Numerical Analysis of Natural Convection of Thermo-Electrically Conducting Fluids in a Square Cavity under a Uniform Magnetic Field* (Calculated Results, Frequency Characteristics)

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In our previous research, primitive variable methods were proposed in the numerical scheme of magnetic field of thermo-electrically conducting fluids. The point to be emphasized in the above-mentioned scheme is to satisfy the conservation law of magnetic flux efficiently. In the present paper, we compare two numerical schemes of magnetic field, namely the cross helicity method and the numerical residual method, concentrating on the convergence of Poisson's equations of the velocity and magnetic fields. Moreover, the relationship between the periodicity of low Prandtl number fluids and the inhibitory effect of convection due to Lorentz force must be clarified quantitatively. The frequency characteristics of kinetic energy, enstrophy, palinstrophy and hydromagnetic cross helicity are calculated using the maximum entropy method.

Key Words: Thermo-Electrically Conducting Fluid, Natural Convection, Frequency Characteristics, Maximum Entropy Method, GSMAC-FEM, MHD

1. Introduction

It is difficult to develop a numerical scheme which satisfies the solenoidal condition of a magnetic field (the conservation law of magnetic flux) sufficiently. Most of the past numerical schemes include either the magnetic vector potential method ($\mathbf{A}$ -method)^{(1),(2)} or the scalar potential method (ϕ -method)^{(3),(4)}.

In Ref. (5), we propose a magnetic field analysis scheme which satisfies the solenoidal condition of a magnetic field sufficiently, by using the hydromagnetic cross helicity density ($\equiv \mathbf{B} \cdot \mathbf{u}$) for a rotational zero field in a hydromagnetic induction equation ($\mathbf{B}$ -expression)⁽⁵⁾.

In Ref. (6), in order to carry out numerical calculations in spite of the existence of fluid motion, a rotational zero field is derived by including the numer-

ical residual term ($\equiv \nabla \cdot \mathbf{B} / \text{Rem}$) in a hydromagnetic induction equation, which does not include velocity $\mathbf{u}$. Thus the general purpose of the magnetic field analysis scheme is extended. In the following discussions, the former scheme is called the cross helicity method, and the latter scheme is called the numerical residual method⁽⁶⁾.

The above-mentioned two magnetic field analysis schemes are a kind of primitive variable method, in which a hydromagnetic induction equation is solved with the magnetic field as an unknown variable. This has the following advantages: (1) it is easy to set up the physical boundary conditions for a magnetic field; (2) it is easy to extend from two-dimensional flow problems to three-dimensional ones; and (3) the gauge condition, i.e., Coulomb gauge or Lorentz gauge, is unnecessary.

From the changes in total kinetic energy and total hydromagnetic cross helicity with time, it has been confirmed that a low Prandtl number fluid shifts from a periodic flow to a steady flow due to the convective inhibitory effect of the Lorentz force. Furthermore, from the change in total hydromagnetic cross helicity with time, it also becomes clear that the combination

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intensity between the velocity and the magnetic fields becomes strong under a weak magnetic field.

In the present paper, the relationship between the periodicity of a low Prandtl number fluid and the convective inhibitory effect of the Lorentz force is discussed quantitatively. The spectral resolutions are made by MEM (maximum entropy method)⁽⁷⁾ for time series data of total kinetic energy, total enstrophy, total palinstrophy^{(8),(9)} and total hydromagnetic cross helicity. Hence, the frequency characteristics of a thermo-electrically conducting fluid are clarified. Furthermore, the comparison of the convergence of Poisson's equations for the velocity and the magnetic fields is discussed between the cross helicity method and the numerical residual method.

2. Comparison of Two Magnetic Field Analysis Schemes

2.1 Comparison of induced magnetic field

Figures 1(a) and (b) show the comparisons of the induced magnetic field vectors for $Ha=28.3$ (0.01 T), 56.6 (0.02 T) and 192.6 (0.07 T) between the cross helicity method and the numerical residual method. For the induced magnetic field vectors at the center of a cavity at $Ha=28.3$, the vortex center positions are far apart. In contrast, the vortex center positions are in good agreement at $Ha=56.6$ and 192.6.

At $Ha=28.3$ and 56.6, the twin vortices at the center of a cavity have the effect of convective promotion due to the Lorentz force. The reason for this is as follows: the electric current in the $+z$ direction is

Fig. 1. Comparisons of the induced magnetic field vector between the cross helicity method and the numerical residual method

Fig. 2. Lorentz force vectors of molten tin at $Pr=0.011$, $Ra=3.659 \times 10^5$, $Rem=10^5$ and Ha equal to 28.3, 56.6 and 192.6

produced by Ampere's law for one of the twin vortices upside the cavity. Then the Lorentz force occurs in the $+x$ -direction, and the convection is promoted. In contrast, the electric current in the $-z$ -direction is produced by Ampere's law for the other vortex downside the cavity. Then the Lorentz force occurs in the $-x$ -direction, and convection is promoted. At $Ha=192.6$, the twin vortices vanish completely. Only secondary vortices are left at the four corners of the cavity (see Fig. 1(c)). Here, the signs $+$ and $-$ indicate the directions of an electric current produced in the z direction.

Figures 2(a) to (c) show the Lorentz force vectors at $Ha=28.3$, 56.6 and 192.6, respectively. As is evident from Ohm's law, the Lorentz force is observed in the $-x$ direction upside the cavity. In contrast, the Lorentz force is observed in the $+x$ -direction downside the cavity, and the convection is inhibited. The convective inhibitory effect of the Lorentz force becomes marked with increasing magnetic field intensity. This explains well the numerical results obtained in Fig. 1.

2.2 Comparison of calculation performance

We let the maximum iteration number of Poisson's equations be 1500. In this Section, we let it be 10000. We will discuss the effect of the above-mentioned two schemes of magnetic field analysis on the convergence of Poisson's equations for the velocity and the magnetic fields.

Figure 3 shows the change in the calculation performance with the Hartmann number ($Ha=28.3$, 56.6, 148.1, 192.6, 236.9, 355.3, 444.2, 592.2 and $Rem=10^5$) for the two schemes of magnetic field analysis. Here, the calculation performance $C.P.$ is defined as

$$C.P. = \frac{\text{CPU time(Primitive)} - \text{CPU time(Modified)}}{\text{CPU time(Primitive)}} \times 100(\%) \quad (1)$$

When the total amount of Joule heating is as largest at $Ha=56.6$ (see Fig. 4), a reduction of the calculation

$$* C.P. \equiv \frac{\text{CPU-time(Primitive)} - \text{CPU-time(Modified)}}{\text{CPU-time(Primitive)}} \times 100 (\%).$$

Fig. 3. Change of the calculation performance with the Hartmann number

Fig. 4. Change of total amount of Joule heating with the Hartmann number

performance of the numerical residual method appears clearly. The reason for this is as follows: by rough estimate, the order of Joule heating Qj can be written as $Qj \sim J_z^2 \sim (\sigma_e u_x B_0)^2$. At as a low Hartmann number as $Ha=28.3$, the x -component of a velocity vector becomes large. However, the total amount of Joule heating becomes small due to a low magnetic field intensity. In contrast, at as a high Hartmann number as $Ha=192.6$, the inhibitory effect of

convection by the Lorentz force becomes stronger, and so the total amount of Joule heating becomes small. Hence, the change in the total amount of Joule heating with the Hartmann number has a maximum value at $Ha = 56.6$. Therefore, when the generative term such as Joule heating is large, a nonphysical numerical instability is induced in the energy equation. Hence, in order to eliminate such numerical instability, we must calculate the energy equation by use of the time marching method of more than 2nd-order time accuracy.

Figures 5(a) to (c) show the comparisons of the convergence of Poisson's equations between the two magnetic field analysis schemes. Figure 5(a) shows the comparison of the CPU time up to a time step $n = 50\,000$ between the two magnetic field analysis schemes. The case of a low magnetic field intensity at

$Ha = 28.3$ is discussed. At an initial state ($n = 0 \sim 20\,000$), the calculation performance of the cross helicity method (before modification) is better than that of the numerical residual method (after modification). Furthermore, when the convection reaches a (semi-)steady state ($n = 20\,000 \sim$), the calculation performance of the numerical residual method is better than that of the cross helicity method. In the case of intermediate magnetic field intensity at $Ha = 56.6$. Then, Joule heating is marked at an initial state ($n = 15\,000$) (see Fig. 9(e)). This has a serious influence on the reduction of calculation performance. Furthermore, in the case of high magnetic field intensity at $Ha = 192.6$, since the convective inhibitory effect of the Lorentz force is large, the flow field immediately reaches a steady state. The calculation performance of the numerical residual method is

Fig. 5. Effect of the two magnetic field analysis schemes on the convergence of Poisson's equations for the velocity and magnetic fields

Fig. 6. Joule heating contours of molten tin at $Pr = 0.011$, $Ra = 3.659 \times 10^5$, $Rcm = 10^5$ and Ha equal to 28.3, 56.6 and 192.6 ($\Delta Q_j = 0.97 \times 10^{-16}$)

much better than that of the cross helicity method. Hence, it is confirmed that the calculation performance of the numerical residual method is better than that of the cross helicity method over a wide Hartmann number range except at $Ha = 56.6$, when Joule heating is large.

Figures 5(b) and (c) show the comparisons of the iteration number of Poisson's equations for the velocity and the magnetic fields, N_{ITV} and N_{ITM} , respectively between the two magnetic field analysis schemes. The convergence of Poisson's equations for the velocity and magnetic fields improves with increasing magnetic field intensity. In order to clarify the effect of Joule heating on the calculation performance, the calculation performance is compared in the absence and presence of Joule heating at $Ha = 56.6$. Regardless of the improvement of the magnetic field analysis scheme, the calculation performance in the absence of the Joule heating decreases by about

4.5 %. The reason for this is as follows: Joule heating is markedly produced near the adiabatic vertical walls (see Figs. 6(a) to (c)), and the temperature difference between the adiabatic top and the hot left walls becomes small. Thus, convection slows down due to the inhibition of the buoyancy force. Since the pressure difference decreases, the correction of pressure can likewise be decreased. The iteration number of Poisson's equation for the velocity field decreases, and the calculation performance improves.

Next, the convergence of Poisson's equation for the magnetic field is discussed (see Fig. 5(c)). In the case of $Ha = 28.3$ and 56.6, the convergence of Poisson's equation of the cross helicity method is better than that of the numerical residual method at an initial state ($n = 0 \sim 20\,000$). In contrast, above $n = 20\,000$ the opposite tendency is observed. On the other hand, in the case of $Ha = 192.6$, the convergence of Poisson's equation of the numerical residual method is better than that of the cross helicity method from an initial state.

3. Computed Results and Discussions

3.1 Hydromagnetic cross helicity

The hydromagnetic cross helicity is defined as the physical quantity for evaluating the combination intensity between the velocity and the magnetic fields. Here, by splitting the magnetic field into the applied component ($\mathbf{B}_0$) and the induced component ($\mathbf{B}_i$), the hydromagnetic cross helicity H_c can be rewritten as

$$\begin{aligned} H_c &\equiv \int_{\Omega} \mathbf{B} \cdot \mathbf{u} d\Omega = \int_{\Omega} (\mathbf{B}_0 + \mathbf{B}_i) \cdot \mathbf{u} d\Omega \\ &= \int_{\Omega} \mathbf{B}_0 \cdot \mathbf{u} d\Omega + \int_{\Omega} \mathbf{B}_i \cdot \mathbf{u} d\Omega = H_{c0} + H_{ci}. \quad (2) \end{aligned}$$

This kind of cross helicity has been discussed in detail as the turbulent cross helicity in a MHD turbulence model^{(10),(11)}, for example, large scale MHD flow systems such as the dynamo and the solar wind. This turbulent cross helicity⁽¹²⁾ corresponds to H_{ci} in Eq.(2).

Figures 7(a) to (c) show total hydromagnetic cross helicity contours at $Ha = 28.3$, 56.6 and 192.6, respectively. The combination intensity between the velocity and the magnetic fields becomes smaller with increasing magnetic field intensity. However, since the hydromagnetic cross helicity shown in Fig. 7 includes the applied component of the magnetic field, it is difficult to clearly understand the combination intensity between the induced magnetic field and the velocity field.

Figure 8 shows the induced hydromagnetic cross helicity contours involving only the induced component of the magnetic field. The combination intensity between the induced magnetic and the velocity fields

Fig. 7. Hydromagnetic cross helicity contours of molten tin at $Pr=0.011$, $Ra=3.659 \times 10^5$, $Rem=10^5$ and Ha equal to 28.3, 56.6 and 192.6 ($\Delta Hc=0.26 \times 10^{-5}$)

increases near the adiabatic vertical walls with decreasing magnetic field intensity. Furthermore, the induced hydromagnetic cross helicity Hc_i consists of positive and negative layers. This means that the direction of the induced magnetic field changes along the velocity profiles.

3.2 Frequency characteristics

The MEM (maximum entropy method) was presented by Parker and Burg in 1967⁽⁸⁾. The MEM is a spectral estimation method based on the concept of entropy in an information theory. In comparison with the original Blackmann Tukey method and FFT⁽¹³⁾, the MEM has the following advantages: (1) the spectra can be resolved from short time series data; and (2) the performance of spectral resolution is very good. However, we must determine the lag of autocorrelation coefficient, m . m can be determined when the forecasting error is minimized. In general, the following empirical equation can be expressed by

Fig. 8. Induced hydromagnetic cross helicity contours of molten tin at $Pr=0.011$, $Ra=3.659 \times 10^5$, $Rem=10^5$ and Ha equal to 28.3, 56.6 and 192.6 ($\Delta Hc_i=0.53 \times 10^{-6}$)

the total number of data, N ,

$$m=(2 \sim 3)N, \quad (3)$$

and we used Eq.(3) in the following.

In the spectral resolution, the time series data after $t=100\,000$ have been used. The sampling time interval between data is a comparatively large value, 200. The reason for this is as follows: in general, the life of a vortex lump decreases by the fluid viscosity inversely as the square of the frequency. The vortices dissipate rapidly as the scale of a vortex decreases. Here, the process of the dissipation of the primary vortex due to the convective inhibitory effect of the Lorentz force is treated. Since a low frequency range is discussed as a large vortex scale, we let the sampling interval be 200.

Figures 9(a) to (c) show the changes in total kinetic energy, total enstrophy and palinstrophy with time in the range of 0 to 400 000. With increasing magnetic field intensity in the Hartmann number

Fig. 9. Changes in total kinetic energy, total enstrophy, total palinstrophy, total hydromagnetic cross helicity, total Joule heating and total magnetic energy with time

range of 0 to 56.6, the convective inhibitory effect of the Lorentz force becomes stronger, and the flow field shifts from periodic to stable.

Figures 10(a) to (c) show the power spectra of total kinetic energy, total enstrophy and palinstrophy, respectively. In the absence of a magnetic field, the power spectra change to the continuous spectra containing many components of the frequency. In contrast, with increasing magnetic field intensity, the width of spectra becomes narrower, and they approach line spectra. The peaks of power spectra for total kinetic energy and total enstrophy shift to the low frequency range, A1 $\rightarrow$ A2 $\rightarrow$ A3 and B1 $\rightarrow$

B2 $\rightarrow$ B3, respectively (see Figs. 10(a) and (b)). This means that the twin vortices in the primary vortex dissipate due to the convective inhibitory effect of the Lorentz force, and the scale of vortices becomes large. As the Hartmann number changes from 0.0, 28.3 to 56.6 in that order, their spectra decrease remarkably in rather a low-frequency side than a high-frequency side as the boundary of the maximal peaks. This means that the primary vortex dissipates due to the increase in the viscous dissipation with increasing magnetic field intensity.

As shown in Fig. 10(c), the peaks of spectra for total palinstrophy, C1 and C2, shift to the high-fre-

Fig. 10. Frequency characteristics of total kinetic energy, total enstrophy, total palinstrophy and total hydromagnetic cross helicity

quency side in the Hartmann number range of 0.0 to 28.3. This is due to the reduction of twin vortices in the primary flow at $Ha=0.0$ by the convective inhibitory effect of the Lorentz force. Since the energy is transported from large-scale vortices to small-scale vortices by the dissipation of the twin vortices, the peak, C 2, shifts to a high-frequency side, in which the vortex scale is very small.

Figures 9(d) to (f) show the changes in total hydromagnetic cross helicity, total Joule heating and total magnetic energy with time in the range of 0 to 400 000, respectively. As is evident from the changes in total kinetic energy and total magnetic energy at $Ha=28.3$ and 56.6 with time in the range of 0 to 100 000, the time scale of the magnetic field is much smaller than that of the velocity field (see Figs. 9(a) and (f)). This may also be clearly explained by the changes in total kinetic energy and total magnetic energy for $Ha=28.3$ with time after $t=100\,000$. Then, the flow field is periodic with time. In contrast, the magnetic field shifts to a steady state because the time of its propagation is very short. As is evident from the change in total hydromagnetic cross helicity for $Ha=28.3$ with time, which is very different from that for $Ha=56.6$, the combination intensity between the

velocity and the magnetic fields not only becomes larger with the development of natural convection but also changes periodically (see Fig. 9(d)).

Figure 10(d) shows the power spectra of the hydromagnetic cross helicity for $Ha=28.3$ and 56.6. It becomes clear that the spectrum shifts from a continuous spectrum involving many components to a line spectrum, whose width is very narrow with increasing magnetic field intensity.

4. Conclusions

In this study, the features of the two magnetic field analysis schemes, i.e., the cross helicity method and the numerical residual method, have been clarified using the computed results. Furthermore, in order to examine the periodic phenomena peculiar to a low Prandtl number fluid quantitatively, spectral analyses were done using the maximum entropy method. The conclusions are summarized as follows:

(1) In the twin vortices of the induced magnetic field at the center of a cavity, the Lorentz force acts in the direction of convective promotion. In contrast, these twin vortices become dissipated with increasing magnetic field intensity, and secondary vortices are left at the four corners of a cavity, where the Lorentz

force acts in the only direction of convective inhibition.

(2) From the frequency characteristics of total kinetic energy and total hydromagnetic cross helicity, it is confirmed quantitatively that the flow field shifts from a periodic state to a steady state with increasing magnetic field intensity. In particular, the spectra in the low frequency side decrease more remarkably than those in the high-frequency side as the boundary of the power spectral peak. This means that the primary flow is dissipated by the convective inhibitory effect of the Lorentz force.

(3) The convergence of Poisson's equations for the magnetic and the velocity fields is discussed for the two magnetic field analysis schemes, i.e., the cross helicity method and the numerical residual method. Under a weak magnetic field, the numerical method is superior to the cross helicity method except at an initial state. In contrast, under a strong magnetic field, the numerical residual method is superior to the cross helicity method from an initial state.

(4) Regardless of the magnetic field analysis scheme, the calculation performance may be improved by Joule heating. However, the effect of the Joule heating on the calculation performance is negligible in the numerical residual method. Namely, when Joule heating is large, the calculation performance of the numerical residual method is inferior to that of the cross helicity method. Then, the energy equation must be calculated by more than the 2nd-order time accuracy. The high-order calculation of the energy equation is left as future problem.

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