Solutions of the Integro-Differential Equation for a Thin Jet-Flapped Aerofoil*

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The integro-differential equation for a thin jet-flapped aerofoil was derived by Spence (Proc. Roy. Soc., London A-238 (1956), p. 46). Its numerical and asymptotic solutions have been obtained, but the existence of solutions has not been discussed yet, since the governing equation is singular. In the present paper, we derive an integral equation of the Fredholm type of the second kind which is not singular from the original integro-differential equation. We further prove the existence and uniqueness of the solution and derive the asymptotic solutions. Comparing with earlier works, we show that earlier asymptotic solutions obtained by the method of matched asymptotic expansions are reasonable.

Key Words: Potential Flow, External Flow, Jet, Jet-Flapped Aerofoil, Integro-Differential Equation

1. Introduction

The jet flap is powerful device for integrating the production of lift and thrust on an aerofoil, by ejecting high-velocity air in a narrow sheet from the trailing edge. The jet sheet is directed from the trailing edge to the mainstream, so that the lift increases, not only as a reaction to the jet thrust, but also as a result of changes induced in the surface pressure distribution of the aerofoil. This phenomenon is called the jet-flapped effect and has been studied by many investigators [e.g., Refs. (1) and (2)]. The most important theoretical work is that of Spence⁽³⁾: the inertia of the jet sheet is balanced by a pressure difference across the jet, so that this problem becomes that of a mixed boundary value and the integro-differential equation for the slope of the jet sheet, which is singular, is derived. It is very difficult to solve this equation

analytically: the position of the jet sheet is not known a priori. A numerical solution of this equation was presented for the first time by Spence⁽³⁾: he used a Fourier series together with a singularity distribution at the trailing edge and its coefficients are obtained. In the recent work by Mateescu and Newman⁽¹⁾, another series, the polynomial series, is used. The jet-flapped effect was applied to supercavitating hydrofoils and many problems were solved numerically. In these works, series obtained numerically do not necessarily guarantee convergence [e.g., Ref. (4)]. This fact implies the necessity of a mathematical study of this singular integro-differential equation.

The mathematical work of this equation was performed for the first time by Spence⁽⁵⁾. He obtained an asymptotic solution for the small jet momentum coefficient C_J : from heuristic discussion, the first-order approximation is satisfied by a simple integro-differential equation and its solution is obtained by using the Mellin transformation. Kida and Miyai⁽⁶⁾ pointed out that Spence's asymptotic solution is only an inner expansion, that is, the solution is not valid far downstream, and they obtained the asymptotic solutions by applying the matched asymptotic expansion

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to this governing integral equation. Their idea was new in terms of the asymptotic approach to integral equations whose kernel contains a small parameter. It was applied to various physical problems⁽⁷⁾⁻⁽¹¹⁾, and its mathematical background was also studied by Kida⁽¹²⁾.

In the present paper, this governing integro-differential equation will be transformed into the nonsingular-type Fredholm equation of the second kind by using complex Laplace transforms, which have been studied by Estrada and Kanwal⁽¹³⁾. The existence of the solution of this equation will be proven by Fredholm's alternative theory⁽¹⁴⁾. Asymptotic solutions will be obtained from this nonsingular equation and compared with earlier results.

2. Basic Relations

We consider a thin two-dimensional aerofoil in a steady incompressible flow whose chord length is c and incidence angle is α . A jet emerges from the trailing edge at an initial deflection τ to the chord line, as shown in Fig. 1. The x axis is taken in the horizontal direction of the undisturbed uniform stream and the y axis is taken to be perpendicular to this. In the linearized theory, that is, α and τ being small enough for their squares to be neglected, the aerofoil and jet sheet are represented by a discontinuity of u , where $U + u$ and v are the velocity components of x and y , respectively, and U is the uniform velocity. The discontinuity of u may be thought of physically as a distribution of vorticity $\gamma(x)$ along the x axis, as shown in Fig. 1. Then, we obtain the induced downwash w by the following relation.

$$w(x) = -\frac{1}{2\pi} \int_0^\infty \frac{\gamma(\xi)}{\xi - x} d\xi \quad (1)$$

where the integral is a principal value at $\xi = x$. The boundary conditions on w are imposed by the flow tangency:

$$w(x) = \begin{cases} U\alpha & (0 < x < c) \\ -Uy_j'(x) & (c < x < \infty) \end{cases} \quad (2)$$

where y_j is the ordinate of the jet sheet and prime “'”

Fig. 1 Physical plane and mathematical model

denotes differentiation with respect to x . The width of the jet sheet is assumed to be very thin but to maintain the momentum of the jet at a finite value. The velocities and lengths are normalized by the uniform velocity and the chord length, respectively. Then the vorticity distribution along the jet sheet is represented by

$$\gamma_j(x) = C_j y_j''(x)/2 \quad (3)$$

where C_j is the jet momentum coefficient for the momentum flux J of the jet

$$C_j = J / \left(\frac{1}{2} \rho c U^2 \right).$$

The boundary condition of y_j at the trailing edge is given by

$$y_j'(1) = -(\alpha + \tau). \quad (4)$$

The simultaneous integral equations given by Eqs. (1)-(3) can be transformed into the integro-differential equation of y_j only. We invert the equation for $0 < x < 1$ using Carleman's method, to solve the vorticity distribution, and its solution is substituted into the basic equation. Then we obtain (see Ref. (3))

$$y_j'(x) = \frac{\lambda}{\pi} \left(\frac{x-1}{x} \right)^{1/2} \int_1^\infty \left(\frac{\xi}{\xi-1} \right)^{1/2} \frac{y_j''(\xi)}{\xi-x} d\xi - \alpha \left(1 - \left(\frac{x-1}{x} \right)^{1/2} \right), \quad (5)$$

where $\lambda = C_j/4$. In the present paper, we will consider the interesting case of $\alpha = 0$ in detail.

3. Analysis

For the case of $\alpha = 0$, the original integro-differential equation is expressed as follows:

$$g(x) = \frac{\lambda}{\pi} \left(\frac{x-1}{x} \right)^{1/2} \int_1^\infty \left(\frac{\xi}{\xi-1} \right)^{1/2} \frac{g'(\xi)}{\xi-x} d\xi,$$

where $y_j'(x) \equiv -\tau g(x)$. Inverting this equation using Carleman's method, we easily obtain

$$g'(x) = -\frac{1}{\pi\lambda} \int_1^\infty \left(\frac{\xi}{x} \right)^{1/2} \frac{g(\xi)}{\xi-x} d\xi$$

$$g(1) = 1 \text{ and } g(\infty) = 0. \quad (6)$$

By setting $x = 1 + \lambda X$, the above equation becomes

$$g'(X) = -\frac{1}{\pi} \int_0^\infty \frac{g(\xi)}{\xi - X} d\xi - \lambda \psi(X), \quad (7)$$

where

$$\psi(X) \equiv \frac{1}{\pi} \frac{1}{(1+\lambda X)^{1/2}} \times \int_0^\infty \frac{g(\xi)}{(1+\lambda X)^{1/2} + (1+\lambda \xi)^{1/2}} d\xi. \quad (8)$$

We here introduce a complex function $F(z)$:

$$F(z) \equiv \frac{1}{2\pi i} \int_0^\infty \frac{g(\xi)}{\xi - z} d\xi. \quad (9)$$

If $F(z)$ is analytic in the region $C \setminus [0, \infty]$, then we can introduce its complex Laplace transform, following Estrada and Kanwal⁽¹³⁾. We define the complex Laplace transform of $F(z)$ for $\omega \in C[0, \infty]$ by

$$\mathcal{L}_C\{F(z); \omega\} \equiv \int_{\Gamma_\omega} e^{-\omega z} F(z) dz, \quad (10)$$

where Γ_ω is the ray $\{t \exp(-i\theta) : 0 < t < \infty\}$ and where $\theta = \arg \omega$. Provided that $F(z)$ is the analytical

representation of an integrable function $g(x)$, we have from the definition of $F(z)$,

$$F_+ + F_- = \frac{1}{\pi i} \int_0^\infty \frac{g(\xi)}{\xi - X} d\xi, \quad F_+ - F_- = g(X), \quad (11)$$

where F_+ and F_- are the boundary values of $F(z)$ from the left and right, respectively. Then, the integral equation (7) is rewritten as

$$F'_+ - F'_- + i(F_+ + F_-) = -\lambda\psi. \quad (12)$$

Letting $\theta \rightarrow 0$ and $\theta \rightarrow 2\pi$ in Eq.(10), we find that the boundary values of the complex transform $G(\omega) \equiv \mathcal{L}\{F(z); \omega\}$ are related to those of $F(z)$ as

$$G_\pm(s) = \mathcal{L}\{F_\pm(X); s\} = \int_0^\infty F_\pm(X) e^{-xs} dX. \quad (13)$$

We define the Laplace transform of $\psi(X)$ as

$$\mathcal{L}\{\psi(X); s\} \equiv \Psi(s). \quad (14)$$

Then the basic relation (12) becomes

$$-(s-i)G_+ + (s+i)G_- = F_+(0) - F_-(0) - \lambda\Psi(s).$$

Taking into account the boundary condition (6), we have from the above relation,

$$(s-i)G_+ - (s+i)G_- = -1 + \lambda\Psi(s). \quad (15)$$

Let us consider the homogeneous equation of Eq. (15):

$$(s-i)X_+ - (s+i)X_- = 0. \quad (16)$$

The solution of this equation is given by Boersma⁽¹⁶⁾, changing the variable s to z :

$$X(z) = (-z)^{-1/2}(1+z^2)^{1/4} e^{-m^*(z)} \\ m^*(z) = -\frac{1}{\pi} \int_0^\infty \frac{z \ln(-t)}{1+t^2} dt. \quad (17)$$

We take this homogeneous solution into account in Eq.(15). Then we easily obtain the solution $G(z)$:

$$\frac{G(z)}{X(z)} = \frac{1}{2\pi i} \int_0^\infty \frac{-1 + \lambda\Psi(\omega)}{(\omega-i)X_+(\omega)} \frac{1}{\omega-z} d\omega.$$

Substituting $X(z)$ into this equation and taking into account Eq.(17), we finally arrive at

$$\frac{G(z)}{X(z)} = \frac{1}{2\pi i} \int_0^\infty \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{3/4}} [-1 + \lambda\Psi(\omega)] d\omega \\ + \frac{z}{2\pi i} \int_0^\infty \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{3/4}} \frac{-1 + \lambda\Psi(\omega)}{\omega-z} d\omega, \quad (18)$$

where

$$m(\sigma) \equiv -\frac{1}{\pi} \int_0^\sigma \frac{\ln t}{1+t^2} dt.$$

Since the second term of the right-hand side of Eq.(18) is still of the Cauchy type, we discuss this term further in order to obtain an alternative expression of $g(X)$. We have from the integration over a closed contour (see Appendix 1),

$$\frac{1}{\pi i} \int_0^\infty \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{3/4}} \frac{d\omega}{\omega-z} \\ = \frac{1}{z^{1/2}(1+z^2)^{1/4}} \exp\left(-\frac{1}{\pi} \int_0^z \frac{\ln s - i\pi}{1+s^2} ds\right). \quad (19)$$

Therefore, we rewrite the above governing equation (18) as

$$\frac{G(z)}{X(z)} = C_o - \frac{1}{2} \frac{z^{1/2}}{(1+z^2)^{1/4}} \exp\left(-\frac{1}{\pi} \int_0^z \frac{\ln s - i\pi}{1+s^2} ds\right) \\ + \frac{\lambda}{2\pi i} z \int_0^\infty \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{3/4}} \frac{\Psi(\omega)}{\omega-z} d\omega, \quad (20)$$

where

$$C_o = \frac{1}{2\pi i} \int_0^\infty \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{3/4}} [-1 + \lambda\Psi(\omega)] d\omega.$$

As p approaches to 0 in Eq.(B-13) which was obtained by Spence⁽⁵⁾, we obtain

$$\frac{1}{\pi} \int_0^\infty \frac{e^{m(x)}}{x^{1/2}(1+x^2)^{3/4}} dx = 1. \quad (21)$$

Therefore, we obtain

$$C_o = \frac{1}{2\pi i} \left(-\pi + \lambda \int_0^\infty \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{3/4}} \Psi(\omega) d\omega \right).$$

We define $I(z)$ by

$$I(z) \equiv \frac{1}{2\pi i} \int_0^\infty \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{3/4}} \frac{\Psi(\omega)}{\omega-z} d\omega. \quad (22)$$

Then we obtain from Eq.(20), using the results of Boersma⁽¹⁵⁾ and Eq.(17),

$$G_\pm(\sigma) = C_o \frac{\sigma \pm i}{\sigma^{1/2}(1+\sigma^2)^{1/4}} e^{-m(\sigma)} - \frac{i}{2} \\ + \lambda \frac{\sigma^{1/2} e^{-m(\sigma)}}{(1+\sigma^2)^{1/4}} (\sigma \pm i) I_\pm(\sigma). \quad (23)$$

From the definition of $I(z)$ in Eq.(22), we arrive at

$$G_+(\sigma) - G_-(\sigma) = 2iC_o \frac{e^{-m(\sigma)}}{\sigma^{1/2}(1+\sigma^2)^{1/4}} \\ + \lambda \frac{\sigma}{1+\sigma^2} \Psi(\sigma) + \frac{\lambda}{\pi} \frac{\sigma^{1/2}}{(1+\sigma^2)^{1/4}} e^{-m(\sigma)} \\ \times \int_0^\infty \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{3/4}} \frac{\Psi(\omega)}{\omega-\sigma} d\omega. \quad (24)$$

Let us consider the last term of the right-hand side of Eq.(24). We here consider the integral:

$$H(s) = \int_0^\infty \frac{e^{-sX}}{(1+\lambda X)^{1/2}((1+\lambda X)^{1/2}+a)} dX, \quad (25)$$

where $a = (1+\lambda\xi)^{1/2}$. Then we obtain (see Appendix 2)

$$H_+(s) = -\int_0^{1/\lambda} \frac{e^{sX}}{(1-\lambda X)^{1/2}((1-\lambda X)^{1/2}+a)} dX \\ - i \int_{1/\lambda}^\infty \frac{e^{sX}(a+i(\lambda X-1)^{1/2})}{(\lambda X-1)^{1/2}(a^2+\lambda X-1)} dX \\ \text{for Real } s < 0 \\ H_-(s) = -\int_0^{1/\lambda} \frac{e^{sX}}{(1-\lambda X)^{1/2}((1-\lambda X)^{1/2}+a)} dX \\ + i \int_{1/\lambda}^\infty \frac{e^{sX}(a-i(\lambda X-1)^{1/2})}{(\lambda X-1)^{1/2}(a^2+\lambda X-1)} dX \\ \text{for Real } s < 0.$$

Using these results, we obtain (see Appendix 3)

$$\frac{1}{2} \frac{\sigma e^{m(\sigma)}}{\sigma^{1/2}(1+\sigma^2)^{3/4}} H(\sigma) \\ = -\frac{1}{2\pi} \int_0^\infty \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{3/4}} \frac{H(\omega)}{\omega-\sigma} d\omega \\ - \frac{1}{2\pi} \int_0^\infty \frac{e^{-m(\omega)}}{\omega^{1/2}(1+\omega^2)^{1/4}} \frac{d\omega}{\omega+\sigma} \\ \times \int_{1/\lambda}^\infty \left(\frac{1+\lambda\xi}{\lambda X-1} \right)^{1/2} \frac{e^{-\omega X}}{\lambda(X+\xi)} dX. \quad (26)$$

Substituting Eq.(26) into the last term of Eq.(24) and taking into account the definition of $\Psi(\omega)$, we obtain

$$\frac{\lambda}{\pi} \frac{\sigma^{1/2}}{(1+\sigma^2)^{1/4}} e^{-m(\sigma)} \int_0^\infty \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{3/4}} \frac{\Psi(\omega)}{\omega-\sigma} d\omega \\ = \frac{1}{\pi} \int_0^\infty g(\xi) d\xi \left(-\lambda \frac{\sigma}{1+\sigma^2} H(\sigma) \right) \\ - \frac{1}{\pi} \frac{\sigma^{1/2}}{(1+\sigma^2)^{1/4}} e^{-m(\sigma)} \int_0^\infty d\omega \int_{1/\lambda}^\infty \frac{e^{-m(\sigma)-\omega X}}{\omega^2(1+\omega^2)^{1/4}}$$

$$\times \frac{1}{\omega + \sigma} \left(\frac{1 + \lambda \xi}{\lambda X - 1} \right)^{1/2} \frac{1}{\xi + X} dX. \quad (27)$$

Thus, we arrive at the following from Eq. (24):

$$\begin{aligned} G_+ - G_- &= 2iC_o \frac{e^{-m(\sigma)}}{\sigma^{1/2}(1+\sigma^2)^{1/4}} + \lambda \frac{\sigma}{1+\sigma^2} \Psi(\sigma) \\ &\quad - \lambda \frac{\sigma}{1+\sigma^2} \frac{1}{\pi} \int_0^\infty g(\xi) d\xi H(\sigma) \\ &\quad - \frac{\sigma^{1/2}}{(1+\sigma^2)^{1/4}} e^{-m(\sigma)} \frac{1}{\pi^2} \int_0^\infty d\omega \int_{1/\lambda}^\infty dX \\ &\quad \times \int_0^\infty g(\xi) \left(\frac{1 + \lambda \xi}{\lambda X - 1} \right)^{1/2} \frac{e^{-m(\sigma) - \omega X}}{\omega^{1/2}(1+\omega^2)^{1/4}} \\ &\quad \times \frac{d\xi}{(\omega + \sigma)(\omega + X)}. \end{aligned} \quad (28)$$

From the inverse Laplace transform, we finally obtain (see Appendix 4)

$$\begin{aligned} g(X) &= -2iC_o f_o(X) + \frac{1}{\pi^2} \int_0^\infty d\omega \int_{1/\lambda}^\infty d\eta \int_0^\infty g(\eta) \\ &\quad \times \frac{e^{-m(\omega) - \omega \eta}}{\omega^{1/2}(1+\omega^2)^{1/4}} \left(\frac{\lambda \xi + 1}{\lambda \eta - 1} \right)^{1/2} \frac{1}{\xi + \eta} \\ &\quad \times \left(f_o(X) - \omega \int_0^X f_o(X-t) e^{-\omega t} dt \right) d\xi. \end{aligned} \quad (29)$$

This final expression is a Fredholm integral equation without any singularity.

4. Uniqueness of solution

We discuss here the uniqueness of the solution if it exists. We will show the contradiction if there are two solutions.

Suppose that there are two solutions, $g_1(x)$ and $g_2(x)$. We define their difference by $\Delta g(x) = g_1(x) - g_2(x)$. Then the following boundary conditions are satisfied:

$$\Delta g(1) = 0, \Delta g(x) \rightarrow 0 \text{ as } x \rightarrow \infty.$$

The function $\Delta g(x)$ is also governed by Eq. (6), so that we multiply $x\Delta g(x)$ and integrate from 1 to infinity:

$$\begin{aligned} \int_1^\infty x \Delta g(x) \Delta g'(x) dx &= -\frac{1}{\pi \lambda} \int_1^\infty \Delta g(x) x^{1/2} dx \\ &\quad \times \int_1^\infty \frac{\xi^{1/2} \Delta g(\xi)}{\xi - x} d\xi. \end{aligned}$$

The right-hand side becomes zero, if $\Delta g(x)x^{1/2}$ is continuous and integrable over $(1, \infty)$. We therefore have, by means of integration by parts:

$$\int_1^\infty \Delta g^2(x) dx = 0.$$

Thus, $\Delta g(x)$ must be zero. From this fact, if a solution exists, it must be unique.

5. Existence of solutions

In order to use Fredholm's alternative theory⁽¹⁴⁾, we consider the kernel function.

$$\begin{aligned} K(X, \xi) &= \frac{1}{\pi^2} \int_0^\infty d\omega \int_{1/\lambda}^\infty d\eta \frac{e^{-m(\omega) - \omega \eta}}{\omega^{1/2}(1+\omega^2)^{1/4}} \\ &\quad \times \left(\frac{\lambda \xi + 1}{\lambda \eta - 1} \right)^{1/2} \frac{1}{\xi + \eta} \left(f_o(X) \right. \\ &\quad \left. - \omega \int_0^X f_o(X-t) e^{-\omega t} dt \right). \end{aligned} \quad (30)$$

The kernel $K(X, \xi)$ is bounded by

$$\begin{aligned} |K(X, \xi)| &\leq \frac{1}{\pi^2} \int_0^\infty d\omega \int_{1/\lambda}^\infty d\eta \frac{e^{-m(\omega) - \omega \eta}}{\omega^{1/2}(1+\omega^2)^{1/4}} \\ &\quad \times \left(\frac{\lambda \xi + 1}{\lambda \eta - 1} \right)^{1/2} \frac{1}{\xi + \eta} \left| f_o(X) - \omega \int_0^X f_o(X-t) e^{-\omega t} dt \right|. \end{aligned}$$

We use the relation:

$$\begin{aligned} \int_{1/\lambda}^\infty \frac{e^{-\omega \eta}}{(\lambda \eta - 1)^{1/2}(\xi + \eta)} d\eta &= (\lambda \omega)^{1/2} e^{-\omega/\lambda} \\ &\quad \times \int_0^\infty X^{-1/2} e^{-X} \frac{dX}{\lambda X + \omega + \lambda \xi \omega} \\ &\leq (\lambda \omega)^{1/2} e^{-\omega/\lambda} \int_0^\infty \frac{X^{-1/2}}{\lambda X + \omega + \lambda \omega \xi} dX \\ &= e^{-\omega/\lambda} \frac{\pi}{(1 + \lambda \xi)^{1/2}}. \end{aligned}$$

Then we have

$$\begin{aligned} |K(X, \xi)| &\leq \frac{1}{\pi} \int_0^\infty e^{-m(\omega) - \omega/\lambda} \frac{1}{\omega^{1/2}(1+\omega^2)^{1/4}} \\ &\quad \times \left| f_o(X) - \omega \int_0^X f_o(X-t) e^{-\omega t} dt \right| d\omega. \end{aligned}$$

Taking into account Spence's result⁽⁵⁾ of $f_o(x)$, we obtain

$$\begin{aligned} f_o(X) - \omega \int_0^X f_o(X-t) e^{-\omega t} dt &\simeq O(1) \\ \text{for } \omega \rightarrow 0 \text{ and } X \ll 1 \text{ or } X \gg 1 \\ &\simeq O(1/\omega) \text{ for } \omega \rightarrow \infty \text{ and } X \ll 1 \\ &\simeq O(e^{-\omega X}/\omega^{1/2}) \text{ for } \omega \rightarrow \infty \text{ and } X \gg 1. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |K(X, \xi)| &\leq \frac{\lambda^{1/2}}{\pi} \int_0^\infty \frac{e^{-m(\lambda p) - p}}{p^{1/2}} \left| f_o(X) \right. \\ &\quad \left. - \lambda p \int_0^X f_o(X-t) e^{-\lambda p t} dt \right| dp \leq C_o \lambda^{1/2} \end{aligned} \quad (31)$$

where C_o is constant and independent of λ . There exists at most one solution of the homogenous equation of the governing relation (29), which is zero. From this fact and Eq. (31), Fredholm's alternative theorem proves that a solution for a small λ exists. From the uniqueness proved in section 4, we see that the governing integro-differential equation has only one solution for small λ .

6. Asymptotic solution

We try to obtain an asymptotic solution for large and small values of X , in the case of small λ , in order to compare with earlier results.

For $X \ll 1$, we use Spence's result⁽⁵⁾ of $f_o(X)$. Then $g(X)$ becomes

$$\begin{aligned} g(X) &\simeq -2iC_o \left(1 + \frac{X}{\pi} (\ln X + \gamma - 2) + \dots \right) \\ &\quad + \frac{1}{\pi^2} \int_0^\infty d\omega \int_{1/\lambda}^\infty d\eta \int_0^\infty g(\xi) \frac{e^{-m(\omega) - \omega \eta}}{\omega^{1/2}(1+\omega^2)^{1/4}} \\ &\quad \times \left(\frac{\lambda \xi + 1}{\lambda \eta - 1} \right)^{1/2} \frac{1}{\xi + \eta} \left(1 + \frac{X}{\pi} (\ln X + \gamma - 2) + \dots \right. \\ &\quad \left. - \omega e^{-\omega X} \int_0^X \left\{ 1 + \frac{t}{\pi} (\ln t + \gamma - 2) + \dots \right\} e^{\omega t} dt \right) d\xi \end{aligned}$$

where γ is the Euler constant. From the boundary condition, $g(0) = 1$, we have

$$2iC_o = -1 + \frac{1}{\pi^2} \int_0^\infty d\omega \int_{1/\lambda}^\infty d\eta \int_0^\infty g(\xi) \\ \times \frac{e^{-m(\omega) - \eta\omega}}{\omega^{1/2}(1+\omega^2)^{1/4}} \left(\frac{\lambda\xi+1}{\lambda\eta-1} \right)^{1/2} \frac{1}{\xi+\eta} d\xi.$$

Therefore, we have

$$g(X) \simeq 1 + \frac{X}{\pi} (\ln X + \gamma - 2) - \frac{1}{\pi^2} \int_0^\infty d\omega \int_0^\infty g(\xi) \\ \times \frac{e^{-m(\omega)}}{\omega^{1/2}(1+\omega^2)^{1/4}} (\lambda\xi+1)^{1/2} \\ \times (1 - e^{-\omega X}) h(\omega, \xi) d\xi$$

where $h(\omega, \xi)$ is given by

$$h(\omega, \xi) = \int_{1/\lambda}^\infty \frac{e^{-\omega\eta}}{(\lambda\eta-1)^{1/2}(\xi+\eta)} d\eta \\ = 2e^{-\omega/\lambda} \int_0^\infty \frac{e^{-\omega\eta^{2/\lambda}}}{\eta^2 + \lambda\xi + 1} d\eta.$$

From these relations, we finally have (see Appendix 5)

$$g(X) \simeq 1 + \frac{X}{\pi} (\ln X + \gamma - 2) \\ - \frac{8}{3} \lambda^{3/2} \Gamma(3/2) \frac{1}{\pi^2} \int_0^\infty g(\xi) d\xi X. \quad (32)$$

We here note that the third term of the right-hand side is not determined by the asymptotic expansion for $X \ll 1$.

Applying Spence's result⁽⁵⁾ of $f_o(X)$ also, we obtain for $X \gg 1$:

$$g(X) \simeq -2iC_o \left(\frac{1}{\pi^{1/2} X} + \frac{1}{2(\pi X)^{3/2}} (\ln(4X) + \gamma - 1) + \dots \right) \\ - \frac{1}{\pi^2} \int_0^\infty d\omega \int_0^\infty g(\xi) \frac{e^{-m(\omega)}}{\omega^{1/2}(1+\omega^2)^{1/4}} (\lambda\xi+1)^{1/2} \\ \times \left(-\frac{1}{2\pi^{1/2}\omega X^{3/2}} \right) h(\omega, \xi) d\xi.$$

Furthermore, we have

$$\int_0^\infty \frac{e^{-m(\omega)}}{\omega^{3/2}(1+\omega^2)^{1/4}} h(\omega, \xi) d\omega \\ \simeq \frac{1}{\pi^{1/2}} \int_0^\infty \frac{d\omega}{\omega^{3/2}} \int_0^\infty \frac{e^{-\omega(1+\eta^2)}}{1+\eta^2} d\eta \\ \simeq -\left(\frac{\pi}{\lambda}\right)^{1/2} \int_0^\infty \frac{e^{-\omega}}{\omega} d\omega.$$

Therefore, we finally have

$$g(X) \simeq -2iC_o \left(\frac{1}{(\pi X)^{1/2}} + \frac{1}{2(\pi X)^{3/2}} \right. \\ \left. \times (\ln(4X) + \gamma - 1) \right) + \frac{\gamma}{2\pi^2} \int_0^\infty g(\xi) d\xi \frac{\lambda^{-1/2}}{X^{3/2}}.$$

From these results, we see that the matching process is necessary as shown by Kida and Miyai⁽⁶⁾.

7. Conclusions

This paper analyzes the integro-differential equation with a Cauchy-type kernel for a thin jet-flapped aerofoil. We summarize the main results obtained in this analysis. (1) The integral equation of the Fredholm type of the second kind which is not singular is derived by using the complex Laplace transform; (2) the existence of the solution is shown by using the Fredholm's alternative theory; (3) the unique-

ness of the solution is proven; (4) the asymptotic solutions for $X \ll 1$ and $X \gg 1$ are obtained; and (5) the need for a matching process is shown.

The present results imply that numerical results, which have been obtained in various problems on a jet-flapped aerofoil, are reasonable from the mathematical point of view. However, strictly speaking, we will have to analyze the consistency and stability of each numerical method.

Appendix 1

From Boersma⁽¹⁵⁾, $X(z)$ has the following features:

(i) $X(z) \simeq (-z)^{1/2}$ as $z \rightarrow 0$, $X(z) \rightarrow 1$ as $z \rightarrow \infty$

(ii) $m^*(\sigma \pm i0) = m(\sigma) \pm i \tan^{-1} \sigma$

$$m(\sigma) = -\frac{1}{\pi} \int_0^\sigma \frac{\ln t}{1+t^2} dt \quad (A-1)$$

(iii) $X_\pm(\sigma) = (\sigma \pm i) e^{-m(\sigma)/\sigma^{1/2}(1+\sigma^2)^{1/4}}$

(iv) $X(z)$ is only cut along the real positive axis.

From these, we consider the following integral over the contour shown in Fig. 2.

$$I(z) = \frac{1}{2\pi i} \oint_C \frac{1}{\omega^{1/2}(1+\omega^2)^{1/4}} \\ \times \exp\left(-\frac{1}{\pi} \int_0^\omega \frac{\ln s - i\pi}{1+s^2} ds\right) \frac{1}{\omega - z} d\omega.$$

Then we have from Cauchy residue theorem:

$$I(z) = \frac{1}{z^{1/2}(1+z^2)^{1/4}} \exp\left(-\frac{1}{\pi} \int_0^z \frac{\ln s - i\pi}{1+s^2} ds\right).$$

On the other hand, $I(z)$ becomes

$$I(z) = \frac{1}{2\pi i} \int_\delta^\infty \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{1/4}} \frac{1+i\omega}{(1+\omega^2)^{1/2}} \frac{1}{\omega - z} d\omega \\ + \frac{1}{2\pi i} \int_\delta^\infty \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{1/4}} \frac{1-i\omega}{(1+\omega^2)^{1/2}} \frac{1}{\omega - z} d\omega, \\ (\delta \rightarrow 0).$$

Therefore, we arrive at Eq. (19).

Appendix 2

From the definition, we have

$$H(se^{i\theta}) = \int_0^\infty \frac{e^{-se^{i\theta} X}}{(1+\lambda X)^{1/2}((1+\lambda X)^{1/2} + a)} dX.$$

Here, we vary the integral variable as $\tilde{X} = Xe^{i\theta}$ then the integrand is analytic except the cut $-\infty < \tilde{X} < -1/\lambda$. We therefore obtain

Fig. 2 Integral contour C

Fig. 3 Integral contour C

$$H(se^{i\pi}) = - \int_0^{1/\lambda} \frac{e^{-s\hat{X}}}{(1-\lambda\hat{X})^{1/2}((1-\lambda\hat{X})^{1/2}+a)} d\hat{X} \\ + \frac{1}{i} \int_{1/\lambda}^{\infty} \frac{e^{-s\hat{X}}}{(\lambda\hat{X}-1)^{1/2}(a-i(\lambda\hat{X}-1)^{1/2})} d\hat{X}.$$

Thus, $H_+(s)$ is obtained. Letting $\theta \rightarrow -\pi$, we also obtain $H_-(s)$.

Appendix 3

We consider the following integral over the contour shown in Fig. 3:

$$I(z) = \frac{1}{2\pi i} \oint_C \frac{H(\omega)}{\omega^{1/2}(1+\omega^2)^{1/4}} \\ \times \exp\left(-\frac{1}{\pi} \int_0^{\omega} \frac{\ln s - i\pi}{1+s^2} ds\right) \frac{1}{\omega - z} d\omega.$$

From the Cauchy residue theorem, if $z \in D$, we obtain

$$I(z) = \frac{H(z)}{z^{1/2}(1+z^2)^{1/4}} \exp\left(-\frac{1}{\pi} \int_0^z \frac{\ln s - i\pi}{1+s^2} ds\right).$$

Integrating along the contour, we obtain

$$I(z) = \frac{1}{2\pi i} \int_0^{\infty} \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{1/4}} \frac{1+i\omega}{(1+\omega^2)^{1/2}} \frac{H(\omega)}{\omega - z} d\omega \\ + \frac{1}{2\pi} \int_0^{\infty} \frac{e^{-m(\omega)}}{\omega^{1/2}(1+\omega^2)^{1/4}} \frac{H(\omega e^{i\pi})}{\omega + z} d\omega.$$

Thus, we arrive at

$$\frac{1}{2} \frac{e^{m(\sigma)}}{\sigma^{1/2}(1+\sigma^2)^{1/4}} \frac{1+i\sigma}{(1+\sigma^2)^{1/2}} H(\sigma) \\ = \frac{1}{2\pi i} \int_0^{\infty} \frac{e^{m(\omega)}}{\omega^{1/2}(1+\omega^2)^{3/4}} \frac{1+i\omega}{\omega - \sigma} H(\omega) d\omega \\ + \frac{1}{2\pi} \int_0^{\infty} \frac{e^{-m(\omega)}}{\omega^{1/2}(1+\omega^2)^{1/4}} \frac{1}{\omega + \sigma} \\ \times \left(- \int_0^{1/\lambda} \frac{e^{-\omega X}}{(1-\lambda X)^{1/2}((1-\lambda X)^{1/2}+(1+\lambda\xi)^{1/2})} dX \right. \\ \left. - \int_{1/\lambda}^{\infty} \frac{e^{-\omega X} i(1+\lambda\xi)^{1/2} - (\lambda X - 1)^{1/2}}{\lambda(\lambda X - 1)^{1/2}(\xi + X)} dX \right) d\omega.$$

Taking the imaginary part of the above equation and changing the order, we finally arrive at Eq.(26).

Appendix 4

We first consider the inverse Laplace transform of the following function:

$$J(s) = \frac{e^{-m(s)}}{s^{1/2}(1+s^2)^{1/4}}.$$

Changing the integral contour, we obtain

$$\mathcal{L}^{-1}(J(s)) = \frac{1}{2\pi i} \left(- \int_{0e^{-i\pi}}^{\infty e^{-i\pi}} + \int_{0e^{i\pi}}^{\infty e^{i\pi}} \right) J(s) e^{sX} ds.$$

From the definition of $m(s)$, we obtain

$$m(\sigma e^{\pm i\pi}) = -m(\sigma) \pm i \tan^{-1} \sigma.$$

Therefore, we obtain

$$\mathcal{L}^{-1}(J(s)) = \frac{1}{\pi} \int_0^{\infty} \frac{e^{m(\sigma)}}{\sigma^{1/2}(1+\sigma^2)^{3/4}} e^{-sX} d\sigma = f_o(X).$$

Second, we consider the inverse Laplace transforms:

$$\mathcal{L}^{-1}(s\Psi(s)/(1+s^2)) = \int_0^X \psi(t) \cos(X-t) dt$$

$$\mathcal{L}^{-1}\left(\frac{s^{1/2}e^{-m(s)}}{(1+s^2)^{1/4}(\omega+s)}\right) \\ = f_o(X) - \omega \int_0^X f_o(X-t) e^{-\omega t} dt$$

$$\mathcal{L}^{-1}\left(\frac{s}{1+s^2} H(s)\right) = \int_0^X \cos(X-t) \\ \times \frac{dt}{(1+\lambda X)^{1/2}((1+\lambda t)^{1/2}+(1+\lambda\xi)^{1/2})}.$$

Thus, we arrive at Eq.(29).

Appendix 5

We see that the final expression of $g(X)$ for $X \ll 1$ is

$$g(X) \simeq 1 + \frac{X}{\pi} (\ln X + \gamma - 2) - X \left(\frac{2}{\pi^2} \int_0^{\infty} g(\xi) d\xi \right) \\ \times \int_0^{\infty} \frac{\omega^{1/2} e^{-m(\omega)}}{(1+\omega^2)^{1/4}} e^{-\omega/\lambda} d\omega \int_0^{\infty} \frac{e^{-\omega\eta^{2/\lambda}}}{1+\eta^2} d\eta.$$

Varying the integral variable ω as $\omega = \lambda\bar{\omega}$, we see that the integrations with respect to ω and η of the last term of the right-hand side become

$$\simeq \lambda^{3/2} \int_0^{\infty} \bar{\omega}^{1/2} \exp\left(\frac{1}{\pi} \lambda \bar{\omega} (\ln \lambda \bar{\omega} - 1)\right) \\ \times e^{-\bar{\omega}} \int_0^{\infty} \frac{e^{-\bar{\omega}\eta^2}}{1+\eta^2} d\eta d\bar{\omega}.$$

Using the integral expression of the Gamma function, we arrive at

$$\simeq \lambda^{3/2} \int_0^{\infty} \frac{d\eta}{1+\eta^2} \int_0^{\infty} \omega^{1/2} e^{-\omega(1+\eta^2)} d\omega \\ \simeq \lambda^{3/2} \int_0^{\infty} \frac{d\eta}{(1+\eta^2)^{5/2}} \int_0^{\infty} x^{1/2} e^{-x} dx \\ = \lambda^{3/2} \Gamma(3/2) \int_0^{\infty} \frac{d\eta}{(1+\eta^2)^{5/2}}.$$

Therefore, we arrive at Eq.(32).

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