Analytic Investigation of the Increase in Kinetic Energy at the Shear-Free Constant-Pressure Boundary of a Fluctuating Flow*

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Physical and numerical experiments and computations reported in the literature conclude that the kinetic energy of free-surface bounded turbulent flows appears to increase as the free surface is approached from below. There is in the previous literature no satisfactory explanation of the mechanism underlying this feature. In order to provide insight revealing the mechanism, an analytic investigation of an oscillating laminar flow bounded on the lower side by a horizontally moving no-slip wall and on the upper side by a shear-free constant-pressure boundary is presented herein. Although the motion is periodic, rather than turbulent, essential features including momentum diffusion are present and hence the flow is a valid candidate for indicating the mechanism. A mathematical solution for the velocity and the kinetic energy is analyzed to determine the physical mechanism leading to the increase in kinetic energy. The finding is that the relaxation of the shear stress at the free surface produces enhanced viscous acceleration of the fluid in a thin layer adjacent to the free surface. The enhanced acceleration produces a nearly constant kinetic energy in the layer. It is concluded that the effect of the shear-free constant-pressure boundary of a fluctuating flow is to produce a nearly uniform flow in a thin layer adjacent to the surface, and that the effects of the free surface penetrate far into the bulk fluid with exponentially decreasing significance.

Key Words:

1. Introduction

Recent investigations of turbulence near and at shear-free free-surfaces have demonstrated that the turbulent kinetic energy density increases in comparison to similar flows without a free surface as the free surface is approached from within the liquid interior. Sarpkaya & Neubert⁽¹⁾ observed this feature during an experimental investigation of the turbulent flow in the interaction of a single tip vortex in water with a free surface at the air/water boundary. They relate similar observations for measured open-channel free-surface flow⁽²⁾, for a computed rigid free-surface channel⁽³⁾ and for a measured free-surface bounded subsurface jet⁽⁴⁾. In addition to those cases, there are several additional computation cases in the literature: turbulence box decay with a rigid shear-free surface⁽⁵⁾⁻⁽⁸⁾, a submerged jet bounded by a shear-free

constant-pressure free surface⁽⁹⁾, the boundary layer in the intersection corner of a no-slip boundary and a shear-free constant-pressure free surface⁽¹⁰⁾ and the boundary layer in the intersection corner of a no-slip boundary and a shear-free rigid surface⁽¹¹⁾. In all of these cases the turbulent kinetic energy density either increased or remained constant⁽⁴⁾ as the shear-free surface was approached from below (from the liquid side).

A comparison of the computations and the experiments is clouded in many cases by the free-surface pressure and surface tension conditions. The measurements were all conducted with an air-water boundary for the free surface. Such a boundary is characterized by negligible shear stress on the air side, constant external pressure, and deformations of the boundary. The computations were made in some cases with a rigid boundary, which, because it does not deform, is characterized by instantaneous pressure gradients. These gradients may produce results similar to those for the experiments in the case that the experiment surface tension (not included in the

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computations) is sufficient to produce significant local pressure gradients in response to infinitesimal boundary deformations. Additional discrepancies between measurements and computations may arise because of possible unmeasured contaminant surface tension gradients producing vorticity in the experiments. Nevertheless, the available data indicate that the kinetic energy density increases as the free surface is approached.

There is some discussion in the literature concerning the kinetic energy at the free surface. The free surface at low Froude numbers provides a blocking effect, causing the isotropic fluctuations to “damp” in the direction normal to the free surface, and to simultaneously increase in the two lateral directions. The offered reason for this is that the kinetic energy must be re-distributed. The free-surface at high Froude numbers is observed to maintain the isotropy in the velocity fluctuations, and this isotropy is hypothesized to contribute to the conversion of turbulent kinetic energy to wave energy⁽¹²⁾. Neither of these mechanistic explanations accounts for the observed increase in kinetic energy. Sarpkaya & Neubert⁽¹⁾ indicate that “two-dimensional turbulence” is associated with the behavior of the kinetic energy near the free surface. They reason that isotropic turbulence in the fluid bulk is transformed by the free surface to two-dimensional eddies on the surface. The implication is that amalgamation of the two-dimensional eddies produces an increase in the turbulent kinetic energy. Various descriptions of this two-dimensional model have been expressed; the most reasonable is that vorticity lines terminate on the free surface but follow chaotic paths in the subsurface flow⁽¹³⁾. Although the concept of two-dimensional turbulence is a helpful visual construct, it does not account for the increase in kinetic energy. Perot & Moin⁽⁵⁾⁻⁽⁷⁾ provide the additional information from their computations that the increase in kinetic energy is associated with reduced dissipation. They also speculate that the kinetic energy increase at the free surface is “possibly” a result of turbulence changing to a “2-component state at the surface”. Missing from their discussion is a description of the mechanism producing the decreased dissipation. Presumably the mechanism would be related to the reduced stretching and turning of vorticity filaments attached to the free surface.

The conclusion is that although it is observed in all of these numerical and physical experiments that the kinetic energy of the turbulence increases or remains constant as the free surface is approached, no satisfactory explanation has been forthcoming. A simple image analogy at the free surface may reproduce features similar to those for a shear-free con-

stant-pressure boundary, such as the increase of velocities on the surface. However, the analogy is probably not appropriate for natural flows in that those flows are not represented by “images” because of the constant-pressure condition at the fluid boundary and because of the actual contact of the subsurface vorticity with the free surface.

The scientific question concerns the physical mechanisms in actual flows. In their review, Sarpkaya & Neubert⁽¹⁾ pose the question “Does (the kinetic energy) indeed increase very near the free surface, are there other flow situations……that exhibit similar characteristics near the free surface……”? The purpose of the research described herein is to shed light on some answers to this intriguing question by investigating an analytic problem: the periodically steady flow between an oscillating horizontal flat wall and a parallel free surface.

The above observations from measurements and computations can be separated into two groups depending on whether or not the flow exhibits mean advection. Such three-dimensional flows as jets and tip vortices are expected to feature mean velocity transport of kinetic energy resulting in an increase of the kinetic energy density at the free surface. For example, a swirling tip vortex flow will continuously pump turbulent kinetic energy into the free surface region. It is of particular scientific interest to investigate the underlying physics leading to the kinetic energy density increase at the free surface for flows in which mean advection is absent. The decaying initially isotropic turbulence boxes investigated in Ref. (5)-(8) fall into this category. In these simulations the turbulence is subjected to the sudden insertion of the shear-free boundary condition, and the evolution (decay) with time is studied for turbulence properties. The flow is statistically uniform in planes parallel to the shear-free boundary. Increases of kinetic energy density must take place as results of changing local conditions rather than as a mean transport of kinetic energy from one location in the flow to another. However, the pressure boundary condition for these flows appears to be a homogeneous condition immediately prior to insertion of the rigid boundary, and a relaxation of that pressure condition immediately after insertion. In other words, the pressure is not maintained constant on the rigid boundary as the turbulent flow evolves. The rationale for this relaxation is that the pressure redistributes the flow to maintain conservation of mass.

A difficulty in interpreting the physical and numerical data in the literature is that the data is a discrete set of flow variables, whereas the need in this case is for an analytic expression for the flow. An

analytic expression can be examined for functional behavior in the context of the physics represented by the governing equations. Such an analytic expression is impractical if not impossible to obtain for a complex turbulent flow.

An alternative approach is followed here to develop insights into the mechanisms underlying the behavior of the turbulent kinetic energy at the shear-free constant-pressure surface boundary of a deterministic fluctuating flow. This alternative approach is to *analytically* solve the problem of the kinetic energy density of the flow bounded on the bottom by an oscillating rigid (plane) wall and on the top by a shear-free constant-pressure surface, and to interpret the solution to understand the mechanisms producing the changes in the kinetic energy density at the free surface. This alternative problem contains the physics associated with diffusion processes transporting vorticity to the free surface. The boundary condition on the bottom of the flow is a no-slip condition, and the boundary condition on the top of the flow is a shear-free (or full-slip) condition with constant pressure. The motion of the rigid wall produces surface-parallel vorticity. Vorticity generated in the neighboring fluid by the oscillating wall diffuses to the free surface. Analytic solutions for the kinetic energy density are investigated as functions of oscillation amplitude and frequency, and depth of submergence of the wall.

The analytic results show that the kinetic energy density monotonically decreases as the free surface is approached, but at a much slower rate than without the free surface present. That is, the kinetic energy is greatly enhanced near the free surface in comparison to the case without the free surface present. The analytic solution shows that as the vorticity disturbances approach the shear-free surface, from below, the original shear wave is joined by a second shear wave which combines to satisfy the free surface condition. This second wave is generated by the free-surface acceleration, produced by the stress-release at the free surface, and propagates downward from the shear-free surface and towards the oscillating wall. A third shear wave is also observed. This third wave propagates towards the free surface with a very small amplitude, essentially nonexistent at the free surface, and is the mechanism to cancel the second wave at the oscillating wall to satisfy the no-slip condition.

The three shear (vorticity) waves are designated: the **p-wave**, which is the only wave that exists in the absence of the free surface, and represents a shear (vorticity) wave moving outwards from the oscillating wall and towards the shear-free surface,

the **ω -wave**, which cancels the p-wave vorticity at the shear-free surface to satisfy the stress-free condition, and which propagates from the free surface and towards the oscillating wall; it is generated by the enhanced viscous acceleration of the stress-release free surface, and the **u-wave**, which is a low-amplitude wave with velocity that cancels the ω -wave velocity at the oscillating wall to satisfy the no-slip condition, and is a wave moving from the oscillating wall and towards the free surface; it is damped to negligible amplitude before impacting the free surface.

Nomenclature

e_k	: kinetic energy density
h	: distance from wall to free-surface
k	: wave number
t	: time
u	: velocity in x -direction
u_c	: wave propagation speed
U_o	: velocity of the wall
x	: distance along wall
y	: distance normal to wall
β	: frequency for oscillatory motion
λ	: viscous length scale
μ	: dynamic viscosity
ν	: kinematic viscosity
ρ	: density

2. The Shear-Free Surface Bounded Oscillating Wall Problem

Figure 1 shows the relative locations of the boundaries and defines the coordinate system for the oscillating (plane) wall flow. The curving contour for the fluctuating velocity depicts the viscosity-damped sinusoidal features to be expected in this flow as the

Fig. 1 Schematic of the shear-free constant-pressure surface and oscillating (plane) wall geometry

shear waves, or vorticity waves, propagate between the oscillating wall and the shear-free surface. The velocity of the fluid at the wall has the velocity of the wall (because of the no-slip condition) and the slope of the velocity at the free surface is zero (because of the full slip, or shear-free, condition, equivalent to a zero tangential stress condition). The pressure at the surface $y=h$ is a constant, and the gravitational potential is $(-gy)$. As shown below, the velocity profile is more complicated than the single sinusoidal wave observed for the case of semi-infinite depth fluid ($h=\infty$)⁽¹⁴⁾.

The equation for the flow is the equation of motion for two-dimensional planar flow:

$$\frac{\partial u(y, t)}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad (1)$$

where “ u ” is the unidirectional velocity in the x -direction, “ t ” is time, “ y ” is the normal distance from the oscillating wall, and “ ν ” is the kinematic viscosity.

3. The Velocity

Consider the case in which the starting transients from the initial acceleration of the wall have disappeared, and the periodically steady horizontal velocity of the wall is given by:

$$u = U_0 e^{i\beta t} \quad (2)$$

Equation (2) anticipates that the frequency of the velocity at any position in the flow is the same as that for the oscillating wall.

The boundary conditions for the problem are:

$$u(0, t) = U_0 e^{i\beta t} \quad (3)$$

which expresses the no-slip condition at the rigid oscillating wall, and

$$\mu \frac{\partial u(h, t)}{\partial y} = 0 \quad (4)$$

which expresses the shear-free condition on the fluid at the free surface. The solution to the problem produces the following equation for the velocity field:

$$\begin{aligned} u(y, t) = & -U_0 \frac{e^{i\beta t - (1+i)\sqrt{\frac{\beta}{2\nu}}(y+h)}}{e^{+(1+i)\sqrt{\frac{\beta}{2\nu}}h} + e^{-(1+i)\sqrt{\frac{\beta}{2\nu}}h}} \\ & + U_0 \frac{e^{i\beta t + (1+i)\sqrt{\frac{\beta}{2\nu}}(y-h)}}{e^{+(1+i)\sqrt{\frac{\beta}{2\nu}}h} + e^{-(1+i)\sqrt{\frac{\beta}{2\nu}}h}} \\ & + U_0 e^{i\beta t - (1+i)\sqrt{\frac{\beta}{2\nu}}y} \end{aligned} \quad (5)$$

Equation (5) is a linear sum of three damped waves which propagate between the oscillating wall and the shear-free surface. The equation exhibits two length scales: the free-surface depth “ h ” and the viscous length scale “ λ ”, defined by the expression:

$$\lambda \doteq 2\pi \sqrt{\frac{2\nu}{\beta}} \quad (6)$$

The real part of Eq.(5) is rewritten to second order in the wave number, k , defined as follows:

$$k = \frac{2\pi}{\lambda} = \sqrt{\frac{\beta}{2\nu}} \quad (7)$$

to produce the following equation:

$$\begin{aligned} \frac{u(y, t)}{U_0} = & e^{-2kh} e^{ky} \cos(\beta t - 2kh + ky) \\ & - e^{-2kh} e^{-ky} \cos(\beta t - 2kh - ky) \\ & + e^{-ky} \cos(\beta t - ky) \end{aligned} \quad (8)$$

This equation can be rewritten to explicitly show the dependence on the ratio of depth-to-viscous length scales, h/λ :

$$\begin{aligned} \frac{u(y, t)}{U_0} = & e^{-4\pi \frac{h}{\lambda}} e^{2\pi \frac{h}{\lambda} \frac{y}{h}} \cos\left(\beta t + 2\pi \frac{h}{\lambda} \frac{y}{h} - 4\pi \frac{h}{\lambda}\right) \\ & (\omega\text{-wave}) \\ & - e^{-4\pi \frac{h}{\lambda}} e^{-2\pi \frac{h}{\lambda} \frac{y}{h}} \cos\left(\beta t - 2\pi \frac{h}{\lambda} \frac{y}{h} - 4\pi \frac{h}{\lambda}\right) \\ & (\omega\text{-wave}) \\ & + e^{-2\pi \frac{h}{\lambda} \frac{y}{h}} \cos\left(\beta t - 2\pi \frac{h}{\lambda} \frac{y}{h}\right) \\ & (p\text{-wave}) \end{aligned} \quad (9)$$

The velocity given by Eq.(9) consists of the linear superposition of three waves. The actual velocity at any instant is a summation of the three waves, all at the same frequency but with different phases. The fluctuating velocity tapers exponentially as the free surface is approached from below as shown in Fig. 1.

Figure 2 shows the velocity amplitude profiles for each of the three waves for the case $h/\lambda=2$. The p-wave, or primary wave, is the solution that exists if there is no free surface (i.e., $h=\infty$); it was discussed above. The ω -wave produces a velocity gradient at the free surface which cancels the velocity gradient of the primary wave to produce the required zero-stress condition at the free surface. The phase velocity for the ω -wave is negative; that is, the ω -wave propagates from the shear-free surface and towards the

Fig. 2 Velocity amplitude profile for the three shear waves propagating between the oscillating wall and the shear-free surface ($h/\lambda=2$)

wall. The u -wave, which has negligible velocity at the shear-free surface, cancels the ω -wave at the oscillating wall to satisfy the no-slip boundary condition. Although only three waves are considered here, in fact there are an infinite number of higher order waves which have been neglected as a consequence of truncating Eq. (8) to second order. These additional waves would continue the "zigzag" of profiles to the left in Fig. 2. The amplitudes of these higher order waves diminish exponentially with increasing order.

The propagation speeds of the three waves are:

$$u_c = \left(-\frac{\beta\lambda}{2\pi}, \frac{\beta\lambda}{2\pi}, \frac{\beta\lambda}{2\pi} \right) \quad (10)$$

for the ω -, u -, and p -waves, respectively. Both the u - and p -waves propagate from the wall and towards the free surface. The ω -wave propagates from the free surface and towards the wall.

The velocity at the free-shear surface consists almost wholly of the sum of the p -wave and the ω -wave. Equation (9) evaluated at $y/h=1$ yields the following equation:

$$\left(\frac{u(y, t)}{U_0} \right)_{y/h=1} = 2e^{-2\pi\frac{h}{\lambda}} \cos \left(\beta t - 2\pi\frac{h}{\lambda} \right) - e^{-6\pi\frac{h}{\lambda}} \cos \left(\beta t - 6\pi\frac{h}{\lambda} \right) \quad (11)$$

The first term on the right hand side is the sum of the p -wave and the ω -wave, which are identical in amplitude and phase at the shear-free surface ($y/h=1$). The value for the velocity on the shear-free surface is twice the value at that position if the shear-free surface is located infinitely far from the oscillating wall. The effects of the shear-free surface rapidly decrease as the oscillating wall is approached, and the velocity at the oscillating wall is identical to that for the case with the surface placed infinitely far from the wall.

The increase in velocity at the free surface is attributed to the relaxation of the shear stress on the shear-free boundary. In comparison to the case of a semi-infinite fluid, the free surface causes particles on the free surface to be "unloaded" with the result that the viscous stresses on the liquid side are no longer working against viscous stress on the external boundary. The result is enhanced acceleration and increased velocity of the particles.

Figure 3 shows the velocity amplitude profiles for each of the three waves for two values of the depth-to-wave length ratio, h/λ . It is observed in each case that the amplitudes are identical for the p - and ω -waves at the free surface, and for the ω - and u -waves at the oscillating wall. The effect of changing h/λ is to expand (or compress) the wave profiles in accordance with the ratios of the values of h/λ .

Figure 4 shows that the velocity amplitude

Fig. 3 Velocity amplitude profiles for the three waves for $h/\lambda=2$ (thick lines) and $h/\lambda=0.5$ (thin lines)

Fig. 4 Velocity amplitude profiles for the oscillating wall flow with free surface at $y/h=1$ ($h/\lambda=2$) and for a semi-infinite fluid

behaves asymptotically as the free surface is approached from below. For comparison, the velocity profile for a semi-infinite fluid is also shown. The free surface produces a doubling of the velocity at the free surface. The amplitude profiles cross each other a short distance below the free surface.

4. The Kinetic Energy Density

The dramatically increased acceleration of the free-surface fluid leads to corresponding increased velocities. It is expected as a consequence that the kinetic energy would also increase. It is shown below that the effect of the shear-free constant-pressure surface on the free-surface kinetic energy is a three-fold increase in comparison to the case of a semi-infinite fluid.

The kinetic energy density (kinetic energy per unit volume), is calculated as follows:

$$e_k = \rho \frac{1}{2} u_{\text{mean square}}^2 \quad (12)$$

The mean square of the velocity fluctuations is

Fig. 5 Kinetic energy density profiles for the ω -wave, p -wave, the ω - p wave, and the total wave ($h/\lambda=2$)

obtained by integrating the square of the velocity given by Eq. (9) over one period and dividing by the period. The resulting equation, neglecting high order terms, is as follows :

$$\begin{aligned} \frac{e_k}{\frac{1}{2}\rho U_0^2} &= \left(\frac{u}{U_0} \right)_{\text{mean square}}^2 = \\ &= \frac{1}{2} e^{4\pi \frac{h}{\lambda} \left(\frac{y}{h} - 2 \right)} + \frac{1}{2} e^{-4\pi \frac{h}{\lambda} \frac{y}{h}} \\ &+ e^{-4\pi \frac{h}{\lambda}} \cos \left(4\pi \frac{h}{\lambda} \left(\frac{y}{h} - 1 \right) \right) \\ &\quad (\omega\text{-}p \text{ coupling}) \end{aligned} \quad (13)$$

The p -wave component is the kinetic energy density produced by the wave in the absence of the free surface. The ω -wave component is the additional energy produced by the flux of vorticity from the shear-free surface. The ω - p coupling is additional energy contributed by the coupling of the p - and ω -waves in the nonlinearity of the velocity for the kinetic energy. Figure 6 shows that the effect of the shear-free constant-pressure surface is to increase the kinetic energy density at the free surface (compare the "total" profile to the p -wave profile). It is observed from Eq. (13) that the value of the kinetic energy density at the free surface is four times the value for a semi-infinite fluid (p -wave alone).

The slope of the kinetic energy density is given by the following expression :

$$\begin{aligned} \frac{h}{\frac{1}{2}\rho U_0^2} \frac{\partial e_k}{\partial y} &= \frac{1}{2} \frac{4\pi h}{\lambda} e^{4\pi \frac{h}{\lambda} \left(\frac{y}{h} - 2 \right)} \\ &- \frac{1}{2} \frac{4\pi h}{\lambda} e^{-4\pi \frac{h}{\lambda} \frac{y}{h}} \\ &- 4\pi \frac{h}{\lambda} e^{-4\pi \frac{h}{\lambda}} \sin \left(4\pi \frac{h}{\lambda} \left(\frac{y}{h} - 1 \right) \right) \leq 0 \end{aligned} \quad (14)$$

The slope is observed to monotonically increase as the free surface is approached from below, becoming

equal to zero at the free surface ($y/h=1$). There is no minimum for the mean square kinetic energy density in the fluid interior. Hence the kinetic energy density is not observed to be larger at the free surface than in the flow interior. Nevertheless, there is a large increase in kinetic energy density at the free surface as a result of the free surface (stress release).

5. Summary

The analytic investigation of an oscillating flow consisting of surface-parallel vorticity bound between a no-slip boundary and a parallel shear-free constant-pressure boundary (a free surface) reveals a mechanism producing "two dimensional" flow at the free surface which is characterized by nearly constant kinetic energy. The mechanism is the enhanced viscous acceleration of the free surface fluid resulting from the stress release at the free surface. Instead of rapidly decaying with distance from the plate, the energy and vorticity asymptotically approach their values at the free surface. The understanding of this simple yet complex planar oscillating flow, with its three-dimensional features, is instrumental for describing and understanding complex free surface flows. In particular, the stress release mechanism is a possible explanation for observed increases in the kinetic energy of free-surface turbulent flows.

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