Effects of Bingham Viscosity on Flow in Curved Pipes*

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Bingham plastic fluid flows through coiled pipes of circular cross section were experimentally studied. In the investigations, kaolin clay slurries were used as the Bingham plastic fluid. The effects of the Reynolds number Re or the Dean number De on the friction factor λ_c in the laminar steady flow of the Bingham plastic fluid flowing through coiled pipes were clarified. The apparent Reynolds number Re_{ap} , expected to play the equivalent role of Re in Newtonian fluids, was introduced using the similarity law of energy dissipation which was derived from a generalized equation of non-Newtonian fluid. It was clarified that the effects of the curvature of the pipes are greater in the case of greater concentration of the slurry, and that there exists a good correlation between the friction factor and the apparent Dean number De_{ap} (defined according to Re_{ap}).

Key Words: Internal Flow, Non-Newtonian Fluid, Curved Pipe, Plastic Fluid, Pipe Flow, Experimental Study, Similarity Law

1. Introduction

Up to now, many studies of the non-Newtonian fluid flow in a curved pipe have been carried out for the sake of industrial uses or application to practical engineering problems⁽¹⁾. More recently, analytical or experimental methods for curved pipe flow problems of non-Newtonian fluid have been applied to the problem of blood flow, e.g., the pulsatile flow in a model of the human aortic arch or the flow in a booster lung, because blood has non-Newtonian viscosity⁽²⁾. In the medical or the physiological fields, research to clarify the characteristics and the structure of blood has been conducted progressively for a long time⁽³⁾⁻⁽⁶⁾. It has been revealed that from the rheological point of view blood belongs to the category of Casson fluid⁽⁷⁾ or Bingham plastic fluid⁽⁸⁾ with

finite yielding stress.

As is pointed out by Yoshioka and Adachi⁽⁹⁾, the flow field of a Bingham plastic fluid consists of a flowing region and a stationary region. In other words, it includes the discontinuous surface of the velocity which can't be defined a prior. Thus when the basic equations of the Bingham plastic fluid are used in an analytical investigation, the boundary surface and the flow field must be solved simultaneously. It is hardly possible to determine the motion of the fluid, because of the undefined boundary.

Therefore, despite the fact that the constitutive equations of Bingham plastic fluid were found in an earlier time in the history of rheology, details of the flow field of the Bingham plastic fluid are hardly known even today. Some particular cases of study, such as the investigation of the friction factor of the straight pipe flow or of the flow field of the creeping flow past a sphere, using the variational principles or energy equations were reported.

Now, we refer to some previous studies, in which the inner flows of the Bingham plastic fluids were treated. In the case of the steady laminar flow of the Bingham plastic fluid in a straight pipe, the exact

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solution has been found. Ito⁽¹⁰⁾ applied the analytical solutions to the similarity law of energy dissipation, and he successfully explained their experimental investigations. However, in the case of turbulent flow, theoretical solutions such as in the case of the Newtonian fluid flows, have not been found yet. Only experimental studies have been reported by Ito, using the same method that they applied to laminar flow. The details of the flow field of the turbulent flow of Bingham plastic fluid have not been revealed. Only a few researchers have attempted to write out the analytical expression of the turbulent flow of the Bingham plastic fluid⁽¹¹⁾. Further experimental or theoretical investigations will still be necessary.

On the other hand, standard analytical techniques, e.g., the perturbation method⁽¹²⁾ or the matched asymptotic expansion method, can't be applied in attacking the problem of the steady flow of Bingham plastic fluid in a curved pipe as described previously. In the case of the curved pipe flow, only a few theoretical studies have been reported even for the laminar flow, except a particular analytical investigation⁽¹³⁾ in which the plug flow region (Generally it moves and changes its shape in the cross section of the curved pipe.) was fixed in its shape and its position at the center of the cross section of the curved pipe. Experimental studies in which Bingham plasticity itself was treated have hardly been seen, except for a reports⁽¹⁴⁾ in which Bingham plastic fluid was regarded as the power law fluid and experimental data were rearranged by using a power law model.

In this study the experiment on Bingham plastic fluid flow in curved pipes was carried out in the case of laminar flow to clarify the characteristics of the flow field. Friction factors were measured for several values of the curvature of the coiled pipes and the density of the Bingham plastic fluids, and the correlations of the experimental results were explained from the viewpoint of the similarity law of energy dissipation.

Nomenclature

- a : pipe radius
- e^{ij} : rate of deformation tensor
- De : Dean number $(=Re\sqrt{\delta})$
- He : Hedstroem number $(=4a^2p_y\rho/\eta_B^2)$
- N_B : Bingham number $(=He/Re)$
- P : pressure
- p_y : yield stress
- R : curvature of the pipe
- Re : Reynolds number $(=2aW\rho/\eta_B)$
- v^i : element of velocity in coordinate i
- W : cross-sectional average velocity

ΔP : differential pressure

δ : curvature ratio $(=a/R)$

ϵ_{ijk} : contravariant permutation tensor

ϵ_{ijk} : Levi Chivita's symbol

λ : friction factor

η_B : plastic viscosity

ρ : density

τ^{ij} : deviatoric stress tensor

II_A : 2nd invariant of A^{ij} $(=\frac{1}{2}\{(A^{ij}g_{ij})^2 - A^{ij}A_{ij}\})$

III_A : 3rd invariant of A^{ij}
 $(=\frac{1}{3!}\epsilon_{ijk}\epsilon_{lmn}A^{il}A^{jm}A^{kn})$

subscripts and superscripts

ap : apparent value

c : curved pipe

N : Newtonian fluid

s : straight pipe

$;$: covariant derivative

$,$: partial derivative

$*$: dimensionless value

2. Experimental Set Up and Instrumentation

2.1 Test fluid and measurement of its physical parameters

Kaolin clay slurries were used in this study for their ease of handling and usefulness. As the clay slurries generate precipitations and the sigma effect occurs easily in the capillary tubes, viscosity of the clay slurries can't be measured correctly using the usual capillary viscometers. The other measurement method, i.e., the McMillen's method⁽¹⁵⁾ or various kinds of cylindrical viscometers were proposed for the survey. Nakamura and Sawada's method⁽¹⁶⁾ was employed in the present study from the viewpoint of accuracy, usefulness and cost of the equipment. This is a modified capillary viscometer with larger diameter d of the capillary tube length L of capillary and

Fig. 1 Viscometer

lower head H as shown in Fig. 1, so that viscosity can be measured easily even in the low-shear-rate flow field without using a narrow capillary tube.

The modified viscometer was composed of an acrylic cylinder and a copper capillary tube. The inlet part of the tube was smoothly rounded to a bell-shaped mouse, to avoid excess head loss. In the present study, d and L are fixed to 2.4 mm and 100 mm, respectively. At the beginning of the experiment the accuracy of the apparatus was verified by using water as the test fluid, and the volumetric flow rate of the slurries which had been kept for six months was measured for three concentration values. The shear stress of the slurries shown in Fig. 2 was calculated from the resulting data by using Tanaka's procedure⁽¹⁷⁾. From these results rheological values of the slurries and Hedstroem number He were derived as shown in Table 1.

2.2 Experimental set up

A schematic illustration of the experimental set up is depicted in Fig. 3. The test fluid was supplied to the test circuit from the reservoir (1) by the pump (3), and was returned into the reservoir through the test section (10) and the back pressure tank (11). Flow rate was controlled by a needle valve (4) installed downstream of the pump. Flow rate was measured by the weighing methods (12). The pipes made of copper were 4 mm in inner diameter, and a straight pipe 600 mm in length was jointed upstream of the test section so that fully developed flow could be obtained at the entrance of the curved region of the test section. The pressure in the curved pipe was measured by using the

Table 1 Physical parameters of the kaolin clay slurry

concentration	ρ [kg/m ³]	p_y [Pa]	η_B [Pa · s]	He
5 wt. %	1028	0.304	4.22×10^{-3}	2897
15 wt. %	1101	0.608	6.08×10^{-3}	4636
25 wt. %	1183	1.667	9.12×10^{-3}	6070

Fig. 2 Shear stresses of the kaolin clay slurries

transducer operating on the principle of semiconductor strain gages. The pressure data were converted to digital signals by A/D converter (6) and transmitted to the measuring computer (8). Three values of the curvature ratio R/a : 51.5, 33.1, 16.5 (The curvature has little influence on the flow field over this range) and two values of the density of the slurries: 5, 15 wt.%, i.e., altogether six conditions were selected in the experiment.

Flow rate and the frictional pressure loss of the curved pipe and of the straight pipe were measured. When the flow rate was small or a high concentration slurry was used as the test fluid, it was feared that precipitation would gradually be generated in the pipes or the pressure taps would be choked by the precipitation. Therefore, the needle valve was opened to its full opening for 1 - 2 minutes to wash away the precipitations before measurement and the pressure transducers were sometimes removed to wash away the slurries which adhered to the pressure taps during the experiment.

3. Results and Discussion

3.1 Friction factor of the curved pipe

Measurement of the friction factors of the water was performed as the preliminary experiment. Figure 4 shows the variation of the friction factor of the curved pipes with Dean number for the three values of the curvatures. The friction factors are normalized by those of the straight pipes. The solid line in Fig. 4 represents the values derived from the quasi-theoretical equation.

$$\frac{\lambda_c}{\lambda_s} = 0.100 8De^{1/2} (1 + 3.945De^{-1/2} + 7.782De^{-1} + 9.097De^{-3/2} + 5.608De^{-2} + \dots) \quad (1)$$

proposed by Ito⁽¹⁸⁾. As is shown in Fig. 4, the values of the friction factors obtained from the present experimental results are in good agreement with those evaluated by Ito. From this figure the accuracy of the equipment was confirmed, and this equipment

Fig. 3 Schematic illustration of the experimental set up

was used throughout the investigations. The same experiments mentioned above were carried out in the case of 2 values of the concentrations: 5, 15 wt.% of the kaolin clay suspension.

Figure 5 shows the correlations of the friction factor of the curved pipe having the curvature ratio of 51.5. The solid line in Fig. 5 denotes the empirical results of Newtonian fluid

Fig. 4 Normalized friction factor of water in curved pipes

Fig. 5 Distribution of the friction factor of the slurries in a curved pipe of curvature 51.5

Fig. 6 Distribution of the normalized friction factor of slurries

$$\lambda_c = \frac{1376}{(1.56 + \log De)^{5.73}} \sqrt{\frac{a}{R}} \quad (2)$$

reported by Ito⁽¹⁹⁾. It is found that the friction factors of the slurries decrease with increasing Dean number, and increase with concentration. Figure 6 shows the normalized friction factors. The experimental data show similar correlations to that in Fig. 5 regardless of the curvature of the pipes. With the increase in the concentration of the slurries, the effect of the curvature appears. From these results the flow field of the slurries in the curved pipe is considerably different from that of Newtonian fluid. The steady flow of Bingham plastic fluid in a curved pipe is dominated by three dimensionless parameters De , He and Re . Namely it can be expressed as

$$f(\lambda_c, De, He, Re) = 0. \quad (3)$$

However, it is inefficient to measure the physical parameters in every experimental run. Ito suggested the similarity law of the apparent physical parameters for steady flows in straight pipes, so that complicated work to calculate the Hedstroem number can be avoided. In this paper Ito's theory was employed to explain the resulting data obtained. Moreover, the similarity law for the curved pipe derived from the general form of the governing equations was proposed to predict the experimentally observed trend of clay slurries.

3.2 Similarity law of the apparent physical parameters

The amount of kinetic energy of an incompressible fluid of volume V can be written as

$$E_{\text{tot}} = \frac{1}{2} \rho \iiint_V v^j v_j \epsilon_{a\beta\gamma} dq^a dr^\beta ds^\gamma \quad (4)$$

where

$$v^i = v^i(q, r, s, t) \quad (5)$$

where q, r, s denote the base vector of space. Differentiating both sides of Eq. (4) with respect to time, and using the relation $\partial(1/2 \rho v^j v_j) / \partial t = \rho \dot{v}^j v_j$, and substituting the Cauchy's momentum equations

$$\dot{v}^j + v^k v_{j;k} = -\frac{1}{\rho} P_{,k} g^{jk} + \frac{1}{\rho} \tau_{,k}^{jk} \quad (6)$$

into $\dot{v}^j$, the following expression is obtained.

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^j v_j \right) = - \left\{ \rho v^k \left(\frac{1}{2} v^j v_j + \frac{P}{\rho} \right) - v_j \tau^{jk} \right\}_{;k} - \tau^{jk} v_{j;k} \quad (7)$$

Now we integrate both sides of Eq. (7) over the volume V . Rewriting the volume integration in the right-hand side of Eq. (6) to a surface integration by applying the divergence theorem; hence

$$\begin{aligned} & \frac{\partial}{\partial t} \iiint_V \frac{1}{2} \rho v^j v_j \epsilon_{a\beta\gamma} dq^a dr^\beta ds^\gamma \\ &= - \iint_S \left\{ \rho v^k \left(\frac{1}{2} v^j v_j + \frac{P}{\rho} \right) - v_j \tau^{jk} \right\} \epsilon_{a\beta\gamma} dq^a dr^\beta ds^\gamma \\ & \quad - \iiint_V \tau^{jk} v_{j;k} \epsilon_{a\beta\gamma} dq^a dr^\beta ds^\gamma \end{aligned} \quad (8)$$

where the direction of $dA_k = \epsilon_{\alpha\beta k} dq^\alpha dr^\beta$ is outward normal to the surface of V . The term on the left-hand side of Eq. (8) denotes the energy flux per unit time, the underlined term on the right-hand side of Eq. (8) denotes the energy flux owing to the fluid motion and the double underlined term denotes the entering energy flux owing to the viscous dissipation. Here the dissipation term in the second term of the right-hand side of Eq. (8) can be rewritten by using symmetrical tensor e^{ij} as follows.

$$\tau^{jk} v_{j;k} = \frac{1}{2} \tau^{jk} e_{jk} \quad (9)$$

On the other hand, by using the constitutive equations of the Stokes fluid, τ^{ij} is expressed as follows:

$$\tau^{jk} = 2\eta e^{jk} \quad (10)$$

where

$$\eta = \eta(II_e, III_e). \quad (11)$$

Therefore the integration can be expressed as

$$\begin{aligned} \iiint_V \tau^{jk} v_{j;k} \epsilon_{\alpha\beta\gamma} dq^\alpha dr^\beta ds^\gamma \\ = \iiint_V 2\eta |II_e| \epsilon_{\alpha\beta\gamma} dq^\alpha dr^\beta ds^\gamma \end{aligned} \quad (12)$$

so Eq. (12) apparently means the efflux of the kinetic energy per unit time owing to the viscous dissipation. Then we consider the steady flows of the Bingham plastic fluid. Considering the steady flow now, Eq. (8) can be expressed as

$$\begin{aligned} \iint_S \left\{ \rho v^k \left(\frac{1}{2} v^j v_j + \frac{P}{\rho} \right) - v_j \tau^{jk} \right\} \epsilon_{\alpha\beta k} dq^\alpha dr^\beta \\ = - \iiint_V \tau^{jk} v_{j;k} \epsilon_{\alpha\beta\gamma} dq^\alpha dr^\beta ds^\gamma \end{aligned} \quad (13)$$

Then substituting the viscous equation of Bingham plastic fluid

$$\eta = \left(\eta_B + \frac{p_y}{\sqrt{4|II_e|}} \right) \quad (14)$$

into Eq. (10) and rearranging Eq. (13) by using modified equation of Eq. (12), the following expression is obtained.

$$\begin{aligned} \iint_S \left\{ \rho v^k \left(\frac{1}{2} v^j v_j + \frac{P}{\rho} \right) - v_j \tau^{jk} \right\} \epsilon_{\alpha\beta k} dq^\alpha dr^\beta \\ = - \eta_B \iiint_V 2|II_e| \epsilon_{\alpha\beta\gamma} dq^\alpha dr^\beta ds^\gamma \\ - p_y \iiint_V \sqrt{|II_e|} \epsilon_{\alpha\beta\gamma} dq^\alpha dr^\beta ds^\gamma \end{aligned} \quad (15)$$

Equation (15) means the energy balance of the Bingham plastic fluid. The meaning of Eq. (15) from the point of view of industrial use is as follows. For convenience of discussion, we restrict the problem to the steady laminar flow in a straight pipe. Let the integration region take the form of a straight pipe of length l and radius a . Then only the surface integral of the pressure term remains in Eq. (15), namely

$$\begin{aligned} \iint_S \left\{ \rho v^k \left(\frac{1}{2} v^j v_j + \frac{P}{\rho} \right) - v_j \tau^{jk} \right\} \epsilon_{\alpha\beta k} dq^\alpha dr^\beta \\ = - \pi a^2 W \Delta P \end{aligned} \quad (16)$$

Moreover, substituting Eq. (16) into the left-hand side

of Eq. (15) and taking into account that $\eta_B = \eta_N$, $p_y = 0$ and making the physical parameters dimensionless by using a , W and ρ , the following expression is obtained.

$$\lambda_s = \frac{1}{Re} \left(\frac{4a}{\pi l} \iiint_V 2|II_e^*| \epsilon_{\alpha\beta\gamma}^* dq^{*\alpha} dr^{*\beta} ds^{*\gamma} \right) \quad (17)$$

The integration can easily be carried out, because the target problem is the Poiseuille flow. Finally Eq. (17) becomes

$$\lambda_s = \frac{64}{Re} \quad (18)$$

It is clear that Eq. (18) indicates the relation between the friction factor and Reynolds number.

Based on the fact discussed above, now let us consider the steady flow of the Bingham plastic fluid in a curved pipe. That is, we rewrite Eq. (15) using the unknown function λ_c similar to the form of Eq. (17)

$$\lambda_c = \frac{C}{Re} \left(1 + N_B \frac{\iiint_B}{\iiint_A} \right) \iiint_A \quad (19)$$

where C is a constant,

$$\lambda_c = \lambda_c(v^{*i}, P^*) \quad (20)$$

$$\begin{aligned} \iiint_A = \iiint_V 2|II_e^*| \epsilon_{\alpha\beta\gamma}^* dq^{*\alpha} dr^{*\beta} ds^{*\gamma} \\ \iiint_B = \iiint_V \sqrt{|II_e^*|} \epsilon_{\alpha\beta\gamma}^* dq^{*\alpha} dr^{*\beta} ds^{*\gamma} \end{aligned} \quad (21)$$

Supposing that Eq. (19) approaches the form of the Newtonians asymptotically when $p_y \rightarrow 0$ then the apparent Reynolds number of the Bingham plastic fluid can be written as

$$Re_{ap} = \frac{1}{1 + N_B \frac{\iiint_B}{\iiint_A}} Re \quad (22)$$

Therefore the number of the dimensionless parameter of Eq. (3) decreases and Eq. (3) can be represented as

$$g(\lambda_c, De, N_B) = 0 \quad (23)$$

It is difficult to solve Eq. (21) analytically, so the values of the integration in Eq. (21) are acquired experimentally. One of the first steps to be taken is to rewrite Eq. (22) to the expression

$$\frac{Re}{Re_{ap}} - 1 = h(N_B) \quad (24)$$

where

$$h(N_B) = N_B \frac{\iiint_B}{\iiint_A} \quad (25)$$

and to obtain the relation between N_B and $h(N_B)$ experimentally. If the assumed equation (22) predicts a correct correlation for the apparent Reynolds number, $h(N_B)$ can be expressed as a single-valued function of N_B , which is independent of the concentration of the slurries or the pipe radius. The solid line in Fig. 7 denotes the variation with respect to function $h(N_B)$ that is obtained from the experimental results as indicated by the symbols in the Fig. 7. Then by rearranging the experimental data, and replacing the relation between De and λ_c by the relation between

Fig. 7 Variation of the function $h(N_B)$

Fig. 8 Relationship between apparent Dean number and friction factor of slurries

De_{ap} and λ_c using the obtained function, N_B is calculated using He and Re which were obtained experimentally. Moreover, the values of the function $h(N_B)$ are obtained from Fig. 7 by using the calculated values of N_B .

Finally substituting the values of $h(N_B)$ and De into the modified Eq. (22) which is Eq. (22) denoted by Dean number $De (= Re\sqrt{a/R})$ in place of Reynolds number Re ,

$$De_{ap} = \frac{1}{1 + h(N_B)} De \quad (26)$$

is obtained. In Fig. 8 the symbols indicate the rearranged results and the solid line indicates values obtained from the correlation

$$\lambda_c \sqrt{\frac{R}{a}} = \frac{1376}{(1.56 + \log De_{ap})^{5.73}} \quad (27)$$

$(13.5 < De_{ap} < 2 \times 10^3)$

which is based on Eq. (2). Both are in good agreement.

4. Concluding Remark

In this study, steady flows of Bingham plastic fluid

in curved pipes are investigated experimentally by using kaolin clay slurries as the test fluid, and the following results are obtained.

(1) The effect of the curvature of pipes becomes more apparent with increasing concentrations of slurries.

(2) The similarity law for the Bingham plastic fluid flow in a curved pipe was derived from the general form of the governing equations of non-Newtonian fluid.

(3) The variation of the friction factor of the Bingham plastic fluid flow in a curved pipe for apparent Dean number can be predicted by the proposed equation (27).

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