finally, some applications of this implicit scheme to unsteady flows through a two-dimensional cascade are shown.

Nomenclature

- C : blade chord length
- C_p : pressure coefficient
- g_{ij}, h_{ij} : metric tensor, Eq. (3)
- J : Jacobian, Eq. (3)
- p : static pressure
- Re : Reynolds number
- U_i : contravariant velocity components, Eq. (4)
- u, u_i : physical velocity and its components
- x, x_i : cartesian coordinate and its components
- Z_i : contravariant vorticity components, Eq. (5)
- Δt : time step
- ϵ_{ijk} : permutation tensor
- ζ, ζ_i : physical vorticity and its components
- ν : kinematic viscosity
- ξ, ξ_i : general curvilinear coordinate and its components
- ϕ : pressure correction
- Superscripts
 - m : m -th approximation of Newton iteration
 - n : n -th time level
 - $*$: intermediate time level

2. Fundamental Equations and SMAC Scheme

The fundamental equations of unsteady incompressible viscous flow can be written in conservative form in general curvilinear coordinates as⁽¹¹⁾

$$\frac{\partial}{\partial t}(JU_i) + \frac{\partial}{\partial \xi_i}(JU_i U_i) - \mathbf{u} \cdot \mathbf{J} \left(U_i \frac{\partial}{\partial \xi_i} \right) \nabla \xi_i = -Jg_{ii} \frac{\partial p}{\partial \xi_i} - \nu \epsilon_{ijl} \frac{\partial}{\partial \xi_i} h_{jl} Z_l \quad (l=1, 2, 3) \quad (1)$$

$$\frac{\partial}{\partial \xi_i}(JU_i) = 0, \quad (2)$$

where ϵ_{ijl} is the permutation tensor, and, for example, the vorticity component $\zeta_l = \epsilon_{lij} \partial u_j / \partial x_i$. The Jacobian J and the metrics g_{ij} and h_{ij} of the transformation are

$$\left. \begin{aligned} J &= \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \\ g_{ij} &= \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_j}{\partial x_k}, \quad h_{ij} = \frac{\partial x_k}{\partial \xi_i} \frac{\partial x_k}{\partial \xi_j} \quad (i, j=1, 2, 3) \end{aligned} \right\} \quad (3)$$

The contravariant velocities (U_1, U_2, U_3) are defined as

$$U_i = (\nabla \xi_i) \cdot \mathbf{u}, \quad (u_j = (\partial x_j / \partial \xi_i) U_i) \quad (4)$$

using the summation convention, and similarly, the contravariant vorticities (Z_1, Z_2, Z_3) are defined as

$$Z_i = (\nabla \xi_i) \cdot \boldsymbol{\zeta}, \quad (\zeta_j = (\partial x_j / \partial \xi_i) Z_i). \quad (5)$$

The third term on the left-hand side of Eq. (1) is an

additional term introduced to express the convection term in conservative form, and the final term on the right-hand side is the viscous diffusion term expressed in the simplest form using the contravariant vorticities.

The SMAC scheme for the fundamental equations (1) and (2) in general curvilinear coordinates can be derived as follows. First, applying in general the forward Euler method to Eq. (1) but the implicit Euler method to the pressure term, and then the fractional step (time splitting) method, we obtain

$$JU_i^* = JU_i^n - \Delta t \left\{ \frac{\partial}{\partial \xi_i} (JU_i U_i) + Jg_{ii} \frac{\partial p}{\partial \xi_i} - \mathbf{u} \cdot \mathbf{J} \left(U_i \frac{\partial}{\partial \xi_i} \right) \nabla \xi_i + \nu \epsilon_{ij} \frac{\partial}{\partial \xi_i} h_{jk} Z_k \right\}^n \quad (6)$$

$$JU_i^{n+1} = JU_i^* - \Delta t Jg_{ii} \frac{\partial \phi}{\partial \xi_i}. \quad (7)$$

Then, taking the divergence of Eq. (7) and using Eq. (2) for JU_i^{n+1} , we obtain the elliptic equation of ϕ

$$\frac{\partial}{\partial \xi_i} \left(Jg_{ii} \frac{\partial \phi}{\partial \xi_i} \right) = \frac{1}{\Delta t} \frac{\partial}{\partial \xi_i} (JU_i^*) \quad (8)$$

$$p^{n+1} = p^n + \phi, \quad (9)$$

where the asterisk * denotes the intermediate time level, and ϕ is the pressure increment.

In this scheme, the dependent variables JU_i and p are defined at different points of the staggered square grid in ξ -space as shown in Fig. 1. The use of such a staggered grid is advantageous in that spurious errors are completely removed as in the SMAC scheme. The computational procedure for the advancing solution with a time increment Δt is as follows.

I. Determine JU_i^* from Eq. (6)

II. Compute ϕ from the boundary value problem of Eq. (8)

III. Determine JU_i^{n+1} from Eq. (7)

This boundary value problem is usually solved by means of the iterative calculation of simultaneous linear equations. If the solution ϕ converges sufficiently, then the advanced values JU_i^{n+1} sufficiently satisfy

Fig. 1 2-D staggered grid and definition points of variables for cell in ξ -space

the continuity condition.

In the generalized MAC-type schemes using the curvilinear coordinate grid, the contravariant and physical velocity components⁽¹²⁾, the volume fluxes^{(5),(13)}, the contravariant physical velocity components⁽¹⁴⁾ or the covariant physical velocity components⁽¹⁵⁾ are used as unknown variables. These physical velocity components have the same magnitude as the physical velocity, and are not as strongly dependent on the nonuniformity of grid spacing. In the present scheme we use the volume fluxes JU_i as the dependent variables, which are rewritten in contravariant velocities for JU as

$$JU = J\mathbf{u} \cdot \nabla \xi = \mathbf{u} \cdot \mathbf{x}_\eta \times \mathbf{x}_\zeta.$$

That is, JU represents the flow rate through the ξ =const. cell side divided by the cell side area. Therefore, the values of JU are independent of the grid spacing in the ξ -direction, but depend directly on the grid spacing in the η - and ζ -directions.

3. Implicit SMAC Scheme for Unsteady 2-D Incompressible Flow

The explicit SMAC scheme thus far described is applicable not only to steady flow but also to the unsteady flow. By the use of the Adams-Bashforth method instead of the forward Euler method, it is possible to improve the accuracy in time. However, in these explicit schemes the time increment Δt , that is, the Courant number, is strictly limited. Here the implicit scheme developed by the authors⁽¹¹⁾ is extended to unsteady flow.

We first describe the time integration of the present scheme for the one-dimensional scalar equation.

$$u_t + f_x = 0, \quad (10)$$

where $f(u) = au$. In this scheme the implicit finite-difference equation of Eq. (10), that is,

$$u^{(m)} - u^n + \frac{1}{2} \Delta t [(f_x)^n + (f_x)^{(m-1)}] + \Delta t \theta (\Delta f_x)^{(m)} = 0, \quad (11)$$

is solved by the Newton iteration, where $u^{(m)} = u^{(m-1)} + \Delta u^{(m)}$, $f^n = f(u^n)$, $\Delta f^{(m)} = f^{(m)} - f^{(m-1)}$, and the superscript (m) denotes the m -th approximation of u^{n+1} . Rewriting Eq. (11) into the delta form, we obtain

$$\left(1 + \Delta t \theta a \frac{\partial}{\partial x} \right) \Delta u^{(m)} = -(u^{(m-1)} - u^n) - \frac{1}{2} \Delta t [(f_x)^n + (f_x)^{(m-1)}]. \quad (12)$$

For $\theta \geq 1$, this implicit form of the equation is TVD stable irrespective of the Courant number. If $m=1$, then $u^{(0)} = u^n$, and Eq. (12) for $\theta=1$ becomes the implicit Euler method as

$$(1 + \Delta t a \partial / \partial x) \Delta u^{(1)} = -\Delta t (f_x)^n.$$

On the other hand, when the solution converges, then

$u^{(m)} = u^{n+1}$, $\Delta u^{(m)} = 0$, and Eq. (12) is always reduced to the second-order Crank-Nicholson method.

$$u^{n+1} - u^n = -\frac{1}{2} \Delta t [(f_x)^n + (f_x)^{n+1}]$$

We next propose an implicit SMAC scheme for unsteady two-dimensional flow. Applying Eq. (12) to Eq. (1), and then the fractional step method, and further, the approximate-factorization method⁽¹⁶⁾, we obtain

$$\left\{ 1 + \Delta t \left(\frac{\partial}{\partial \xi} JU^n - \nu \frac{\partial}{\partial \xi} \frac{1}{J} \frac{\partial}{\partial \xi} h_{22} \right) \frac{1}{J} \right\} \cdot \left\{ 1 + \Delta t \left(\frac{\partial}{\partial \eta} JV^n - \nu \frac{\partial}{\partial \eta} \frac{1}{J} \frac{\partial}{\partial \eta} h_{11} \right) \frac{1}{J} \right\} \Delta JU^{*(m)} = RHS_1^{(m-1)} \quad (13 \cdot a)$$

$$\left\{ 1 + \Delta t \left(\frac{\partial}{\partial \xi} JU^n - \nu \frac{\partial}{\partial \xi} \frac{1}{J} \frac{\partial}{\partial \xi} h_{22} \right) \frac{1}{J} \right\} \cdot \left\{ 1 + \Delta t \left(\frac{\partial}{\partial \eta} JV^n - \nu \frac{\partial}{\partial \eta} \frac{1}{J} \frac{\partial}{\partial \eta} h_{11} \right) \frac{1}{J} \right\} \Delta JV^{*(m)} = RHS_2^{(m-1)} \quad (13 \cdot b)$$

where

$$\left. \begin{aligned} RHS_1^{(0)} &= -\Delta t \left\{ F_1 - \mathbf{u} \cdot \left(JU \frac{\partial}{\partial \xi} + JV \frac{\partial}{\partial \eta} \right) \nabla \xi \right\}^n \\ RHS_1^{(m)} &= RHS_1^{(m-1)} + \Delta RHS_1^{(m)} \\ \Delta RHS_1^{(m)} &= -(JU_l^{(m)} - JU_l^{(m-1)}) \\ &\quad - \frac{1}{2} \Delta t (F_1^{(m)} - F_1^{(m-1)}), \quad (l=1, 2, m \geq 1) \end{aligned} \right\} \quad (14)$$

$$\left. \begin{aligned} F_1 &= \frac{\partial}{\partial \xi} JU^2 + \frac{\partial}{\partial \eta} JUV + \nu \zeta_\eta + Jg_{11} p_\xi + Jg_{12} p_\eta \\ F_2 &= \frac{\partial}{\partial \xi} JUV + \frac{\partial}{\partial \eta} JV^2 - \nu \zeta_\xi + Jg_{21} p_\xi + Jg_{22} p_\eta \\ \zeta &= \frac{1}{J} \left\{ \frac{\partial}{\partial \xi} (h_{21} U + h_{22} V) - \frac{\partial}{\partial \eta} (h_{11} U + h_{12} V) \right\} \end{aligned} \right\} \quad (15)$$

and

$$\left. \begin{aligned} JU^{(m)} &= JU^{*(m)} - \frac{1}{2} \Delta t (Jg_{11} \phi_\xi^{(m)} + Jg_{12} \phi_\eta^{(m)}) \\ JV^{(m)} &= JV^{*(m)} - \frac{1}{2} \Delta t (Jg_{21} \phi_\xi^{(m)} + Jg_{22} \phi_\eta^{(m)}) \end{aligned} \right\} \quad (17)$$

where

$$\left. \begin{aligned} JU_l^{*(m)} &= JU_l^{(m-1)} + \Delta JU_l^{*(m)} \quad (l=1, 2) \\ p^{(m)} &= p^{(m-1)} + \phi^{(m)} \end{aligned} \right\} \quad (18)$$

The additional term of Eq. (1) is considered only on the right-hand side of Eq. (13) and is treated simply, because it does not change as much as the other terms. The viscous term is considered only the main component on the left-hand side. These terms are necessary in order to increase the convergency and obtain a reasonable solution.

The equation of the pressure correction $\phi^{(m)}$ is derived from Eq. (7) as

$$\begin{aligned} &\frac{\partial}{\partial \xi} (Jg_{11} \phi_\xi^{(m)} + Jg_{12} \phi_\eta^{(m)}) + \frac{\partial}{\partial \eta} (Jg_{21} \phi_\xi^{(m)} + Jg_{22} \phi_\eta^{(m)}) \\ &= \frac{2}{\Delta t} \left(\frac{\partial}{\partial \xi} JU^{*(m)} + \frac{\partial}{\partial \eta} JV^{*(m)} \right). \end{aligned} \quad (19)$$

In this implicit SMAC scheme for unsteady flow, the

solutions at each time step JU_l^{n+1} and p^{n+1} are obtained from Eqs. (13) to (19) of the Newton iteration. The right-hand sides of Eqs. (13) and (19) represent the residuals of Eqs. (1) and (2), respectively, and $\Delta JU_l^{*(m)}$ and $\phi^{(m)}$ of the left-hand sides are the corrections of JU_l and p at each iteration. If the values of these residuals are close to zero, then the corrections also approach zero and the solution converges. The accuracy of the solution only depends on the right-hand sides, and is independent of the left-hand sides. Therefore, for the convection terms of the RHS of Eq. (13) the second- or third-order upstream difference with third-order Lagrangian interpolation should be used, while for the LHS, the first-order upstream difference with linear interpolation can be used.

The computational procedure of the present implicit SMAC scheme is as follows.

- I. $n = n+1$, $m=0$, compute $RHS_1^{(0)}$.
- II. $m = m+1$.
- III. Compute $\Delta JU_l^{*(m)}$ from Eq. (13) using the Gaussian elimination, determine $JU_l^{*(m)}$.
- IV. Compute $\phi^{(m)}$ from Eq. (19) using the Tschebysheff SLOR method, determine $p^{(m)}$.
- V. Compute $JU_l^{(m)}$ from Eq. (17).
- VI. Compute $RHS_1^{(m)}$, if $\phi^{(m)}$ and $RHS_1^{(m)}$ are sufficiently small, GO TO I.
- VII. GO TO II.

The solution at each time t^{n+1} usually converges within a few subiterations.

4. Finite-Difference Equations and Boundary Conditions

Let subscripts U , V , P and X show the points where JU , JV , p and x are defined, respectively. The finite-difference approximation of Eq. (15) is

$$\begin{aligned} (F_1)_{0,0U} &= \{ (J\tilde{U}^2)_{0,0P} - (J\tilde{U}^2)_{-1,0P} \\ &\quad + (J\tilde{U}V)_{0,1X} - (J\tilde{U}V)_{0,0X} \\ &\quad + \nu (\zeta_{0,1} - \zeta_{0,0}) + (\tilde{g}_{11})_{0,0} (p_{0,0} - p_{-1,0}) \\ &\quad + (\tilde{g}_{12})_{0,0} (p_{0,1X} - p_{0,0X}) \} / h \end{aligned} \quad (20)$$

where $\tilde{g}_{ij} = Jg_{ij}$, $h = \Delta \xi (= \Delta \eta)$ and $p_{0,0X} = (p_{-1,-1} + p_{0,-1} + p_{-1,0} + p_{0,0})/4$. If we use the modified QUICK scheme, then

$$\begin{aligned} (J\tilde{U}^2)_{0,0P} &= JU_{0,0P} \{ -(1+\alpha)U_{-1,0} + (9+3\alpha)U_{0,0} \\ &\quad + (9-3\alpha)U_{1,0} - (1-\alpha)U_{2,0} \} / 16 \\ (J\tilde{U}V)_{0,0X} &= JV_{0,0X} \{ -(1+\beta)U_{0,-2} + (9+3\beta)U_{0,-1} \\ &\quad + (9-3\beta)U_{0,0} - (1-\beta)U_{0,1} \} / 16 \end{aligned} \quad (21)$$

where $\alpha = 4 \text{ sign } (JU_{0,0P})$, and $\beta = 4 \text{ sign } (JV_{0,0X})$. The values of $JU_{0,0P}$ and $JV_{0,0X}$ are evaluated using third-order Lagrangian interpolation. Equation (21) becomes the QUICK scheme⁽¹⁷⁾ for $\alpha = \beta = \pm 1$, and is formed using the character such that JU and JV do

not change largely with respect to ξ and η , respectively. The finite-difference approximation of Eq. (16) is

$$\zeta_{0,0} = \{(\tilde{h}_{21}JU + \tilde{h}_{22}JV)_{0,0V} - (\tilde{h}_{21}JU + \tilde{h}_{22}JV)_{-1,0V} - (\tilde{h}_{11}JU + \tilde{h}_{12}JV)_{0,0V} + (\tilde{h}_{11}JU + \tilde{h}_{12}JV)_{0,-1V}\} / hJ_{0,0} \quad (22)$$

where $\tilde{h}_{ij} = h_{ij}/J$, and $JU_{0,0V} = (JU_{0,-1} + JU_{1,-1} + JU_{0,0} + JU_{1,0})/4$.

Equation (13•a) is solved by dividing it into two steps, and the finite-difference equations are

$$\left. \begin{aligned} & -\{\lambda U_{1,0}^+ + \tilde{\nu}(\tilde{h}_{22})_{-1,0V}\} \Delta JU_{-1,0}^{** (m)} \\ & + \{1 + \lambda(U_{0,0}^+ - U_{0,0}^-) + 2\tilde{\nu}(\tilde{h}_{22})_{0,0V}\} \Delta JU_{0,0}^{** (m)} \\ & + \{\lambda U_{1,0}^- - \tilde{\nu}(\tilde{h}_{22})_{1,0V}\} \Delta JU_{1,0}^{** (m)} = (RHS_1)_{0,0}^{(m-1)} \end{aligned} \right\} \quad (23 \cdot a)$$

$$\left. \begin{aligned} & -\{\lambda V_{0,-1}^+ + \tilde{\nu}(\tilde{h}_{11})_{-1,0V}\} \Delta JU_{0,-1}^{* (m)} \\ & + \{1 + \lambda(V_{0,0}^+ - V_{0,0}^-) + 2\tilde{\nu}(\tilde{h}_{11})_{0,0V}\} \Delta JU_{0,0}^{* (m)} \\ & + \{\lambda V_{0,1}^- - \tilde{\nu}(\tilde{h}_{11})_{0,1V}\} \Delta JU_{0,1}^{* (m)} = \Delta JU_{0,0}^{** (m)} \end{aligned} \right\}, \quad (23 \cdot b)$$

where $\lambda = \Delta t/h$, $\tilde{\nu} = \nu \Delta t/h^2$, $C^\pm = (C \pm |C|)/2$ and $\tilde{h}_{ij} = h_{ij}/J^2$. The simultaneous linear equations of Eq. (23) have a diagonally dominant tridiagonal matrix, and can be solved easily by the Gaussian elimination. The finite-difference equations of Eq. (17) are

$$\begin{aligned} JU_{0,0} &= JU_{0,0}^* - \lambda \{(\tilde{g}_{11})_{1,0}(\phi_{0,0} - \phi_{-1,0}) \\ &+ (\tilde{g}_{12})_{0,0}(\phi_{0,1} - \phi_{0,0})\} / 2 \\ JV_{0,0} &= JV_{0,0}^* - \lambda \{(\tilde{g}_{22})_{0,0}(\phi_{0,0} - \phi_{0,-1}) \\ &+ (\tilde{g}_{21})_{0,0}(\phi_{1,0} - \phi_{0,0})\} / 2. \end{aligned} \quad (24)$$

The finite-difference equation of Eq. (19) is derived by substituting Eq. (24) into the equation,

$$(JU)_{0,0}^{(m)} - (JU)_{0,0}^{(m-1)} + (JV)_{0,0}^{(m)} - (JV)_{0,0}^{(m-1)} = 0, \quad (25)$$

of Eq. (2); that is,

$$\begin{aligned} & \{(\tilde{g}_{11})_{1,0}(\phi_{1,0} - \phi_{0,0}) \\ &+ (\tilde{g}_{12})_{1,0}(\phi_{0,1} + \phi_{1,1} - \phi_{0,-1} - \phi_{1,-1})/4 \\ &- \{(\tilde{g}_{11})_{0,0}(\phi_{0,0} - \phi_{-1,0}) \\ &+ (\tilde{g}_{12})_{0,0}(\phi_{-1,1} + \phi_{0,1} - \phi_{-1,-1} - \phi_{0,-1})/4 \\ &+ \{(\tilde{g}_{22})_{0,1}(\phi_{0,1} - \phi_{0,0}) \\ &+ (\tilde{g}_{21})_{0,1}(\phi_{1,0} + \phi_{1,1} - \phi_{-1,0} - \phi_{-1,1})/4 \\ &- \{(\tilde{g}_{22})_{0,0}(\phi_{0,0} - \phi_{0,-1}) \\ &+ (\tilde{g}_{21})_{0,0}(\phi_{1,-1} + \phi_{1,0} - \phi_{-1,-1} - \phi_{-1,0})/4\} \\ &= 2(JU_{1,0}^* - JU_{0,0}^* + JV_{0,1}^* - JV_{0,0}^*)/\lambda. \end{aligned} \quad (26)$$

Implicit boundary conditions are used for all of the boundaries, which makes possible the use of large time steps, particularly in the steady flow simulation. On the solid wall boundary, the nonslip condition $U = V = 0$ is prescribed. The inlet velocity profile and constant pressure are applied at the upstream and downstream boundaries, respectively. In the case of simulation of cascade flows, periodic conditions are imposed on the cyclic boundaries in the upstream and the downstream regions. The pressure equation near the boundary where the values of velocity are prescribed must be modified as follows. For example, at the upstream boundary $\xi = \text{const.}$, using $JU_{0,0}^{n+1}$ instead of the $JU_{0,0}^{* (m)}$, we obtain

$$\begin{aligned} & \{(\tilde{g}_{11})_{1,0}(\phi_{1,0} - \phi_{0,0}) \\ &+ (\tilde{g}_{12})_{1,0}(\phi_{0,1} + \phi_{1,1} - \phi_{0,-1} - \phi_{1,-1})/4 \\ &+ \{(\tilde{g}_{22})_{0,1}(\phi_{0,1} - \phi_{0,0}) \\ &+ (\tilde{g}_{21})_{0,1}(-3\phi_{0,0} - 3\phi_{0,1} \\ &+ 4\phi_{1,0} + 4\phi_{1,1} - \phi_{2,0} - \phi_{2,1})/4\} \\ &- \{(\tilde{g}_{22})_{0,0}(\phi_{0,0} - \phi_{0,-1}) \\ &+ (\tilde{g}_{21})_{0,0}(-3\phi_{0,-1} - 3\phi_{0,0} + 4\phi_{1,-1} \\ &+ 4\phi_{1,0} - \phi_{2,-1} - \phi_{2,0})/4\} \\ &= 2(JU_{1,0}^* - JU_{0,0}^{n+1} + JV_{0,1}^* - JV_{0,0}^*)/\lambda. \end{aligned} \quad (27)$$

The simultaneous linear equations (26) and (27) are solved here by the Tschebyscheff SLOR method, and the convergency can be accelerated by alternating the line direction.

5. Applications to Unsteady Cascade Flow Problems

A numerical simulation of unsteady 2-D decelerating cascade flow was performed using the present unsteady implicit scheme. The blade section in a cascade with a pitch-chord ratio of 0.667 is NACA 65 (12) 10. The H-shaped grid shown in Fig. 2 was generated algebraically in a single passage, where the grid was concentrated near the blade surface according to the Reynolds number in order to simulate the boundary layer and the wake flows accurately. The upstream and downstream boundaries are located at the chord length and at 1.3 times the chord length of the blade from the leading edge and the trailing edge, respectively. In the present unsteady cascade flow computation, for fast convergence, the results of a preliminary steady calculation up to certain time steps are used as the starting values. Then, the unsteady-state solution can be obtained by the iterative calculation at each time step. The time step Δt was set to 0.01 for the preliminary steady calculation, while in the unsteady calculation Δt of 0.001 was used to obtain the time evolution of computation results. Here the solutions are regarded as convergent when the right-hand side of Eq. (23•a) and the corrections

Fig. 2 Computational grid for $Re=10^4$ (121×45)

of velocity and pressure are less than 5×10^{-5} . Also, the continuity condition, Eq. (25), has been confirmed to be satisfied within the accuracy of 10^{-5} . The inlet flow angle is fixed at $\beta = 45^\circ$ in all of the present computations of two-dimensional cascade flows.

Figure 3 shows the computational results for a 2-D decelerating cascade flow with the angle of attack $\alpha = 14.5^\circ$ at the Reynolds number $Re = 1\,000$ which is based on the blade chord. With this Re , the steady-state solution was obtained. Boundary layer flow on the blade and wake flow near the trailing edge are simulated well. Figure 4 shows the time evolution of velocity vectors and the surface pressure distribution in an unsteady cascade flow for $Re = 10\,000$. The $121 \times$

45 grid points in Fig. 2 are used. The present iterative computation converged within $m=5$, hence, stable calculation was carried out without a sharp increase in CPU time. In this Re , the cascade flow becomes unsteady as indicated in the figure, which shows the instantaneous aspects of separation and wake flow in detail. A vortex generated by rolling-up from the pressure side to the suction side at the trailing edge at the dimensionless time $T=0$ causes the sharp pressure drop near the trailing edge. After the vortex sheds from the trailing edge the pressure at the rear part of the blade tends to increase. During this period, another vortex grows from the separated shear layer on the suction surface. After that, a new rolled-up vortex induced by this separation vortex appears. As a result, periodic fluctuating flow similar to the Kármán vortex street is formed in the wake. The

Fig. 3 Steady decelerating cascade flow at $\alpha = 14.5^\circ$ and $Re = 10^3$ (61×21 grid points)

Fig. 4 Unsteady decelerating cascade flow at $\alpha = 14.5^\circ$ and $Re = 10^4$: (a) time evolution of velocity vector and (b) surface pressure distribution

period is about 0.5, i.e., the Strouhal number defined by the blade thickness becomes 0.2. The vortices shedding into the wake have been simulated well in the downstream region.

Figures 5 and 6 show other numerical results of time-averaged pressure distributions in the cascade flow for $Re=2.45 \times 10^5$ at $\alpha=12.1^\circ$ and 27.6° , respectively. The cascade flows at this Re become unsteady again. In this computation, the unsteady solution was also obtained within $m=5$ at each time step, and two kinds of grids, coarse and fine grids, were used. The coarse grid has 165×55 grid points, while the fine grid has 255×55 grid points and is almost twice as refined as the coarse one in the streamwise direction from the leading edge to the downstream boundary. Both grids have 55 grid points in the cascade direction, but the fine grid is more clustered near the blade surfaces. In the computed results, with increasing angle of attack, small-scale vortices emerge successively on both suc-

tion and pressure sides of the rear part of the blade, as demonstrated in Fig. 4. These vortices shedding into the wake cause large pressure fluctuations on the blade. Such a cascade wake flow becomes wavy and seems to be nearly periodic. Figures 5(b) and 6(b) show the time-averaged surface pressure distribution. From comparisons of surface pressure distribution with the experiment⁽¹⁸⁾, the present unsteady flow simulation using the coarse grid is considered to be fairly accurate, although the computed time-averaged values differ somewhat from the experimental ones on the rear part of the blade where the unsteady vortex flow exists. On the other hand, better agreement with experimental data is shown in the results obtained using the fine grid. It can be seen clearly that the results of the fine grid for the rear part of the blade were superior to those of the coarse grid. This indicates that a certain degree of grid refinement is necessary for simulating of strong unsteady vortex flow accurately. In the comparison of two results for $\alpha=12.1^\circ$, a slight difference is shown near the leading

Fig. 4 --- Continued

Fig. 5 Time-averaged pressure distribution at $\alpha=12.1^\circ$ and $Re=2.45 \times 10^5$: (a) pressure contours and (b) surface pressure distribution

Fig. 6 Time-averaged pressure distribution at $\alpha=27.6^\circ$ and $Re=2.45 \times 10^5$: (a) pressure contours and (b) surface pressure distribution

edge. This may be due to the inadequateness of the H-type grid used near the leading edge. In Fig. 5(b), the fine grid may be considered to capture weak eddies, so that small pressure fluctuations appear on the suction side of the front part of the blade where small separation bubbles probably exist. A turbulent flow analysis for above high Reynolds number will be made in the next report in which a turbulent flow scheme with some turbulence model will be introduced.

6. Conclusions

An implicit SMAC scheme has been developed for solving the unsteady incompressible Navier-Stokes equations in general curvilinear coordinates. The present implicit scheme has been confirmed to be accurate and efficient from numerical simulations of unsteady flows through a two-dimensional decelerat-

ing cascade. Accordingly, applications to other unsteady flow problems will be expected.

The results of the present work can be summarized as follows:

(1) The present implicit scheme has the advantage of applicability to actual flow such as three-dimensional flow as well as turbulent flow, since the primitive variables are used as the unknown variables.

(2) The continuity condition is sufficiently satisfied by solving the governing equations of contravariant velocities in the staggered curvilinear coordinate grid. Also, the appearance of spurious error in the pressure field is suppressed completely in a manner similar to that in the original SMAC scheme.

(3) The unsteady-state solution can be obtained by a few iterations at each time step, so that the unsteady flow calculation can be carried out without a sharp increment of the computing time.

(4) The unsteady flow computation at $Re=10\,000$ shows that the periodic fluctuating cascade flow with vortex shedding from the trailing edge into the wake can be simulated clearly, and the Strouhal number is predicted to be 0.2.

(5) The computed results of surface pressure distribution were in good agreement with the experimental data at the high Reynolds number $Re=2.45 \times 10^5$.

(6) The present unsteady flow scheme was verified to be successfully applied to unsteady cascade flow with strong vortices.

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