Stability of Steady Flow in Collapsible Tubes*

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In this investigation, the authors studied the stability of steady flows in collapsible tubes of which static characteristics were theoretically calculated in the previous paper. The dynamics of the collapsible tube is approximately represented by a simplified lumped parameter model. On the basis of the model, the instability condition has been derived and the mechanism of the instability of the steady flow has been clarified. The instability region obtained theoretically is qualitatively in good agreement with the experimental results.

Key Words: Collapsible Tube, Vibration Coupled with Fluid Motion, Blood Flow, Stability, Biomechanics

1. Introduction

Collapsible tubes (c-tubes) are compliant tubes which are easily collapsed when the transmural pressure (internal pressure minus external pressure) is negative. The flows in the c-tubes are very complicated, since they exhibit marked nonlinearity due to interference between the changes of the flow and the cross-sectional shape. There are numerous vessels in the human body similar to c-tubes like veins, tracheae, and urethrae, to name a few. It is well known that strange phenomena are brought about in collapsible tubes owing to the nonlinearity; for instance, negative resistances in static characteristics, flow saturation phenomena, self-excited vibrations occurring in the vena cava during cardiopulmonary bypass, and Korotkoff sound during blood pressure measurement. Since the problems are physiologically very important, many studies have hitherto been done on the flow problems⁽¹⁾⁻⁽⁶⁾. In particular, on the mechanism of self-excited vibrations, there is much experi-

mental work^{(1),(3),(5),(6)} and theoretical work using lumped parameter models or distributed parameter models^{(2),(4)}. However, there is no theory which can comprehensively explain the phenomena in the c-tube.

In the previous paper⁽⁷⁾, one of the authors theoretically studied static characteristics of the c-tubes, clarified the cause of the negative resistance in the ΔP - Q characteristic curves, and indicated that no negative resistance appears for no load resistance.

In this study, we derive a lumped parameter mathematical model describing the dynamic behavior of the flow in the c-tubes, investigate the stability of the steady flow and clarify the mechanism of self-excited vibrations.

Nomenclature

- A_0 : cross-sectional area of rigid pipes
- C_1, C_2 : capacitances of upstream section I and the downstream section III of c-tube
- F_P, F_Q : derivatives of function $F(Q, P_{t0})$
- F_T : frictional coefficient of fluid
- $f(P_2)$: characteristic function of c-tube
- K_P, K_E : rigidities of c-tube for collapsed and expanded states
- L_1, L_2 : inertances of upstream and downstream rigid pipes ($= \rho l_1/A_0, \rho l_2/A_0$)
- L : length of c-tube
- l_1, l_2 : lengths of rigid pipes

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- P_e : external pressure
 P_s : supply pressure
 P_{u1}, P_{t0} : transmural pressure at upstream and downstream ends of c-tube ($=P_1-P_e, P_2-P_e$)
 P_1, P_2 : pressures at upstream and downstream ends of c-tube
 Q : flow in collapsed section II of c-tube
 Q_0 : flow in steady state
 Q_1, Q_2 : flow at upstream and downstream ends of c-tube
 R : resistance of downstream pipe and restrictor
 r : resistance of upstream pipe
 T : tension acting on c-tube
 ρ : density of working fluid
 0 : subscript denoting steady state

2. Governing Equations

In order to carry out the stability analysis of steady flow in a collapsible tube (c-tube), the tube is divided into three sections, that is, upstream capacitive section I, collapsed resistive section II and a downstream capacitive section III, as illustrated in Fig. 1. The characteristics of each section are assumed to be expressed by lumped parameter models.

The fluid flowing from an overflow tank passes through rigid pipe L_1 , flows into the collapsible tube, passes through downstream rigid pipe L_2 and discharges into the atmosphere from a restrictor attached to rigid pipe L_1 . The diameter of the rigid pipes equals that of the c-tube when the transmural pressure $P_{t0}=0$.

Governing equations describing dynamic behavior of the fluid circuit including the c-tube are derived as follows.

The equation of fluid motion in pipe L_1 is

$$L_1 \frac{dQ_1}{dt} = P_s - P_1 - r|Q_1|Q_1. \quad (1)$$

The capacitive characteristic of section I is

$$C_1 \frac{dP_1}{dt} = Q_1 - Q. \quad (2)$$

The resistive characteristic of section II is

$$\Delta P = P_1 - P_2 = F(Q, F_{t0}), \quad (3)$$

Fig. 1 Schematic diagram of collapsible tube

which is the static characteristics of the c-tube.

The capacitive characteristic of section III is

$$C_2 \frac{dP_2}{dt} = Q - Q_2. \quad (4)$$

The equation of fluid motion in pipe L_2 is

$$L_2 \frac{dQ_2}{dt} = P_2 - R|Q_2|Q_2. \quad (5)$$

$F(Q, P_{t0})$ in Eq. (3) is a static characteristic of the c-tube for $R=0$, which has been studied in detail in the previous paper⁽⁷⁾. Figure 2 shows the relationship between the flow (Q) and the pressure loss (ΔP) of the c-tube, which are numerically calculated from a distributed parameter model describing the steady flow in the c-tube.

Figure 3 shows the relationship between the tube loss (ΔP) and the external pressure (P_e) under the condition of $Q=\text{constant}$ and $R=0$. In the calculation, the following values were used as the tube parameters; the tube length $L=0.1$ m, the frictional factor $F_T=0.0355$, the tube rigidities $K_p=1.0 \times 10^2$ Pa (for collapsed state), $K_e=9.2 \times 10^4$ Pa (for expanded state), and tension $T=2.61 \times 10^2$ kg/m

3. Stability of Steady Flow

A steady flow in the c-tube corresponds to a steady state of Eqs. (1)-(5). The steady point is determined from the following equations by putting $dQ_1/dt = dQ_2/dt = dP_1/dt = dP_2/dt = 0$ in Eqs. (1)-(5):

$$\left. \begin{aligned} P_s - P_{t0} - r|Q_{t0}|Q_{t0} &= 0 \\ \Delta P &= P_{t0} - P_{20} = F(Q_0, P_{t0}), \\ P_{20} - R|Q_{20}|Q_{20} &= 0 \end{aligned} \right\} \quad (6)$$

Fig. 2 Static characteristics of collapsible tube (ΔP - Q curves)

Fig. 3 Static characteristics of collapsible tube (ΔP - P_e curves)

where $P_{t0} = P_{20} - P_e$.

Taking account of $Q_0 = Q_{10} = Q_{20}$, Eq. (6) is rearranged as

$$\begin{aligned} \Delta P &= F(Q_0, RQ_0^2 - P_e) \\ \Delta P &= P_s - (r + R)Q_0^2 \end{aligned} \quad (7)$$

The steady operating point is given as intersections of the above equations.

Linearizing Eqs. (1)–(5) using the small perturbations ΔQ_1 , ΔP_1 , ΔP_2 and ΔQ_2 , we obtain

$$\left. \begin{aligned} \frac{d\Delta Q_1}{dt} &= -\frac{2rQ_0}{L_1}\Delta Q_1 - \frac{1}{L_1}\Delta P_1 \\ \frac{d\Delta P_1}{dt} &= \frac{1}{C_1}\Delta Q_1 - \frac{1}{C_1F_Q}\Delta P_1 + \frac{1+F_P}{C_1F_Q}\Delta P_2 \\ \frac{d\Delta P_2}{dt} &= \frac{1}{C_2F_Q}\Delta P_1 - \frac{1+F_P}{C_2F_Q}\Delta P_2 - \frac{1}{C_2}\Delta Q_2 \\ \frac{d\Delta Q_2}{dt} &= \frac{1}{L_2}\Delta P_2 - \frac{2RQ_0}{L_2}\Delta Q_2 \end{aligned} \right\} \quad (8)$$

where $F_Q = \partial F / \partial Q$ and $F_P = \partial F / \partial P_{t0}$; F_Q means the rate of increase of tube loss to increasing Q under the condition $P_{t0} = \text{constant}$, and F_P means the rate of tube loss with increasing P_{t0} under the condition $Q = \text{constant}$. Since the cross-sectional area of the c-tube increases with increase of the transmural pressure P_{t0} , the tube loss decreases with increasing P_{t0} . Therefore, F_P is always negative.

The values of F_Q and F_P are given as the gradients of the curves plotted in Fig. 2 and Fig. 3, respectively. Taking into account that the curves are obtained under the condition of $R=0$ and $P_{t0} = -P_e$, the gradient of the curve in Fig. 3 equals $-F_P$.

The factors F_Q and F_P can be related to the gradient of ΔP - Q characteristics as follows:

$$\frac{d\Delta P}{dQ} = F_Q + F_P \left(\frac{dP_{t0}}{dQ} \right). \quad (9)$$

The characteristic equation of Eq. (8) is

$$\begin{vmatrix} s + \frac{2rQ_0}{L_1} & \frac{1}{L_1} & 0 & 0 \\ -\frac{1}{C_1} & s + \frac{1}{C_1F_Q} & -\frac{1+F_P}{C_1F_Q} & 0 \\ 0 & -\frac{1}{C_2F_Q} & s + \frac{1+F_P}{C_2F_Q} & \frac{1}{C_2} \\ 0 & 0 & -\frac{1}{L_2} & s + \frac{2RQ_0}{L_2} \end{vmatrix} = 0. \quad (10)$$

It is difficult to investigate the stability of the c-tube analytically by means of the above 4th-order equation which has numerous system parameters. Hence, Eq. (10) is simplified by reducing the order on the basis of the assumptions that (1) the inertance and the resistance of L_1 can be neglected, and (2) the capacitance C_1 is so large that P_1 is regarded as constant. The assumptions are justified by the fact that L_1 is short, and that the variation of P_1 is markedly small compared to that of P_2 when the amplitude of P_2 is relatively small (Appendix 1).

Under these assumptions, Eq. (10) is reduced to the second-order equation

$$(C_2L_2F_Q)s^2 + \{2RQ_0F_QC_2 + (1+F_P)L_2\}s + \{F_Q + 2RQ_0(1+F_P)\} = 0. \quad (11)$$

The instability condition of the steady state is easily derived as

$$\left. \begin{aligned} D_1 &= F_Q + 2RQ_0(1+F_P) < 0 \\ \text{or} \\ D_2 &= 2RQ_0F_QC_2 + (1+F_P)L_2 < 0 \end{aligned} \right\} \quad (12)$$

As seen from Eq. (12), F_Q and $-F_P$ are always positive. Thus, when $(1+F_P)$ is negative, the steady state becomes unstable, depending upon the value of other parameters.

The first condition can be interpreted that, if the total pressure loss including the loss of the load resistance decreases with increasing Q , the steady state becomes statically unstable. On the other hand, the second inequality including C_2 and L_2 implies that the instability is caused dynamically.

Now, we consider the instability condition from a different point of view. Under the assumptions of $L_1 = 0$ and $C_1 = \infty$, Eqs. (1)–(5) are reduced to Eqs. (3), (4) and (5). Although there are three variables ΔP_2 , Q_2 and Q , the equations are rearranged into two equations with P_2 and Q_2 by substituting the solution of Eq. (3) with respect to Q into Eq. (4).

The flow Q is expressed as a function of P_1 , P_2 and P_e . If P_1 is chosen as a pressure reference, the functional form is

$$Q = f(P_2 - P_1, P_1 - P_e). \quad (13)$$

Figure 4 shows the Q - P_2 characteristic curves for $P_1 = 4$ kPa, where the transmural pressure $P_t (= P_1 - P_e)$ at the upstream end of the c-tube is taken as a

Fig. 4 Static characteristics of collapsible tube (Q - P_e curves)

parameter. The curves are none other than the so-called “flow saturation” curves which were pointed out by Brecher in his paper on the flow in blood vessels⁽⁸⁾.

Under the condition of $L_1=0$ and $C_1=\infty$, P_1 is equal to P_s , thus Eq. (13) is reduced to the form

$$Q = f(P_2), \quad (14)$$

and Eqs. (4), (5) and (14) are simplified as follows:

$$\left. \begin{aligned} C_2 \frac{dP_2}{dt} &= f(P_2) - Q_2 \\ L_2 \frac{dQ_2}{dt} &= P_2 - R|Q_2|Q_2 \end{aligned} \right\} \quad (15)$$

Rearranging Eq. (15) with respect to Q_2 , we obtain the following nonlinear second-order equation.

$$L_2 C_2 \frac{d^2 Q_2}{dt^2} + (2RC_2|Q_2|) \frac{dQ_2}{dt} + Q_2 - f'(P_2) \left(L_2 \frac{dQ_2}{dt} + R|Q_2|Q_2 \right) = 0 \quad (16)$$

The linearized form of Eq. (16) is

$$L_2 C_2 \frac{d^2 \Delta Q_2}{dt^2} + (2RC_2 Q_0) \frac{d\Delta Q_2}{dt} + \Delta Q_2 - f'(P_{20}) \left(L_2 \frac{d\Delta Q_2}{dt} + 2RQ_0 \Delta Q_2 \right) = 0, \quad (17)$$

where $f'(P_2)$ means the derivative of Eq. (14) and is closely related to the factors F_Q and F_P ^{**1}. The characteristic equation of Eq. (17), of course, coincides with Eq. (11).

The cause of the instability of the c-tube lies in $f'(P_{20})$, or $-(1+F_P)$ in Eq. (17). Taking account of $\Delta P_2 = L_2 dQ_2/dt + 2RQ_0 \Delta Q_2$, the first condition in Eq. (12) is interpreted as a static instability with an increase of $\Delta P_2 (=2RQ_0 \Delta Q_2)$ due to a load resistance. On the other hand, the second condition means a dynamical instability caused by $L_2 dQ_2/dt$, that is, an

increase of ΔP_2 as a reaction to the acceleration of the downstream fluid column.

In other words, the first condition indicates that self-excited vibrations occur if the negative resistance exceeds the load resistance, which has been pointed out by many researchers^{(1),(2)}, whereas the negative resistance does not appear in the ΔP - Q curve for a small load resistance. But it has been shown experimentally that self-excited vibrations also occur in this case⁽³⁾. The phenomenon can be explained by means of the second condition. For instance, the gradient of the ΔP - Q curve is always positive for $R=0$, that is, $d\Delta P/dQ = F_Q + F_P dP_{t0}/dQ = F_Q > 0$ because of $dP_{t0}/dQ = 0$. Thus, $D_1 = F_Q > 0$; however, if $(1+F_P) < 0$, $D_2 = (1+F_P)L_2 < 0$. Consequently, in this case, the steady state becomes dynamically unstable.

4. Static Characteristics and Unstable Region

Figures 5 (a) and (b) illustrate F_Q - Q curves and F_P - Q curves obtained from the ΔP - Q curve and ΔP - P_e curves shown in Figs. 2 and 3. The stability boundaries are indicated on the ΔP - Q curves in Figs. 6(a), (b) and (c), which correspond to different load resistances, i.e., $R=0$ Pa/(m³/s)² for Fig. 6(a), $R=3.0 \times 10^{11}$ Pa/(m³/s)² for Fig. 5(b) and $R=6.35 \times 10^{11}$ Pa/(m³/s)² for Fig. 6(c). The load resistance in Fig. 6(c) is the same value as in the previous paper⁽⁷⁾.

In the calculation, the equivalent value of the downstream capacitance of the c-tube is estimated as $C_2 = 5.36 \times 10^{-10}$ m³/Pa by using the downstream pipe inertance $L_2 = 2.55 \times 10^6$ kg/m⁴ (see Appendix 2).

The curves D_1 in Fig. 6 represent $D_1=0$, the inside of which is statically unstable, whereas the D_2 curves represent $D_2=0$, the inside of which $D_2 < 0$. Since the region for $D_1 < 0$ is always included in the region $D_2 < 0$, the D_2 curves are the boundaries of the stable region. The steady state is dynamically unstable between the region $D_2 < 0$ and $D_1 < 0$.

Figure 7 shows a ΔP - Q curve obtained experimentally for $R = 6.31 \times 10^{11}$ Pa/(m³/s)². Self-excited vibrations occur in the regions without measured points along the theoretical curves and the broken line indicates the boundary of self-excitation. As seen in the figure, the self-excited region is qualitatively in good agreement with the theoretical result. In the experiment, with increasing flow Q , the self-excited vibration originates at point B just after the peak of the curve and continues to vibrate until point D just before the turning point of the curve.

With decreasing flow Q , the steady-state point passes through point D stably. Then, it becomes unstable at point C. This hysteresis cannot be explained by the present linear theory which discusses the local

^{**1} The linearized tube characteristic $\Delta P_1 - \Delta P_2 = F_Q \Delta Q + F_P \Delta P_2$ results in $\Delta Q = -\{(1+F_P)/F_Q\} \Delta P_2$ for $P_1 = \text{constant}$. Thus, $f'(P_{20}) = dQ/dP_2 = -(1+F_P)/F_Q$.

stability in the vicinity of the steady operating point ; it should be analyzed taking nonlinearity into consideration. The problem will be studied separately.

(a) F_Q curves

(b) F_P curves

Fig. 5 Gradients F_Q and F_P

Fig. 7 Boundary of self-excited region

5. Conclusions

The dynamic behavior of collapsible tubes is represented by a lumped parameter model, and the

(a) $R=0 \text{ Pa}/(\text{m}^3/\text{s})^2$

(b) $R=3.0 \times 10^{11} \text{ Pa}/(\text{m}^3/\text{s})^2$

(c) $R=6.36 \times 10^{11} \text{ Pa}/(\text{m}^3/\text{s})^2$

Fig. 6 Instability regions of steady flow

stability of steady states corresponding to a steady flow is investigated analytically. On the basis of the model, the mechanism of the instability is clarified and the instability conditions are derived. As a result, it is pointed out that there are two kinds of instability, that is, a statical instability due to the load resistance and a dynamical instability caused by the load inertance. The theoretical instability region qualitatively agrees well with experimental results.

Appendix 1 Experimental vibratory waveforms

Figure 8 shows the vibratory waveforms for P_1 and P_2 which are obtained from an experiment carried out under the conditions $L=0.1$ m, $P_e=3.0$ kPa, $R=7.85 \times 10^{11}$ Pa/(m³/s)² and $L_2=0.22$ m. The amplitude of P_1 is markedly smaller than that of P_2 , and P_1 is almost constant for small Q as seen in Fig. 8(a). It is considered that the upstream flow change caused by the downstream pressure change is absorbed by the tube capacitance C_1 . This justifies the assumption that the variation of P_1 can be neglected for small perturbation from the steady state.

Appendix 2 Determination of Capacitance C_2

By arranging the second stability condition of Eq. (12), we obtain

$$C_2 = \frac{|1 + F_P| L_2}{2RQ_0 F_Q} \quad (18)$$

In the above equation, L_2 and R are the specified parameters and F_Q and F_P are obtained from Fig. 5 where the steady flow Q_0 on the stability boundary is determined.

The equivalent capacitance in this case is $C_2 = 5.36 \times 10^{-10}$ m³/Pa

By approximating the partially collapsed tube by two prisms, as shown in Fig. 9, the capacitance C_2 can be roughly estimated as follows: from the tube law, the volumetric change of the prism due to a pressure change ΔP_2 is

$$\begin{aligned} \Delta V &= \int \Delta A dx = \frac{2\Delta P_2}{3K_p} \int A_0 \left(\frac{A}{A_0} \right)^{5/2} dx \\ &= \frac{8}{21} \left(\frac{V_0}{K_p} \right) \Delta P_2, \end{aligned} \quad (19)$$

where ΔA is a change of the cross-sectional area and $V_0 = A_0 x_0/2$.

The estimated value is

$$C_2 = \Delta V / \Delta P_2 = 8 V_0 / (21 K_p) = 21.4 \times 10^{-10}$$

where $A_0 = 7.85 \times 10^{-5}$ m², $x_0 = 0.015$ m and $K_p = 105$ Pa. The value is four times as large as the experimental value. However, the theoretical value is of the same order as the experimental one.

References

- (1) Conrad, W.A., Pressure-Flow Relationship in

Fig. 8 Vibratory waveforms of self-excited vibrations

Fig. 9 Simplified model of a partially collapsed tube

- Collapsible Tubes, IEEE Trans. Bio-Med. Eng., Vol. 16, No. 4 (1969), p. 284.
- (2) Katz, A.I. et al., Flow Through a Collapsible Tube (Experimental Analysis and Mathematical Model), Biophys. J., Vol. 9 (1969), p. 1261.
- (3) Ohba, K. et al., Flow Oscillation and Wave Propagation in Collapsible Tube, Tech. Rep., Kansai Univ., Vol. 25 (1984), p. 1.
- (4) Cancelli, C.D. et al., A Separated-Flow Model for Collapsible-Tube Oscillations, J. Fluid Mech., Vol. 157 (1985), p. 357.
- (5) Bertram, C.D., A Mathematical Model of Unsteady Collapsible Tube Behavior, Trans. ASME, J. Biomech. Eng., Vol. 111 (1989), p. 185.
- (5) Tuji, T. et al., Study on Hemodynamics During Cardiopulmonary Bypass, Jinkou Zouki, (in Japanese), Vol. 7, No. 2 (1978), p. 435.
- (7) Hayashi, S. et al., Static Characteristics of Collapsible Tubes, 2nd Int. Symp. on Fluid-Control, Measurement, Mechanics and Flow Visualization, Sheffield (1988), p. 234.
- (8) Brecher, G.A., Mechanism of Venous Flow under Different Degrees of Aspiration, Am. J. Physiol., Vol. 169 (1952), p. 423.