Nonlinear Phenomena Induced by Finite-Amplitude Oscillation of Air Column in Closed Duct*

(Analysis of Acoustic Streaming)

Masaaki KAWAHASHI** and Masahiro ARAKAWA***

Acoustic streaming is one of the nonlinear phenomena produced by strong sound waves. This type of streaming is driven by acoustic momentum flux in an attenuating sound field. A strong standing wave in a duct generated by finite-amplitude oscillation of the air column dissipating due to friction at the duct wall produces acoustic streaming. The velocity of streaming is estimated from the steady part of the second-order term of a perturbation expansion of a sinusoidal oscillation. However, finite-amplitude oscillation gives rise to shock-wave propagation in the duct. In order to estimate acoustic streaming produced by finite-amplitude oscillation, it is necessary to analyze the response of the oscillatory boundary layer to shock waves in detail. The present paper deals with numerical analysis of the acoustic streaming described above. The fourth-order spatial difference method is applied to two-dimensional analysis of acoustic streaming in this work. Calculated results show velocity distributions in the oscillatory boundary layer and structures of steady streaming for various amplitudes of oscillation.

Key Words: Oscillatory Flow, Secondary Flow, Numerical Analysis, Wave, Finite-Amplitude Wave, Acoustic Streaming

1. Introduction

Since acoustic streaming induced as a nonlinear phenomenon due to finite-amplitude oscillation of an air column in a closed duct was first described by Rayleigh in his article⁽¹⁾, it has been a matter of interest for many researchers. As seen by the growing interest in nonlinear phenomena in recent years, the investigation of acoustic streaming has entered a new phase. As engineering problems relevant to acoustic streaming, improvement of heat and mass transfer, control of the turbulent boundary layer, and the effect on the performance of a sonic compressor, are attracting considerable attention.

As Lighthill explained in his paper⁽²⁾, acoustic streaming is induced under the action of Reynolds

stress in the sense of mean acoustic momentum flux. Acoustic streaming is caused by the gradient of acoustic Reynolds stress in an attenuated sound field, therefore the features of streaming depend on the mechanism of sound attenuation processes. This paper deals with acoustic streaming induced in a sound field of a standing wave generated by air column oscillation in a closed tube. In this case, the attenuation of sound is caused by wall friction. This type of streaming is called "Rayleigh's streaming".

Assuming that the amplitude of air column oscillation by sound propagation in the duct is sufficiently small, acoustic streaming is obtained as the steady part of the second-order term in perturbation expansion. The magnitude and the profile of streaming depend strongly on the oscillatory velocity profile in the Stokes layer. In engineering applications, acoustic streaming induced by a large-amplitude standing wave with shock wave propagation caused by finite amplitude oscillation of the air column must be analyzed. In our case, the response of the Stokes layer to shock wave propagation and its transition to turbulence are considerable problems. Therefore detailed analysis of finite amplitude oscillation in the duct

* Received 29th May, 1995. Japanese original: Trans. Jpn. Soc. Mech. Eng., Vol. 60, No. 580, B (1994), pp. 4059-4065. (Received 23rd March, 1994)

** Department of Mechanical Engineering, Faculty of Engineering, Saitama University, 255 Shimo-okubo, Urawa, Saitama 338, Japan

*** Faculty of Engineering, Kansai University, 3-3 Yamate-cho, Suita, Osaka 564, Japan

accompanied by a thin oscillatory boundary layer is required. Recent developments in numerical analysis can fulfill this requirement. However the numerical calculation of finite-amplitude wave phenomena including the second-order nonlinear term is restricted by severe conditions on accuracy and stability.

Here, we report two-dimensional analysis of acoustic streaming induced by finite-amplitude oscillation in a closed duct driven by a piston. A finite difference method of fourth-order accuracy in space and second-order accuracy in time⁽³⁾, the accuracy and stability of which have been investigated by calculation for a one-dimensional model in a previous paper⁽⁴⁾, is applied. In this analysis, it is assumed that a laminar boundary layer develops in the duct. The results obtained show the change of the streaming profile owing to the increase in the amplitude of oscillation and the existence of a double layer in the vicinity of the duct wall.

2. Nomenclature

- c^* : velocity of sound
 c : nondimensional velocity of sound ($=c^*/c_0^*$)
 e : nondimensional total energy per unit volume ($=e^*/\rho_0^*c_0^*$)
 f : nondimensional frequency of piston oscillation ($=f/f_0^*$)
 f_0^* : nondimensional acoustic resonance frequency
 h^* : half-width of duct
 h : non dimensional half-width of duct ($=h^*/\Lambda^*$)
 L^* : duct length, L : nondimensional duct length ($=L^*/\Lambda^*$)
 p^* : pressure, p : nondimensional pressure ($=p^*/\rho_0^*c_0^*$)
 t^* : time, t : nondimensional time ($=t^*/(1/\omega^*)$)
 u^* : velocity of x component
 u : nondimensional velocity of x component ($=u^*/c_0^*$)
 v^* : velocity of y component
 v : nondimensional velocity of y component ($=v^*/c_0^*$)
 x, y : physical coordinate systems
 γ : ratio of specific heat
 δ^* : thickness of boundary layer
 δ : nondimensional thickness of boundary layer ($=\delta^*/\Lambda^*$)
 δu : nondimensional velocity amplitude of piston ($=\delta u^*/c_0^*$)
 Λ^* : wavelength at the piston oscillation frequency
 λ^* : second kind of viscosity
 λ : nondimensional second kind of viscosity ($=\lambda^*/\rho_0^*c_0^*\Lambda^*$)
 μ^* : viscosity, μ : nondimensional viscosity

$$(\mu^*/\rho_0^*c_0^*\Lambda^*)$$

ρ^* : density, ρ : nondimensional density
 $(=\rho^*/\rho_0^*)$

ξ, η : curvilinear coordinate system

ω^* : angular frequency ($=2\pi f^*$)

Cr : Courant number

Pr : Prandtl number

Subscripts

0 : standard state

$\langle \rangle$: time average

c : oscillatory component along duct axis

$\sim$: amplitude of oscillatory component

$*$: dimensional quantity

3. Numerical Analysis

3.1 Method of calculation

The calculation model is illustrated in Cartesian coordinates in Fig. 1. A standing wave in a two-dimensional duct closed at one end is induced by air column oscillation excited by a piston oscillating sinusoidally at the other end. The width of the duct is sufficiently smaller than the wavelength determined by the driving frequency of the piston. The region of calculation is half of the duct bounded by the duct axis at which the symmetrical boundary condition is applied. In this calculation, second-mode oscillation of the air column is assumed considering periodical structures of acoustic streaming developing in each quarter-wavelength and the time consumed for calculation. Nondimensional frequency of oscillation $f=1.0$ and nondimensional length of duct $L=1.0$. Nondimensional value of the half-width of the duct used is $h=1.8 \times 10^{-2}$.

When the driving frequency of the piston is close to the resonance frequency of the air column, the increase of oscillation amplitude introduces a finite-amplitude wave motion in the duct. Nonlinear distortion of the waves then generates propagating shock waves.

Shock waves appear in the range of $u \delta u \geq 1 \times 10^{-3}$,

Fig. 1 Calculation model and coordinate system

as shown in the results of one-dimensional analysis at the second-mode resonance in the range of velocity amplitude of the oscillating piston of $\delta u = 1.0 \times 10^{-5} \sim 1.0 \times 10^{-2}$. Calculations in this report are carried out in this same range of amplitude, and the induction of acoustic streaming and the change of its profile due to the effects of shock waves are investigated.

3.2 Basic equations and boundary conditions

The basic equations are two-dimensional, unsteady, compressible Navier Stokes equations, the unsteady equation of continuity and the energy equation. In comparison with experiments, the calculations in this work are restricted under the conditions that both the ratio of wavelength to duct width and the Stokes number (h/δ) are large.

In mesh generation, the mesh size in the vicinity of the duct wall must be fine, because the induction of acoustic streaming depends strongly on the velocity profile in the Stokes layer. A nonuniform mesh in perpendicular to the wall is applied and the basic equations are presented in a curvilinear coordinate system (ξ, η) as follows.

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial \xi} + \frac{\partial F}{\partial \eta} = 0 \quad (1)$$

$$U = \frac{1}{J} \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{Bmatrix}, E = \frac{1}{2\pi J} \begin{Bmatrix} \rho u U + \xi_x(p - \tau_{xx}) - \xi_y \tau_{xy} \\ \rho v U - \xi_x \tau_{yx} + \xi_y(p - \tau_{yy}) \\ (e + p)U - \xi_x \beta_x - \xi_y \beta_y \end{Bmatrix}$$

$$F = \frac{1}{2\pi J} \begin{Bmatrix} \rho V \\ \rho u V + \eta_x(p - \tau_{xx}) - \eta_y \tau_{xy} \\ \rho v V - \eta_x \tau_{yx} + \eta_y(p - \tau_{yy}) \\ (e + p)V - \eta_x \beta_x - \eta_y \beta_y \end{Bmatrix}$$

$$p = (\gamma - 1) \left(e - \frac{m^2 + n^2}{2\rho} \right), m = \rho u, n = \rho v$$

$$\tau_{xx} = \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial u}{\partial x},$$

$$\tau_{yy} = \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial v}{\partial y},$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\beta_x = \frac{\gamma\mu}{Pr} \frac{\partial \varepsilon}{\partial x} + u\tau_{xx} + v\tau_{xy}, \beta_y = \frac{\gamma\mu}{Pr} \frac{\partial \varepsilon}{\partial y} + u\tau_{xy} + v\tau_{yy}$$

$$U = \xi_t + \xi_x u + \xi_y v, V = \eta_x u + \eta_y v, \varepsilon = \frac{p}{(\gamma - 1)\rho}$$

The following boundary conditions are determined. The wall and the closed end of the duct are nonslip, and a symmetric boundary condition is applied along the duct axis. The driving end is the surface of the sinusoidally oscillating piston, and the boundary condition of piston velocity is defined at the position of the piston at each time step of calculation. This means that the calculation region is between the closed end and the instantaneous position of the piston. The thermal boundary condition of walls is adiabatic. The frequency of piston oscillation is fixed

at the second-mode resonance frequency of the air column, and the range of velocity amplitude of the piston is from $\delta u = 1 \times 10^{-5}$ to 1×10^{-2} .

3.3 Method of numerical calculation

Because acoustic streaming is calculated as a second-order small quantity induced by mean acoustic momentum flux, high stability and accuracy in the calculation are required. The explicit MacCormack method of fourth-order accuracy in space and second-order accuracy in time is applied, and the calculations are carried out by means of the time splitting method in which the basic equations are separated into two one-dimensional equations.

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial \xi} = 0 \quad (2)$$

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial \eta} = 0 \quad (3)$$

Equation (2) is discretized as follows. (predictor)

$$\bar{U}_i^n = U_i^n - \frac{1}{6} \frac{\Delta t}{\Delta \xi} (7E_i^n - 8E_{i-1}^n + E_{i-2}^n) \quad (4)$$

(corrector)

$$U_i^{n+1} = \frac{1}{2} \left\{ U_i^n + \bar{U}_i^n + \frac{1}{6} \frac{\Delta t}{\Delta \xi} (7E_i^n - 8E_{i+1}^n + E_{i+2}^n) \right\} \quad (5)$$

The calculation mesh is shown in Fig. 2. The number of meshes divided uniformly in the direction of the duct length is 80. The number of nonuniform meshes in the direction of the duct width is 64. The inside of the Stokes layer is divided uniformly into 14 meshes and the outside, in a geometric ration.

The thickness of the Stokes layer δ^* developed in a sinusoidally oscillating flow is represented by

$$\delta^* \propto (\mu^*/\rho^* \omega^*)^{1/2},$$

where the proportional coefficient in this relation is 5 in the calculation.

Courant numbers in x - and y -directions are 0.01 and 0.4, respectively. Because numerical oscillation due to the very small Courant number in the x -direction occurs in the calculated results, the fourth-order artificial viscosity term of third-order accuracy in space is introduced only in the x -direction.

Fig. 2 Calculation mesh (80 × 64)

4. Results and Discussion

4.1 Pressure history at closed end

Figure 3 shows the variation of pressure histories at the closed end for different velocity amplitudes δu of the piston obtained with fully developed oscillation

Fig. 3 Variation of pressure histories at closed end for different amplitudes of piston velocity

(a) Oscillatory velocity amplitude of x component ($x=0.25$)

(b) Oscillatory velocity amplitude of y component ($x=0.5$)

Fig. 4 Variation of oscillatory velocity amplitude in Stokes layer

at the second-mode resonance frequency of the duct. In comparison with the results of one-dimensional analysis, all waveforms are similar, and there are slight differences in the pressure amplitude as the velocity amplitude of the piston increases. This result suggests that the fundamental wave motion in this calculation is not influenced by the effect of the boundary layer.

4.2 Velocity amplitude distributions of oscillatory flow in Stokes layer

On the condition that both the ratio of wavelength to the duct width and the Stokes number are large enough, it is assumed that plane waves propagate through most of the duct cross section, and that a very thin Stokes layer is developed in the vicinity of the wall. The magnitude and the structure of acoustic streaming depend on the change of the velocity profile in the Stokes layer with the increase of oscillation amplitude. The maximum Reynolds number based on the thickness of the Stokes layer in this calculation is 160 which is less than any critical Reynolds number obtained previously⁽⁵⁾.

The distributions of velocity amplitudes of oscillation in the Stokes layer for different amplitudes of the piston velocity are shown in Fig. 4 as (a) velocity profiles of the x -component at the position corresponding to the loop of the acoustic standing wave and (b) velocity profiles of the y -component at the node. When the velocity amplitude of the piston is very small, the velocity profile agrees precisely with that obtained theoretically using the perturbation method. As the amplitude of wave motion increases, the position indicating the maximum velocity of the x component moves toward the duct wall and the value increases. However the thickness of the Stokes layer does not change. Although the tendency of the variation of velocity profiles in the Stokes layer agrees with that observed upon increasing the Stokes number in the calculation by Tijdeman⁽⁶⁾, it is considered that the variation in this calculation is caused by the increase of nonlinear distortion of wave motion in the duct.

4.3 Profile of mean acoustic momentum flux

Acoustic streaming is induced by acoustic Reynolds stress in the sense of mean acoustic momentum flux, represented as, $\langle \rho u u \rangle \langle \rho u v \rangle$, induced by oscillatory flow.

Figure 5 shows the variation of acoustic Reynolds stress distributions in the direction of the duct length at $y = \delta/2$ for different velocity amplitudes of the piston. The results shown in Figs. 5 (a) and (b) are calculated as the time average of these stresses over 2 cycles of piston oscillation. When the amplitude of piston oscillation is small, the profile of $\langle \rho u u \rangle$ is sinusoidal, but it changes with the increase of amplitude at the node. This is caused by sudden change of

Fig. 5 Variation of acoustic Reynolds stress distribution in the direction of duct axis at position of $y = \delta/2$ for different amplitudes of piston velocity

Fig. 6 Variation of spatial gradients of acoustic Reynolds stress in Stokes layer

velocity due to the interaction between the forward movement and the backward movement of shock waves near the node. The change of the profiles of acoustic Reynolds stress causes the change of its gradient.

The gradient of acoustic Reynolds stresses corresponding to the magnitude of the driving force of acoustic streaming in the x -direction is divided into two terms of $\partial \langle \rho u u \rangle / \partial x$ and $\partial \langle \rho u v \rangle / \partial y$. The variations of the gradient distribution in the direction of the duct length at the different positions in the Stokes layer are shown in Fig. 6. When the velocity amplitude of the piston is small, the profile of the gradient of acoustic Reynolds stress is sinusoidal. As the amplitude increases, the profile becomes steeper at the center of the duct. The ratio of the two terms changes with the distance from the wall in Stokes layer, and the sign of both terms is the same inside the layer, but

Fig. 7 Velocity profile of acoustic streaming

Fig. 8 Velocity profile of acoustic streaming in Stokes layer ($\delta u = 1 \times 10^{-2}$)

different outside the layer. This gives rise to a change in the contribution of each term to the induction of acoustic streaming with distance from the wall.

4.4 Velocity profile of acoustic streaming

The steady component obtained by time integral averaging of oscillatory flow corresponds to acoustic streaming, as shown in Fig. 7. The y component of the velocity of acoustic streaming in the x direction is shown. The velocity profiles change with the amplitude of oscillation. In the case of small amplitude, as shown in Fig. 7 (a), the x -component velocity distributions of acoustic streaming are in phase in the y direction at all positions along the x -direction. This

Fig. 9 Variation of velocity profile of acoustic streaming at edge of Stokes layer

produces steady streaming toward the center from both ends of the duct, the direction of which reverses in the next cycle. This type of streaming was not observed in the former investigations because streaming was very weak. On the other hand, in the case of large amplitude, the direction of streaming turns reverses in the vicinity of the wall. The resultant streaming shows a circulatory flow, as predicted in previous investigations⁽⁷⁾.

To investigate velocity profiles of acoustic streaming in detail, the profiles in the vicinity of the wall are shown in Fig. 8. The results indicate the existence of another structure of circulatory streaming known as the double layer. This type of structure has been observed in an incompressible oscillatory flow^{(8),(9)}, but never in the case of an acoustic standing wave.

The distributions in the x -direction of the x component velocity of acoustic streaming at the outer edge of the Stokes layer are shown in Fig. 9. The magnitude of velocity is normalized by the result obtained by means of the perturbation method. Each

Fig. 10 Contour maps of stream function of acoustic streaming

profile almost coincides with the others independent of the amplitude, but the peak position shifts toward the center of the duct. As a result, the whole structure of acoustic streaming is transformed with the increase of amplitude. Contour maps of the stream function corresponding to the pattern of acoustic streaming are shown in Fig. 10. With increasing of amplitude the structure is transformed into a complicated pattern.

5. Conclusions

The two-dimensional, numerical analysis of acoustic streaming induced as a nonlinear phenomenon due to finite amplitude oscillation of an air column in a duct yield the following conclusions.

(1) As the amplitude of oscillation increases, the effective thickness of the Stokes layer decreases and the peak position of velocity shifts toward the wall.

(2) The calculated results of the distribution of acoustic Reynolds stress as the driving force of acoustic streaming show that the magnitudes of stresses almost agree with that calculated by the perturbation method, but the profiles are transformed at the node for large-amplitude oscillation with shock wave propagation.

(3) The structure of calculated acoustic streaming changes with the amplitude of oscillation. When the amplitude is very small, the circulatory streaming predicted theoretically does not occur. As the amplitude increases, circulatory streaming develops and is then distorted to a complicated structure. The existence of a double structure of streaming in the vicinity of the wall was also presented.

References

- (1) Rayleigh, J.W.S., *Theory of Sound*, Vol. 2 (2nd ed.), (1945), p. 334, Dover.
- (2) Lighthill, M.J., *J. Acoustic Streaming*, *Sound Vib.*, Vol. 61, No. 3(1987), p. 391.
- (3) Turkel, E., *On the Practical Use of High Order Methods for Hyperbolic System*, *J. Comput. Phys.*, Vol. 35(1980), p. 319.
- (4) Arakawa, M. and Kawahashi, M., *Nonlinear Phenomena Induced by Finite Amplitude Oscillation of Air Column in a Closed Tube*. (1st report), *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 60, No. 580, B(1994), p. 4052.
- (5) Merkli, P. and Thomann, H., *Transition to Turbulence in Oscillating Pipe Flow*, *J. Fluid Mech.*, Vol. 68, Part 3(1975), p. 567.
- (6) Tijdeman, H., *On the Propagation of Sound Waves in Cylindrical Tubes*, *J. Sound Vib.*, Vol. 39, No. 1(1975), p. 1.
- (7) Andrade, E. N. Da., *On the Groupings and General Behavior of Solid Particles under the Influence of Air Vibrations in Tubes*, *Phil. Trans. Roy. Soc.*, Vol. A 230(1932), p. 413.
- (8) Stuart, J. T., *Double Boundary Layer in Oscillatory Viscous Flow*, *J. Fluid Mech.*, Vol. 24, Part 4(1966), p. 673.
- (9) Takemitsu, N., Nagare, *History of Secondary Flow Related to Kundt Tube*, (in Japanese), Vol. 12, No. 4(1980), p. 24.