Three-Dimensional Vortex Methods Derived from the Navier-Stokes Equations*

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Vortex methods have been used for many fluid engineering problems to simulate flow fields, in particular high Reynolds number flow fields. These methods have some advantages; (1) are free of grid generation, (2) use a simple algorithm, (3) are inherently free of numerical viscosity, and (4) exhibit self-adaptation for strong shear flow. In three-dimensional flows, a few numerical schemes have been used in actual calculations, but their theoretical foundation is not sufficient in a sense that numerical schemes are not explicitly derived from the Navier-Stokes equations. The present paper aims to give the theoretical foundation of the vortex methods used for three-dimensional flows. In this paper, a basic integral equation concerning the vorticity is derived from the three-dimensional Navier-Stokes equations and typical algorithms of vortex methods, the vorton method, vortex element method and vortex stick method, are then derived from this integral equation. Furthermore, the vorticity field given by the integral equation is shown to be solenoidal, provided the initial one is solenoidal. Finally, the velocity field is analytically obtained for Gaussian distribution of vorticity, which is the typical vorticity distribution in vortex methods, and a new type of Gaussian vorton, which is solenoidal, is proposed and the velocity field induced by this vorton is obtained.

Key Words: Vortex, Computational Fluid Dynamics, Three-Dimensional Flow, Navier-Stokes Equation, Incompressible Flow, Vortex Method, Integral Equation

1. Introduction

Vortex methods, which model flows using discrete vortices, have been used by many researchers to simulate high Reynolds number flows with strong shear such as flows around bluff bodies and jets. The vortex method has some advantages: (1) The flow region with strong shear stress is automatically adaptive without grid generation and (2) the algorithm is simple and the physical interpretation is easy due to the fact that the vorticity field is calculated directly. The features of vortex methods, numerical examples and mathematical theories have been reviewed by many researchers (e.g., Leonard⁽¹⁾, Kamemoto^{(2),(3)} and Sarpkaya⁽⁴⁾).

In two-dimensional high Reynolds number flows, vortices are transported with fluid particles and the vorticity is almost constant due to the high Reynolds

number, so that the flow can be simulated using potential flow together with discrete vortices and the global flow field is easily understood physically. Kamemoto⁽²⁾ stated that this type of vortex method is "level 1". In this level, qualitative rather than quantitative flow features are simulated and various flows have been calculated using this method. The viscous effect is introduced by the core spreading method or the random walk method.

In three-dimensional flows, we have to consider the stretch term in the vorticity field. For this stretch, various algorithms are proposed from the consideration of physical phenomena; the vorton method, vortex element method and vortex stick method (see Kida and Nakajima⁽⁵⁾). For these numerical schemes, Winckelman and Leonard⁽⁶⁾ pointed out that the vorticity field is not solenoidal. Therefore, it is very important to construct these schemes from the Navier-Stokes equations, in order to elucidate their theoretical background.

In the present study we tried to derive the fundamental equation of vortex methods from the Navier-Stokes equations in three-dimensional flows. The present authors reported on the analysis of two-

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dimensional flows in order to give the mathematical foundation to vortex methods^{(7),(8)}. The present study is along the same line and aims to extend the previous analysis to three-dimensional flows. First we construct an integral equation with respect to the vorticity field using the Navier-Stokes equations and elucidate the relation between the Navier-Stokes equations and the vortex method in the case of high Reynolds number flows. Second we discuss the fact that the vorticity field in the vortex method is solenoidal, and we derive typical vortex methods, the vorton method, vortex element method and vortex stick method, from the integral equation. Finally, we analytically derive the velocity field for the Gaussian vorton and a new type of Gaussian vorton, which is solenoidal, is proposed.

2. Derivation of Integral Equation

2.1 Vorticity equation

We consider that the flow is incompressible and three-dimensional. Velocity and vorticity are given as $\vec{u}$ and $\vec{\omega}$, respectively. Reynolds number is defined as Re . The vorticity equation can be derived from the Navier-Stokes equations

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla) \vec{\omega} = \vec{\omega} \cdot \mathbf{D} + \frac{1}{Re} \nabla^2 \vec{\omega}, \quad (1)$$

where $\mathbf{D}$ represents the symmetry tensor given by

$$\mathbf{D} = \frac{1}{2}(\nabla \vec{u} + \nabla^T \vec{u}), \quad (2)$$

where ∇^T is the transposed nabla. In deriving the above equation, we use the relation, $\text{div } \vec{u} = 0$.

In the vortex method, vorticity field is discretized and the discrete vorticity is traced by the Lagrangian method. We here define the advective velocity field as $\vec{u}_a$, and the strain rate tensor in this velocity field is defined as

$$\mathbf{D}_a = \frac{1}{2}(\nabla \vec{u}_a + \nabla^T \vec{u}_a). \quad (3)$$

By using advective velocity $\vec{u}_a$, Eq. (1) is written as

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u}_a \cdot \nabla) \vec{\omega} = \vec{\omega} \cdot \mathbf{D}_a + \frac{1}{Re} \nabla^2 \vec{\omega} + \vec{f}, \quad (4)$$

where $\vec{f}$ is the corrected term due to the velocity difference $\vec{u} - \vec{u}_a$ and is defined as

$$\vec{f} = ((\vec{u}_a - \vec{u}) \cdot \nabla) \vec{\omega} + \vec{\omega} \cdot (\mathbf{D} - \mathbf{D}_a). \quad (5)$$

2.2 Basic relation of isolated vortex

Let us consider an isolated vortex at point A whose initial position is $\vec{a}$ (see Fig. 1), and let its approximate advective velocity be $\vec{u}_a$. Then, we have the trajectory of the vortex: $d\vec{x}/dt = \vec{u}_a(\vec{x}(\vec{a}, t), t)$, $\vec{x} = \vec{a}$ ($t=0$). Taking this fact into account, we introduce new coordinates $\vec{X}$:

$$\vec{x} = \vec{a} + \int_0^t \vec{u}_a(\vec{x}(\vec{a}, s), s) ds + \vec{X}. \quad (6)$$

The coordinates $\vec{X}$ are ones relative to the vortex transported by the velocity $\vec{u}_a$. The physical state in

Fig. 1 Approximate advective velocity

Fig. 2 Relative coordinate system

this coordinate system is shown in Fig. 2. In this coordinate system, Eq. (4) becomes

$$\frac{\partial \vec{\omega}}{\partial t} \Big|_x - \vec{\omega} \cdot \mathbf{D}_a - \frac{1}{Re} \nabla_x^2 \vec{\omega} = \vec{f}, \quad (7)$$

where $|_x$ and suffix X denote the partial differentiation with respect to the $(\vec{X}, t)$ -plane.

Here we denote the (i, j) component of the strain rate tensor $\mathbf{D}_a$ as ε_{ij} . Then Eq. (7) is expressed as

$$\frac{\partial \omega_j}{\partial t} \Big|_x - \varepsilon_{ji} \omega_i - \frac{1}{Re} \nabla_x^2 \omega_j = f_j, \quad (8)$$

where the dummy index denotes Einstein's convention.

Let us consider the following initial case as shown in Fig. 2, in order to apply the analysis used in the previous paper⁽⁷⁾.

$$\omega_j = \omega_{0j} \delta(\vec{X}) \text{ at } t=0. \quad (9)$$

To solve Eq. (8) under this initial condition, we use the Laplace and Fourier transformations. We define the Laplace transformation of ω_j and f_j as

$$\begin{aligned} \omega_j &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} W_j dp \\ f_j &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} F_j dp. \end{aligned} \quad (c > 0) \quad (10)$$

Then, Eq. (8) is expressed as

$$\left(p W_j - \frac{1}{Re} \nabla_x^2 W_j \right) - \varepsilon_{ji} W_i = \omega_{0j} \delta(\vec{X}) + F_j. \quad (11)$$

Furthermore, we apply the Fourier transformation to this equation

$$\left(p + \frac{k^2}{Re}\right) \mathfrak{W}_j - \varepsilon_{ji} \mathfrak{W}_i = \omega_{0j} + \tilde{\mathfrak{F}}_j, \quad (12)$$

where $k^2 = k_i \cdot k_i$. Here $\mathfrak{W}_j$ and $\tilde{\mathfrak{F}}_j$ are defined by

$$W_j = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \mathfrak{W}_j \exp(-k_i X_i) dk, \quad (13)$$

$$F_j = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \tilde{\mathfrak{F}}_j \exp(-k_i X_i) dk,$$

where $i = \sqrt{-1}$ and $dk = dk_1 dk_2 dk_3$. In deriving this equation, we assume that the strain rate tensor $\mathbf{D}_a$ is independent of time t . This means that the velocity field $\tilde{u}_a$ is independent of t . This is not exactly correct, but in the vortex method an explicit method with respect to the time derivative is used, so that within a small time period this assumption is reasonable. We note, therefore, that the final result obtained in this section is only valid for a small time period.

Equation (12) is a system of linear equations with respect to $\tilde{\mathbf{W}}$

$$\tilde{\mathbf{E}} \cdot \tilde{\mathbf{W}} = \tilde{\omega}_0 + \tilde{\mathbf{F}}. \quad (14)$$

The solution of this equation is easily given by

$$\mathfrak{W}_i = (\omega_{0i} + \tilde{\mathfrak{F}}_i) \left(\frac{a_{j1}^i}{X - \alpha} + \frac{a_{j2}^i}{X - \beta} + \frac{a_{j3}^i}{X - \gamma} \right), \quad (15)$$

where X is defined by

$$X = p + \frac{k^2}{Re}. \quad (16)$$

Further, a_{ij}^i is a function of ε_{mn} and α, β , and γ are defined by

$$X^3 - (\varepsilon_{22}\varepsilon_{33} + \varepsilon_{33}\varepsilon_{11} + \varepsilon_{11}\varepsilon_{22} - \varepsilon_{32}^2 - \varepsilon_{12}^2 - \varepsilon_{31}^2)X + \varepsilon_{11}\varepsilon_{22}\varepsilon_{33} - 2\varepsilon_{12}\varepsilon_{23}\varepsilon_{31} \\ = (X - \alpha)(X - \beta)(X - \gamma). \quad (17)$$

This relation is from the determinant of the coefficient matrix $\tilde{\mathbf{E}}$, and $\text{div } \tilde{u}_a = 0$ is used in this derivation. From Eq.(15), we can obtain the following equation by applying the inverse Fourier and Laplace transformations (see Appendix 1).

$$\omega_i = \omega_{0i} \left(\frac{Re}{4\pi t} \right)^{3/2} \exp\left(-\frac{Re}{4t} r^2\right) a_j^i(t) \\ + \left(\frac{Re}{4\pi} \right)^{3/2} \int_0^t \int_{-\infty}^{\infty} \exp\left(-\frac{Re}{4t'} r'^2\right) \frac{a_j^i(t')}{t'^{3/2}} \\ \times f_i(\vec{X}_i - \vec{X}_i, t - t') dX' dt', \quad (18)$$

where

$$dX' = dX'_1 dX'_2 dX'_3, \\ r = |\vec{X}|, \quad r' = |\vec{X} - \vec{X}'| \\ a_j^i(t) = a_{j1}^i \exp(\alpha t) + a_{j2}^i \exp(\beta t) + a_{j3}^i \exp(\gamma t).$$

In this equation, the first term on the right hand side is the stretch of the vorticity with respect to the direction of the eigenvector of the strain rate tensor and the second term is the corrected term due to the difference between the exact velocity field and the approximate one.

2.3 Basic integral equation

Let us consider the case where the vorticity is initially distributed in some region. Then we may treat this case as the sum of isolated vorticity, although this is not rigorously correct.

Let the initial vorticity be $\tilde{\omega}_0(x)(t=0)$ and the vorticity $\tilde{\omega}_c$ at $t=t'$ be generated. Then using Eq. (18), we finally arrive at

$$\omega_i(x, t) = \int_{-\infty}^{\infty} \left[\omega_{0j}(\tilde{a}) \left(\frac{Re}{4\pi t} \right)^{2/3} \right. \\ \times \exp(r(t, r_0)) a_j^i(\tilde{a}, t) \\ + \int_0^t \tilde{\omega}_{cj}(\tilde{a}, t') \left(\frac{Re}{4\pi(t-t')} \right)^{2/3} \\ \times \exp(r(t-t', r_i)) a_j^i(\tilde{a}, t-t') dt' \\ + \left(\frac{Re}{4\pi} \right)^{2/3} \int_0^t \int_{-\infty}^{\infty} \left(\frac{1}{s} \right)^{3/2} \\ \times \exp(r(s, r')) a_j^i(\tilde{a}, s) \\ \left. \times f_j(\vec{X}_i - \vec{X}_i, t-s) dX' ds \right] da, \quad (19)$$

where

$$a_j^i(\tilde{a}, t) = a_{j1}^i \exp(\alpha(\tilde{a})t) + a_{j2}^i \exp(\beta(\tilde{a})t) \\ + a_{j3}^i \exp(\gamma(\tilde{a})t)$$

$$r(t, y) = -\frac{Re}{4t} y^2$$

$$r_0 = \left| \vec{x} - \tilde{a} - \int_0^t \tilde{u}_a(\vec{x}(\tilde{a}, s)) ds \right|$$

$$r'_i = \left| \vec{x} - \tilde{a} - \int_t^t \tilde{u}_a(\vec{x}(\tilde{a}, s)) ds \right|$$

$$r' = \left| \vec{x} - \tilde{a} - \int_s^t \tilde{u}_a(\vec{x}(\tilde{a}, \xi)) d\xi - \vec{X}' \right|$$

and $da = da_1 da_2 da_3$

This basic equation(19) is derived under the assumption that the approximate velocity field $\tilde{u}_a$ is independent of t , as already mentioned. Further more, in this equation it is assumed that the continuous initial vorticity field is the infinite sum of the initial isolated vorticity. This assumption is not rigorously correct, but the analysis for the two-dimensional flow⁽⁷⁾ implies that this assumption is approximately correct.

In this equation, $a_j^i(\tilde{a}, t)$ is not yet explicitly determined. In the following section, we will try to determine it for small time t .

2.4 Relation between a_j^i and ε_{ij}

Let t be a very small time, $t \leq \Delta t \ll 1$, where Δt is the time step in the actual calculation. In Eq.(15), we consider the following relation

$$\frac{a_{j1}^i}{X - \alpha} + \frac{a_{j2}^i}{X - \beta} + \frac{a_{j3}^i}{X - \gamma} \\ = \frac{1}{(X - \alpha)(X - \beta)(X - \gamma)} \\ \times \{ a_{j1}^i(X - \beta)(X - \gamma) + a_{j2}^i(X - \alpha)(X - \gamma) \\ + a_{j3}^i(X - \alpha)(X - \beta) \}. \quad (20)$$

The coefficients of the terms of X and X^2 in the term $\{ \}$ in Eq.(20) are given by

$$\begin{cases} a_{j1}^i + a_{j2}^i + a_{j3}^i \\ -(\beta + \gamma)a_{j1}^i - (\alpha + \gamma)a_{j2}^i - (\alpha + \beta)a_{j3}^i \equiv M_j^i. \end{cases} \quad (21)$$

On the other hand, we obtain $\tilde{\mathbf{W}} = \tilde{\mathbf{E}}^{-1} \cdot (\omega_0 + \tilde{\mathbf{F}})$ from Eq.(14). Therefore, we can obtain

$$\mathfrak{W}_i = [\text{second order polynomial of } X]$$

$$/[(X-\alpha)(X-\beta)(X-\gamma)].$$

Comparing Eq.(20) with the above equation, we finally arrive at(see Appendix 2)

$$a_{ji}^i = \delta_{ij}, \quad M_j^i = \varepsilon_{kij} \delta_{kj}. \quad (22)$$

Further, from $\text{div } \vec{u}_a = 0$, we obtain

$$\alpha + \beta + \gamma = 0.$$

Therefore, we can arrive at

$$M_j^i = \alpha a_{j1}^i + \beta a_{j2}^i + \gamma a_{j3}^i \\ = \varepsilon_{ij}. \quad (23)$$

3. Basic Integral Expression of Vortex Method

We here consider the following third term of Eq. (19) which is the corrected term due to the velocity difference

$$I_i \equiv \int_{-\infty}^{\infty} \left[\left(\frac{Re}{4\pi} \right)^{3/2} \int_0^t \int_{-\infty}^{\infty} \left(\frac{1}{s} \right)^{3/2} \right. \\ \times \exp(r(s, r')) a_{ji}^i(\vec{a}, s) \\ \left. \times f_j(\vec{X}' - \vec{X}, t-s) d\vec{X}' ds \right] d\vec{a}. \quad (24)$$

Since $\Delta t \ll 1$, we obtain for $t < \Delta t$,

$$\int_0^t \vec{u}_a(\vec{x}(s, \vec{a})) ds \cong \vec{u}_a(\vec{x}(0, \vec{a})) t = \vec{u}_a(\vec{a}) t \quad (25)$$

and

$$\vec{x} \cong \vec{X} + \vec{a} + \vec{u}_a t \\ \vec{r}' \cong |\vec{X} - \vec{X}' + s \vec{u}_a|. \quad (26)$$

Using these relations, we finally obtain the following equations for high Reynolds number flows(see appendix 3)

$$I_i \cong \Delta t \int_{-\infty}^{\infty} \{ a_{j1}^i(\vec{a}) + a_{j2}^i(\vec{a}) + a_{j3}^i(\vec{a}) \} \\ \times (((\vec{u}_a - \vec{u}) \cdot \nabla) \vec{\omega} + \vec{\omega} \cdot (\mathbf{D} - \mathbf{D}_a))_j (-\vec{X}, 0) d\vec{a}, \quad (27)$$

where $(\)_j(-\vec{X}, 0)$ denotes the j component of $(\)$ at $(-\vec{X}, 0)$ in the space and time coordinate system. Therefore, Eq.(24) is approximately expressed from Eq.(22) as

$$I_i \cong \Delta t \int_{-\infty}^{\infty} \left\{ (u_{ia} - u_i) \frac{\partial}{\partial a_i} \omega_i + \omega_i (\varepsilon_{ii} - \varepsilon_{ii}) \right\} d\vec{a}, \quad (28)$$

where $\mathbf{D} = \{ \varepsilon_{ii} \}$ which is the strain rate tensor due to $\vec{u}$.

From this equation, we see that the corrected term is taken as being of order of Δt^2 , if the approximate velocity field $\vec{u}_a$ is a good approximation of $\vec{u}$. Therefore, we can derive the following basic integral equation for the vortex method in the case of $\vec{\omega}_c = 0$

$$\omega_i = \left(\frac{Re}{4\pi\Delta t} \right)^{3/2} \int_{-\infty}^{\infty} \omega_{0i}(\vec{a}) \exp \left(-\frac{Re}{4\Delta t} r_0^2 \right) d\vec{a} \\ + \Delta t \left(\frac{Re}{4\pi\Delta t} \right)^{3/2} \int_{-\infty}^{\infty} \omega_{0j}(\vec{a}) \varepsilon_{ij}(\vec{a}) \\ \times \exp \left(-\frac{Re}{4\Delta t} r_0^2 \right) d\vec{a} + O((\Delta t)^2), \quad (29)$$

where

$$r_0 \triangleq |\vec{x} - \vec{a} - \Delta t \vec{u}_a|.$$

Equation (29) indicates that the vorticity field

between time t and $t + \Delta t$ is expressed by summing the following two terms; the first term related to the viscous diffusion and the second term due to the stretch of vorticity field.

4. Discrete Vorticity Field

In three-dimensional flows, a few vortex methods are proposed. Typical methods are the vorton method, the vortex element method and the vortex stick method(see⁽⁵⁾). These methods are dependent on the discrete method of Eq.(29).

Let the integral region be divided by small regions $V_j(j=1, 2, \dots)$. In this small region, we use the trapezoidal rule of integration. Then we have

$$\int_{V_j} \omega_{0i}(\vec{a}) \exp \left(-\frac{Re}{4\Delta t} r_0^2 \right) d\vec{a} \\ \approx \sum_j \Gamma_{0j}^i \exp \left(-\frac{Re}{4\Delta t} r_{0j}^2 \right), \quad (30)$$

where $r_{0j} = |\vec{x} - \vec{a}_j - \Delta t \vec{u}_a(\vec{a}_j, 0)|$ and $\Gamma_{0j} = \int_{V_j} \omega_{0i} d\vec{a}$.

This is the concept of the vorton method. In the actual vorton method, the initial vorticity field is approximated by using a cut-off function, so that several types of expression of vorticity field are obtained but their mathematical analysis is carried out by Beale and Majda^{(9),(10)}.

We take the small region V_j divided along the direction of the eigenvectors of the strain rate tensor in the approximate velocity field and use Chorin's cut-off function. For convenience, we take the x_1 -axis as the direction of the maximum eigenvector of the strain rate tensor. Then, V_j is taken as

$$V_j = \{ (x_1, x_2, x_3) | |x_1 - a_{j1}| \leq l_j/2, \\ ((x_2 - a_{j2})^2 + (x_3 - a_{j3})^2)^{1/2} \leq \sigma_j \}, \quad (31)$$

where σ_j is the cut-off radius and l_j is the length of the small region. We consider the following initial vorticity

$$\vec{\omega}_0(\vec{x}, t) \cong \omega_{01}(\vec{x} - \vec{x}') * f \left(\frac{|\vec{x}'|}{\sigma} \right) \frac{1}{\sigma^3} \mathbf{e}_1 \\ = \begin{cases} \omega_{01}(\vec{x}) \mathbf{e}_1 : \vec{x} \in S_0 \\ 0 : \vec{x} \notin S_0, \end{cases} \quad (32)$$

where $\mathbf{e}_1$ is the unit vector of the x_1 axis and $S_0 = \{ (x_1, x_2, x_3) | |x_1| \leq l/2, (x_2^2 + x_3^2)^{1/2} \leq \sigma \}$. Here we see that l is the length of the vortex element and σ is the radius of its cross section. The symbol $*$ denotes the convolution. Then we have

$$\left(\frac{Re}{4\pi\Delta t} \right)^{3/2} \int_{-\infty}^{\infty} \omega_{01}(\vec{a}) \exp \left(-\frac{Re}{4\Delta t} r_0^2 \right) d\vec{a} \\ \cong \sum_j \left(\frac{Re}{4\pi\Delta t} \right)^{3/2} \int_{V_j} \omega_{01}(\vec{a}) \exp \left(-\frac{Re}{4\Delta t} r_0^2 \right) d\vec{a}. \quad (33)$$

Here, we assume $Re \gg 1$ and $\Delta t \ll 1$. The above equation is approximately given by

$$\begin{aligned} &\cong \sum_j \left(\frac{Re}{4\pi\Delta t} \right) l_j \omega_{01}(\tilde{a}_j) \\ &\quad \times \int_0^{2\pi} \int_0^{\sigma_j} \exp\left(-\frac{Re}{4\Delta t} r^2\right) a' da' d\theta \\ &\quad \times \chi\left(\frac{|x_1 - a_{1j} - \Delta t u_{a1}|}{l_j/2}\right), \end{aligned} \quad (34)$$

where

$$\begin{aligned} r^2 &= R^2 + a'^2 - 2Ra' \cos \theta \\ R^2 &= (x_2 - a_{j2} - \Delta t u_{a2})^2 + (x_3 - a_{j3} - \Delta t u_{a3})^2 \\ a'^2 &= a_2^2 + a_3^2, \quad \chi(x) = \begin{cases} 1 : |x| \leq 1 \\ 0 : |x| > 1. \end{cases} \end{aligned}$$

Therefore, we have from Eq.(29),

$$\begin{aligned} \omega_i &\cong \sum_j \left(\frac{Re}{4\pi\Delta t} \right) (\omega_{01}(\tilde{a}_j) \delta_{i1} + \omega_{01}(\tilde{a}_j) \varepsilon_{i1}(\tilde{a}_j) \Delta t) l_j \\ &\quad \times \int_0^{2\pi} \int_0^{\sigma_j} \exp\left(-\frac{Re}{4\Delta t} r^2\right) a' da' d\theta \\ &\quad \times \chi\left(\frac{|x_1 - a_{1j} - \Delta t u_{a1}|}{l_j/2}\right). \end{aligned} \quad (35)$$

This equation implies that the initial vortex $\omega_{01}(\tilde{a}_j)$ is transported from $\tilde{a}_j$ to $\tilde{a}_j + \Delta t \tilde{u}_a(\tilde{a}_j)$ together with the viscous diffusion.

Let us consider the above integral in the case where the center of the vortex element is shifted to the origin. Chorin's cut-off function is expressed as

$$\begin{aligned} \frac{1}{\sigma^3} f\left(\frac{|\tilde{x}|}{\sigma}\right) &= \frac{1}{l_j} \chi\left(\frac{|\tilde{x}_1|}{l_j/2}\right) g\left(\frac{\sqrt{x_2^2 + x_3^2}}{\sigma_j}\right) \\ g(|\tilde{x}|) &= \begin{cases} 1 : |\tilde{x}| \leq 1 \\ 0 : |\tilde{x}| > 1 \end{cases} \end{aligned} \quad (36)$$

and $g(|\tilde{x}|)$ is approximately expressed as the following series

$$g\left(\frac{|\tilde{x}|}{\sigma}\right) \cong \sum_n a_n \exp\left\{-\frac{1}{\sigma^2} ((x_2 - x_{n2})^2 + (x_3 - x_{n3})^2)\right\}, \quad (37)$$

where a_n and (x_{n2}, x_{n3}) are some contents. Using these relations, we have the following relation

$$\begin{aligned} &\exp\left(-\frac{Re}{4\Delta t} ((x_2 - x'_{2j} - a_{j2})^2 + (x_3 - x'_{3j} - a_{j3})^2)\right) \\ &\quad * \frac{1}{\sigma_{j3}^3} f\left(\frac{|\tilde{x}'|}{\sigma_j}\right) \cong \sum_n a_n \frac{\pi \frac{4\Delta t}{Re} \sigma_j^2}{\sigma_j^2 + \frac{4\Delta t}{Re}} \\ &\quad \times \exp\left(-\frac{(x_2 - a_{j2} - x_{n2})^2 + (x_3 - a_{j3} - x_{n3})^2}{\sigma_j^2 + \frac{4\Delta t}{Re}}\right) \\ &\quad \times \frac{1}{l_j} \chi\left(\frac{|\tilde{x}_1|}{l_j/2}\right) \cong \frac{\pi \frac{4\Delta t}{Re}}{\sigma_j^2 + \frac{4\Delta t}{Re}} \frac{\sigma_j^2}{\sigma_j^3(\Delta t)} f\left(\frac{|\tilde{x} - \tilde{a}|}{\sigma_j(\Delta t)}\right), \end{aligned} \quad (38)$$

where

$$\sigma_j(\Delta t) = \left(\sigma_j^2 + \frac{4\Delta t}{Re}\right)^{1/2} \cong \sigma_j + \frac{2\Delta t}{Re\sigma_j}. \quad (39)$$

Applying this relation to the integral of Eq.(35), the vorticity ω_1 is derived from Eq.(35) as

$$\begin{aligned} \omega_1(\tilde{x}, t) &\cong \sum_j \frac{\sigma_j^2}{\sigma_j^2 + \frac{4\Delta t}{Re}} \omega_{01}(\tilde{a}_j) l_j (1 + \varepsilon_{11} \Delta t) \\ &\quad \times \int_0^{2\pi} \int_0^{\sigma_j} \frac{1}{\sigma_j^3(\Delta t)} f\left(\frac{r}{\sigma_j(\Delta t)}\right) a' da' d\theta. \end{aligned} \quad (40)$$

Here, let us assume the following relation

$$1 \gg \sigma_j \gg 2\sqrt{\frac{\Delta t}{Re}}.$$

Then, we have

$$\begin{aligned} \omega_1(\tilde{x}) &\cong \sum_j \omega_{01}(\tilde{x} - \tilde{x}')(1 + \varepsilon_{11} \Delta t) \\ &\quad * \frac{1}{\sigma_j^3(\Delta t)} f\left(\frac{|\tilde{x}' - \tilde{a}_j - \Delta t \tilde{u}_a(\tilde{a}_j)|}{\sigma_j(\Delta t)}\right). \end{aligned} \quad (41)$$

These final results imply that after the lapse of time step Δt , the vorticity ω_1 is changed from ω_{01} to $\omega_{01}(1 + \varepsilon_{11} \Delta t)$ from Eq.(41) and the radius of the cut-off function is changed from σ_j to $\sigma_j + 2\Delta t/(Re\sigma_j)$ from Eq.(39).

In the vortex stick method, ε_{11} is obtained from the velocity field at the center of the vortex element and the length of the vortex element is obtained using the mass conservation of the vortex element. In the vortex element method, ε_{11} is obtained by calculating the positions of both nodal points of the vortex element from the velocity field and the direction of $\tilde{\omega}$ is also determined. In both methods, $1 \gg \sigma \gg 2\sqrt{\Delta t/Re}$ is assumed.

In the vortex element method, the vorticity field is solenoidal but in the vortex stick method and the vorton method it is not. However, we note that the vorticity field is not solenoidal locally but the convergence of these methods is proved in the norm $L_2^{(9),(10)}$.

5. Solenoidal

As already mentioned, the vorticity field is not solenoidal in some numerical schemes so that it is necessary to discuss whether Eq.(29) is solenoidal. From Eq.(29), we have

$$\begin{aligned} \text{div } \tilde{\omega} &= \frac{\partial \omega_i}{\partial x_i} \\ &\cong \Re(\Delta t) \int_{-\infty}^{\infty} \omega_{0i}(a) \left(-\frac{Re}{2\Delta t} (x_i - a_i - \Delta t u_{ai})\right) \\ &\quad \times \exp(r(\Delta t, r_0)) da \\ &\quad + \Delta t \Re(\Delta t) \int_{-\infty}^{\infty} \omega_{0j}(a) \varepsilon_{ij}(a) \\ &\quad \times \left(-\frac{Re}{2\Delta t} (x_i - a_i)\right) \exp(r(\Delta t, r_0)) da, \end{aligned} \quad (42)$$

where $\Re(\Delta t) = \left(\frac{Re}{4\pi\Delta t}\right)^{3/2}$. Here, the following relation is satisfied

$$\begin{aligned} &\frac{\partial}{\partial a_i} \exp\left(-\frac{Re}{4\Delta t} r^2\right) \\ &= -\frac{Re}{2\Delta t} (x_j - a_j - \Delta t u_{aj})(-\delta_{ij} - \Delta t \frac{\partial u_{aj}}{\partial a_i}) \\ &\quad \times \exp\left(-\frac{Re}{4\Delta t} r^2\right). \end{aligned} \quad (43)$$

Therefore, we have

$$\frac{Re}{2\Delta t}(x_i - a_i - \Delta t u_{ai}) \exp\left(-\frac{Re}{4\Delta t} r_o^2\right) \\ \approx \left(\delta_{ij} - \Delta t \frac{\partial u_{aj}}{\partial a_i}\right) \frac{\partial}{\partial a_j} \exp\left(-\frac{Re}{4\Delta t} r_o^2\right). \quad (44)$$

Substituting this relation into Eq.(42) and using the integration by parts, we have

$$\frac{\partial \omega_i}{\partial x_i} \approx \Re(\Delta t) \int_{-\infty}^{\infty} \frac{\partial \omega_{0i}}{\partial a_i} \exp(r(\Delta t, r_o)) da \\ + \Delta t \Re(\Delta t) \int_{-\infty}^{\infty} \frac{\partial}{\partial a_i} (\omega_{0j} \varepsilon_{ij}) \exp(r(\Delta t, r_o)) da \\ - \Delta t \Re(\Delta t) \int_{-\infty}^{\infty} \frac{\partial}{\partial a_j} \left(\omega_{0i} \frac{\partial u_{aj}}{\partial a_i}\right) \exp(r(\Delta t, r_o)).$$

Let us consider the case where the initial vorticity field is solenoidal, $\text{div } \tilde{\omega}_0 = 0$. In this case, the first term of the right hand side of above equation is zero, since $\text{div } \tilde{\omega}_0 = 0$. Therefore, if the following relation is satisfied, the vorticity field is solenoidal

$$\frac{\partial}{\partial a_i} (\omega_{0j} \varepsilon_{ij}) = \frac{\partial}{\partial a_j} \left(\omega_{0i} \frac{\partial u_{aj}}{\partial a_i}\right). \quad (45)$$

Equation (45) is rewritten as

$$\frac{\partial \omega_{0j}}{\partial a_i} \varepsilon_{ij} + \omega_{0j} \frac{\partial \varepsilon_{ij}}{\partial a_i} = \frac{\partial \omega_{0i}}{\partial a_j} \frac{\partial u_{aj}}{\partial a_i} + \omega_{0i} \frac{\partial^2 u_{aj}}{\partial a_i \partial a_j}.$$

Further, Eq.(45) becomes

$$\omega_{0j} \left(\frac{\partial \varepsilon_{ij}}{\partial a_i} - \frac{\partial}{\partial a_i} \left(\frac{\partial u_{aj}}{\partial a_j} \right) \right) + \left(\varepsilon_{ij} - \frac{\partial u_{aj}}{\partial a_j} \right) \frac{\partial \omega_{0j}}{\partial a_i} = 0. \quad (46)$$

Here, we use the following relations

$$\frac{\partial \varepsilon_{ij}}{\partial a_i} = \frac{\partial}{\partial a_i} \left(\frac{\partial u_i}{\partial a_j} \right) + \frac{1}{2} \frac{\partial}{\partial a_i} \left(\frac{\partial u_j}{\partial a_i} - \frac{\partial u_i}{\partial a_j} \right) \\ \varepsilon_{ij} - \frac{\partial u_i}{\partial a_j} = \frac{1}{2} \left(\frac{\partial u_j}{\partial a_i} - \frac{\partial u_i}{\partial a_j} \right). \quad (47)$$

Then, the left hand side of Eq.(46) becomes, using these above relations,

$$\omega_{0j} \frac{\partial}{\partial a_i} \left(\frac{\partial u_j}{\partial a_i} - \frac{\partial u_i}{\partial a_j} \right) + \left(\frac{\partial u_j}{\partial a_i} - \frac{\partial u_i}{\partial a_j} \right) \frac{\partial \omega_{0j}}{\partial a_i} \\ = \frac{\partial}{\partial a_i} \left(\omega_{0j} \left(\frac{\partial u_j}{\partial a_i} - \frac{\partial u_i}{\partial a_j} \right) \right). \quad (48)$$

Thus, we see that Eq.(46) is satisfied. Therefore, we see that the vorticity field is solenoidal, provided the initial vorticity field is solenoidal. This fact implies that the approximate vorticity field calculated by the vortex stick method and the vorton method is not solenoidal locally but this disadvantage may be resolved by dividing the initial vorticity field into much smaller regions.

6. Gaussian Vorton and Its Velocity Field

Let us consider the case where the initial vorticity field is approximated by the discrete vortices with Gaussian cut-off function.

$$\omega_i(x) \approx \sum_{j=1}^N c_j^o \exp(-\sigma^2 |\tilde{x} - \tilde{x}_j|^2), \quad (49)$$

where x_j is some value and σ is assumed to be very large. Then, we can assume that $\omega_{0j} \varepsilon_{ij}$ is also approximately expressed as

$$\omega_{0j} \varepsilon_{ij} = \sum_{p=1}^N \gamma_p^i \exp(-\sigma^2 |\tilde{x} - \tilde{x}_p|^2). \quad (50)$$

Then, when $\sigma = (Re/(4\delta t))^{1/2}$, from Eq.(29) for $t = \delta t$ we obtain

$$\omega_i = b^3 \int_{-\infty}^{\infty} \omega_{0i}(\tilde{x} - \Delta t \tilde{u} + \tilde{y}) e^{-\sigma^2 |\tilde{y}|^2} d y_1 d y_2 d y_3 \\ + \Delta t b^3 \int_{-\infty}^{\infty} \omega_{0j}(\tilde{x} - \Delta t \tilde{u} + \tilde{y}) \\ \times \varepsilon_{ij}(\tilde{x} - \Delta t \tilde{u} + \tilde{y}) e^{-\sigma^2 |\tilde{y}|^2} d y_1 d y_2 d y_3 \\ \approx \frac{b^3}{\sigma^3} \sum_{p=1}^N \sum_{l,m,n}^N a_p^i w_l w_m w_n \\ \times \exp\left(-\sigma^2 \left| \tilde{x} - \Delta t \tilde{u}(\tilde{x}_p) - \tilde{x}_p + \frac{\tilde{y}_{l,m,n}}{\sigma} \right|^2\right) \\ + \Delta t \frac{b^3}{\sigma^3} \sum_{p=1}^N \sum_{l,m,n}^N \gamma_p^i w_l w_m w_n \\ \times \exp\left(-\sigma^2 \left| \tilde{x} - \Delta t \tilde{u}(\tilde{x}_p) - \tilde{x}_p + \frac{\tilde{y}_{l,m,n}}{\sigma} \right|^2\right), \quad (51)$$

where $b \equiv (Re/(4\pi\Delta t))^{1/2}$. Thus, when the initial vorticity is approximately expressed as the Gaussian vorton, the vorticity field is also expressed as the Gaussian vorton. Here, w_l and $\tilde{y}_{l,m,n}$ in Eq.(51) are given by the relation with the numerical quadrature. Therefore, it is important to obtain the velocity field for the Gaussian vorton method.

In the Gaussian vorton method, the velocity field has not been obtained exactly. In this section, we will show that the velocity field is analytically obtained.

Here we consider a Gaussian vorton $\tilde{\omega}$ whose center is the origin and its strength is $\tilde{\omega}_0 = (\omega_1, \omega_2, \omega_3)$ $\tilde{\omega} = \tilde{\omega}_0 \exp(-\sigma^2 |\tilde{x}|^2)$. (52)

The velocity at $\tilde{x}$ is given by the Biot-Savart formula

$$\tilde{u} = -\frac{1}{4\pi} \int \frac{\tilde{d}(p)}{|\tilde{d}(p)|^3} \times \omega_i \mathbf{e}_i \exp(-\sigma^2 |\tilde{y}|^2) d y, \quad (53)$$

where $\tilde{d}(p) = \tilde{x} - \tilde{y}$ and $d y = d y_1 d y_2 d y_3$. Here, we define the following integral

$$\tilde{I} \equiv \int_V \frac{1}{|\tilde{d}(p)|} \exp(-\sigma^2 |\tilde{y}|^2) d y. \quad (54)$$

Then, the velocity field is given by

$$\tilde{u} = \frac{1}{4\pi} \begin{pmatrix} \frac{\partial \tilde{I}}{\partial x_2} \omega_3 - \frac{\partial \tilde{I}}{\partial x_3} \omega_2 \\ \frac{\partial \tilde{I}}{\partial x_3} \omega_1 - \frac{\partial \tilde{I}}{\partial x_1} \omega_3 \\ \frac{\partial \tilde{I}}{\partial x_1} \omega_2 - \frac{\partial \tilde{I}}{\partial x_2} \omega_1 \end{pmatrix}. \quad (55)$$

In order to solve Eq.(54), we define the polar coordinates

$$\tilde{x} = r_0(\sin \theta_0 \cos \phi_0, \sin \theta_0 \sin \phi_0, \cos \theta_0), \\ \tilde{y} = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$

Further, we use the following relation

$$\frac{1}{|\tilde{x} - \tilde{y}|} = \sum_{n=0}^{\infty} \sum_{m=0}^n \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta_0) P_n^m(\cos \theta) \\ \times \cos m(\phi - \phi_0) \\ \times \begin{cases} \frac{r^n}{r_0^{n+1}} : r_0 > r \\ \frac{r_0^n}{r^{n+1}} : r_0 < r \end{cases}, \quad (56)$$

where

$$\epsilon_m = \begin{cases} 1 : m=0 \\ 2 : m \neq 0 \end{cases}$$

and $P_n^m(x)$ is the Legendre function. Then, we have

$$\begin{aligned} \hat{I} &= 2\pi \int_0^\pi d\theta \int_0^\infty \sum_{n=0}^\infty \frac{(n-m)!}{(n+m)!} P_n(\cos \theta_0) \\ &\quad \times P_n(\cos \theta) \begin{cases} \frac{r^n}{r_0^{n+1}} : r_0 > r \\ \frac{r_0^n}{r^{n+1}} : r_0 < r \end{cases} \\ &\quad \times \exp(-\sigma^2 r^2) r^2 \sin \theta dr. \end{aligned} \quad (57)$$

Here, we have used the relation

$$\int_0^\pi P_n(\cos \theta) \sin \theta d\theta = \begin{cases} 1 : n=0 \\ 0 : n \neq 0 \end{cases}. \quad (58)$$

Thus, we finally arrive at

$$\hat{I} = \frac{2\pi}{\sigma^3} \frac{1}{r_0} \text{Erf}(\sigma r_0), \quad (59)$$

where $\text{Erf}(x) = \int_0^x e^{-t^2} dt$. Therefore, we can obtain the velocity field from Eqs.(55) and (59).

7. Gaussian Vortex Ring Vorton and Its Velocity Field

The vorticity field given by the Gaussian vorton method is not solenoidal locally. In this section, we will introduce a new type of Gaussian vorton, by which the vorticity field is solenoidal. Let vorticity $\vec{\omega}$ be given by the following relation

$$\vec{\omega} = \text{rot } \vec{A}. \quad (60)$$

Then, the following relation is satisfied automatically:

$$\text{div rot } \vec{\omega} \equiv 0. \quad (61)$$

This relation implies that the vorticity field $\vec{\omega}$ given by Eq.(60) is solenoidal; $\text{div } \vec{\omega} = 0$. Therefore, we can construct many vorton methods by choosing various functions $\vec{A}$. In the actual calculation, it is better that $\vec{A}$ be as simple as possible. Therefore, let $\vec{A}$ be given by

$$\vec{A} = \sum_j \vec{a}_j \exp(-\sigma^2 |\vec{x} - \vec{x}_j|^2). \quad (62)$$

In this case, the discrete vorticity field is given by

$$\begin{aligned} \vec{\omega}_j &= \sum_j (\alpha_n(x_m - x_{jm}) - \alpha_m(x_n - x_{jn})) \\ &\quad \times \exp(-\sigma^2 |\vec{x} - \vec{x}_j|^2), \end{aligned} \quad (63)$$

where (l, m, n) is the permutation of $(1, 2, 3)$. To calculate the velocity field from Eq.(63), we here consider the velocity induced from the following single vorton.

$$\omega_i = (-x_j \alpha_k + x_k \alpha_j) \exp(-\sigma^2 |\vec{x}|^2), \quad (64)$$

where (i, j, k) is the permutation. To obtain the velocity field, we define the following integral

$$\hat{I} = \int_v \frac{1}{|d(\vec{p})|} y_i \exp(-\sigma^2 |\vec{p}|^2) dy. \quad (65)$$

Then, the velocity field is obtained by Eq.(55). Applying Eq.(56) to Eq.(65), we obtain the following relation

Fig. 3 The velocity field induced by an isolated Gaussian vortex vorton

$$\begin{aligned} \hat{I}_1 &= \pi \int_0^\pi d\theta \int_0^\infty \sum_n 2 \frac{(n-1)!}{(n+1)!} \\ &\quad \times P_n^1(\cos \theta_0) P_n^1(\cos \theta) \cos \phi_0 \\ &\quad \begin{cases} \frac{r^n}{r_0^{n+1}} : r_0 > r \\ \frac{r_0^n}{r^{n+1}} : r_0 < r \end{cases} r^3 \sin^2 \theta e^{-\sigma^2 r^2} dr. \end{aligned} \quad (66)$$

Here, we use the relation:

$$\int_0^\pi \sin^2 \theta P_n^1(\cos \theta) d\theta = \begin{cases} 0 : 1 < n \\ \frac{4}{3} : n=1 \\ 0 : n=0. \end{cases} \quad (67)$$

Thus, we finally arrive at

$$\hat{I}_1 = \frac{\pi}{\sigma^4} \frac{x_1}{r_0^2} \left(-e^{-\sigma^2 r_0^2} + \frac{\sigma^2}{r_0} \int_0^{r_0} e^{-\sigma^2 r^2} dr \right), \quad (68)$$

where $\text{Erf}(x) = \int_0^x e^{-t^2} dt$. We note that by using this vorton, there is not any question for solenoidal even in the discrete vorticity field.

8. Vorticity and Velocity Field Induced by an Isolated Vorton

The vorticity field is given by

$$\vec{\omega}(\vec{a}) = \exp(-\sigma^2 |\vec{x}|^2) \mathbf{e}_3, \quad (69)$$

where $\mathbf{e}_3$ is the unit vector of x_3 direction. In this case, the velocity field is given by

$$\begin{aligned} u_1 &= \frac{x_2}{2} \frac{F(\sigma r_0)}{\sigma r_0} \\ u_2 &= -\frac{x_1}{2} \frac{F(\sigma r_0)}{\sigma r_0} \\ u_3 &= 0, \end{aligned} \quad (70)$$

where $F(x) = (1/x)e^{-x^2} - (1/x^2)\text{Erf}(x)$. Figure 3 shows the velocity field in the case of $\sigma=1$.

Figure 4 shows the vorticity distribution due to following Gaussian vortex ring vorton

$$\begin{aligned} \omega_1 &= -2x_2 \exp(-\sigma^2 |\vec{x}|^2) \\ \omega_2 &= 2x_1 \exp(-\sigma^2 |\vec{x}|^2) \\ \omega_3 &= 0. \end{aligned} \quad (71)$$

Fig. 4 The vortex field of an isolated Gaussian vortex ring vorton. a: Vorticity vector, b: Distribution of vorticity on $x_3=0$ plane

In this case, the velocity field is given by

$$\begin{aligned} u_1 &= \frac{1}{2} \frac{x_2 x_3}{r_0^3} F_1(\sigma r_0) \\ u_2 &= \frac{1}{2} \frac{x_1 x_3}{r_0^3} F_1(\sigma r_0) \\ u_3 &= \frac{G(\sigma r_0)}{r_0^2} - \frac{1}{2} \frac{x_2^2 - x_1^2}{r_0^3} F_1(\sigma r_0), \end{aligned} \quad (72)$$

where $F_1(x) = (2/x)G(x) - dG(x)/dx$ and $G(x) = -e^{-x^2} + (1/x) \text{Erf}(x)$. Figure 5 shows the velocity field in the case of $\sigma=1$.

9. Conclusions

In the present paper, the fundamental integral equation for the vortex method is derived from the Navier-Stokes equations. In the three-dimensional flows, the stretch term of vorticity is important, and it is shown to be of order of Δt , where Δt is the time step. From this integral equation, typical schemes of the vortex method, the vorton method, the vortex element method and the vortex stick method, are derived and their theoretical background is elucidated.

The present analysis further shows that the vorticity field is solenoidal if the initial vorticity field is solenoidal. In the actual numerical scheme, the vor-

Fig. 5 The velocity field induced by an isolated Gaussian ring vorton. a: Velocity vector, b: Velocity vector on $x_2=0$ plane

ticity field is not solenoidal locally, but we see from the present analysis that this disadvantage is resolved by initially dividing the region into much fine discrete regions. Finally, in this paper, the velocity field in the vorton method is analytically obtained, a Gaussian vortex ring vorton which is itself solenoidal is proposed and the vorticity and velocity fields due to this vorton are shown.

Appendix 1

We see from Eqs.(15) and (16) that we must obtain the following integral

$$\begin{aligned} & \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{e^{-k_l X_{li}}}{k^2 + a^2} dk \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{|\vec{X}|^{1/2}} \int_0^{\infty} \frac{k^{3/2}}{k^2 + a^2} J_{1/2}(k|\vec{X}|) dk, \end{aligned}$$

where $J_{1/2}$ is the Bessel function. Here, we use the following relation

$$\int_0^\infty \frac{x^{3/2}}{x^2 + y^2} J_{1/2}(rx) dx = y^{1/2} K_{1/2}(ry),$$

where $K_{1/2}$ is the modified Bessel function. From these relations, we have

$$\begin{aligned} & \frac{1}{(2\pi)^3} \int_{-\infty}^\infty \frac{e^{-k_i X_{ii}}}{k^2 + \text{Re}(p-a)} dk \\ &= \frac{1}{(2\pi)^{2/3}} \frac{\{\text{Re}(p-a)\}^{1/2}}{|\vec{X}|} K_{1/2}(\{\text{Re}(p-a)\}). \end{aligned}$$

Here, the inverse Laplace transformation of the following relation

$$\frac{2^{3/2} p^{1/4}}{a^{1/2}} K_{1/2}(ap^{1/2})$$

is given by

$$\frac{1}{t^{3/2}} \exp\left(-\frac{a^2}{4t}\right).$$

Therefore, we can obtain Eq. (18) by using the convolution.

Appendix 2

From Eq. (14), we have

$$\tilde{\mathfrak{G}}_1 = \frac{1}{|\mathbf{E}|} \begin{vmatrix} \omega_{01} + \tilde{\mathfrak{F}}_1 & -\varepsilon_{21} & -\varepsilon_{31} \\ \omega_{02} + \tilde{\mathfrak{F}}_2 & X - \varepsilon_{22} & -\varepsilon_{32} \\ \omega_{03} + \tilde{\mathfrak{F}}_3 & -\varepsilon_{23} & X - \varepsilon_{33} \end{vmatrix}.$$

Therefore, the numerator of the above relation is given by

$$\begin{aligned} & (\omega_{01} + \tilde{\mathfrak{F}}_1)\{(X - \varepsilon_{22})(X - \varepsilon_{33}) - \varepsilon_{23}^2\} \\ & - (\omega_{02} + \tilde{\mathfrak{F}}_2)\{(-\varepsilon_{21})(X - \varepsilon_{33}) - \varepsilon_{13}\varepsilon_{23}\} \\ & + (\omega_{03} + \tilde{\mathfrak{F}}_3)\{\varepsilon_{21}\varepsilon_{32} + \varepsilon_{31}(X - \varepsilon_{22})\}. \end{aligned}$$

Comparing this result with Eq. (20), we have

$$\begin{aligned} & a_{j1}^1 + a_{j2}^2 + a_{j3}^3 = \delta_{1j} \\ & (\beta + \gamma)a_{j1}^1 + (\alpha + \gamma)a_{j2}^2 + (\alpha + \beta)a_{j3}^3 \\ & = (\varepsilon_{22} + \varepsilon_{33})\delta_{1j} - \varepsilon_{21}\delta_{2j} - \varepsilon_{31}\delta_{3j}. \end{aligned}$$

Here, we use the relation $\text{div } \tilde{u}_a = 0$, then $\varepsilon_{ii} = 0$.

Thus, Eq. (22) is obtained.

Appendix 3

In the high Reynolds number flow, we can take $\varepsilon = \left(\frac{4s}{Re}\right)^{1/2} \ll 1$ for $0 < s \leq \Delta t$. Here, we define

$$\begin{aligned} \vec{X} + s\tilde{u}_a &= \varepsilon(r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta) \\ \vec{X}' &= \varepsilon(\tilde{r}' \sin \theta' \cos \phi', \tilde{r}' \sin \theta' \sin \phi', \tilde{r}' \cos \theta'). \end{aligned}$$

Then we have

$$\begin{aligned} r'^2 &= \varepsilon^2[r^2 + r'^2 - 2r\tilde{r}' \sin \theta \sin \theta' \cos(\phi - \phi') \\ &\quad - 2r\tilde{r}' \cos \theta \cos \theta']. \end{aligned}$$

Therefore, for $\varepsilon \ll 1$, we have

$$\begin{aligned} I_1 &\cong \left(\frac{1}{\pi}\right)^{3/2} \int_{-\infty}^\infty \int_0^{\Delta t} a_j^i(\tilde{a}, s) f_j(-\vec{X}, t-s) e^{-r^2} da ds \\ &\quad \times 2\pi \int_0^\pi \tilde{r}'^2 \sin \theta' \\ &\quad \times \exp(-\tilde{r}'^2 + 2r\tilde{r}' \cos \theta \cos \theta') \\ &\quad \times I_0(2r\tilde{r}' \sin \theta \sin \theta') d\tilde{r}' d\theta', \end{aligned}$$

where I_0 is the modified Bessel function. Further, we use the following relation

$$\int_0^\pi \sin \theta' \exp(2r\tilde{r}' \cos \theta \cos \theta')$$

$$\begin{aligned} & \times I_0(2r\tilde{r}' \sin \theta \sin \theta') d\theta' \\ &= \sum_{n=0}^\infty \frac{\pi^{1/2}}{n!} r^{-(n-1/2)} \frac{(\sin \theta)^{2n}}{(\cos \theta)^{n+1/2}} \\ &\quad \times \int_0^\infty \tilde{r}'^2 \exp(-r'^2) \tilde{r}'^{-(n-1/2)} I_{n+1/2}(2r\tilde{r}' \cos \theta) d\tilde{r}' \\ &\quad \times \int_0^\infty x^{2n} \exp(-ax^2) dx = \frac{(2n-1)!!}{2^{n+1}} \left(\frac{\pi}{a^{2n+1}}\right)^{1/2}. \end{aligned}$$

Then, we have

$$\begin{aligned} & \int_0^\infty \int_0^\pi \tilde{r}'^2 \sin \theta' \exp(-\tilde{r}'^2 + 2r\tilde{r}' \cos \theta \cos \theta') \\ &\quad \times I_0(2r\tilde{r}' \sin \theta \sin \theta') d\tilde{r}' d\theta \\ &= \frac{\sqrt{\pi}}{2} \exp(r^2). \end{aligned}$$

Therefore, we finally arrive at

$$\begin{aligned} I_i &\cong \left(\frac{1}{\pi}\right)^{3/2} \int_{-\infty}^\infty \int_0^{\Delta t} a_j^i(\tilde{a}, s) f_j(-\vec{X}, t-s) \\ &\quad \times 2\pi \frac{\sqrt{\pi}}{2} da ds \\ &\cong \Delta t \int_{-\infty}^\infty \{a_{j1}^i(a) + a_{j2}^i(a) + a_{j3}^i(a)\} \\ &\quad \times (((\tilde{u}_a - \tilde{u}) \cdot \nabla) \tilde{\omega} + \tilde{\omega} \cdot (\mathbf{D} - \mathbf{D}_a))_j(-\vec{X}, 0) da. \end{aligned}$$

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