Influence of Flow Force on the Flapper of a Water Hydraulic Servovalve*

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In this paper, flow force due to nozzle flow and its influence on the flapper-nozzle system for water hydraulic servovalves are discussed. The flow force acting on the flapper is estimated using the momentum theory, and expressed as a function of the nozzle pressure and flapper-nozzle gap. Analysis shows that the double nozzle flapper gives a quasi-linear characteristic to the relationship among the force on the flapper, the flapper gap and the spool velocity. The numerical result based on the theory is used to estimate the accuracy of the classical linear approximation. The moment due to the flow force resists input moment acting on the armature. Consequently the flow force reduces the magnitude of the change of the nozzle backpressures that drive the spool of the water hydraulic valve. Experimental verification of the theory is also presented.

Key Words: Flapper-Nozzle System, Flow Force, Fluid Power, Hydrostatic Bearing, Nozzle, Servovalve, Valve, Water Hydraulics, Water Hydraulic Servovalve

1. Introduction

The flapper-nozzle system treated in this paper is used as the first stage of a two-stage water hydraulic servovalve^{(1),(2)}. Figure 1 illustrates the structure of the flapper-nozzle system. In this system, an electromagnetic force acting on the armature causes a displacement of the flapper. The flapper displacement induces the change of the backpressures of the nozzles.

The flapper position is influenced by the magnetic torque on the armature, an elastic force due to the supporting element, and fluid forces due to the nozzle jets. As the result of these actions, the moment balance of the flapper system becomes

$$T_A = K_T \psi + (l_F - l_t/2)F. \quad (1)$$

Since the flapper displacement is the result of the horizontal displacement of the supporting point and the deflection of the flapper,

$$x = (l_F - l_t/2)\psi - F(1/K_F + 1/K_M). \quad (2)$$

In the relationship of the armature to the flapper, T_A is the input and x is the output. For the servovalve, x is an intermediate variable determining the nozzle backpressures, p_{nl} and p_{nr} , that drive the spool. We have already reported⁽³⁾ the relationships between x and the nozzle backpressures. Here we will study the

Fig. 1 Armature-flapper nozzle system

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relationship between T_A and x .

Nomenclature

- A : nozzle cross section ($=\pi d_n^2/4$)
 B : flapper half-width
 c_B : bearing discharge coefficient
 c_n : nozzle discharge coefficient
 d_B : throttle diameter of hydrostatic bearing
 d_c : diameter of conduit connecting spool end and nozzle
 d_n : nozzle diameter
 F : fluid force acting on flapper (fluid jet force)
 F_l, F_r : fluid jet force on left and right sides
 f, f_l, f_r : nondimensional forces (e.g., $f=F/Ap_s$)
 H : neutral gap between nozzle and flapper
 h_a, h_b : clearances between spool and sleeve
 K_F : spring constant of flapper rod bending
 K_M : spring constant of flexure tube bending
 K_T : spring constant of flexure tube rotation
 L_a, L_b : spool land lengths on tank side and nozzle side
 l_F : distance between nozzle axis and armature center
 l_t : flexure tube length
 P_{n0}, P_{b0} : nondimensional nozzle pressure and bearing pressure at equilibrium
 P_{bB}, P_{rB} : nondimensional bearing pressure
 p_l, p_r : pressure at spool end (spool backpressure) on left and right sides
 p_{lf}, p_{rf} : pressure at nozzle exit on left and right sides
 p_{nl}, p_{nr} : nozzle inlet pressure on left- and right sides
 p_s : supply pressure
 $P=p_l/p_s$: nondimensional variable (e.g., $P_l=p_l/p_s$)
 q_l, q_r : rate of flow from nozzle on left and right sides
 q_l^* : rate of flow leaving left-side of flapper
 $q_0 = \pi c_B d_B^2 \sqrt{2p_s/\rho}$, unit flow to give dimensionless quantities
 R : spool radius
 T_A : external moment acting on armature
 $T = \pi R^2 H / q_0$, unit time to give dimensionless quantities
 U : flapper velocity
 x : flapper displacement
 z : spool displacement
 $\alpha_a = \pi h_a^3 R p_s / 6 \mu L_a q_0$
 $\alpha_b = \pi h_b^3 R p_s / 6 \mu L_b q_0$
 $\alpha_c = c_n H d_n / c_B d_B^2$
 $\alpha_d = (4 c_n H / d_n)^2$
 $\alpha_l = (l_F - l_t / 2) / H$
 $\alpha_m = (1 / K_F + 1 / K_M) A p_s / H$
 $\alpha_q = 16 c_B^2 \{ \zeta_n (d_B / d_n)^4 + \zeta_c (d_B / d_c)^4 \}$
 $\xi = x / H, \zeta = z / H, \eta = d\zeta / d\tau$

ζ_c, ζ_n : pressure loss coefficients in conduit and nozzle

μ : viscosity of water

ρ : density of water

$\tau = t / T$, nondimensional time

ϕ : armature rotation angle

2. Estimation of Fluid Jet Force

2.1 Width of plain jet and flapper

Figure 2 illustrates the fluid jet near the flapper. Since the flow angle leaving the flapper has not been studied so far, we will discuss the angle (δ) in the following. When we apply a two-dimensional model of the nozzle and the potential flow, we get⁽⁴⁾ (see Appendix)

$$\epsilon = \tan(\delta/2) \quad (3)$$

$$\frac{2H}{d_n} = \frac{1+a^2}{\pi} \log\left(\frac{1+a}{1-a}\right) + \frac{2a}{1+\epsilon^2} \left\{ 1 - \epsilon^2 - \frac{4\epsilon}{\pi} \log\left(\frac{1+\epsilon}{1-\epsilon}\right) \right\} \quad (4)$$

$$\frac{2B}{d_n} = 1 - \frac{2}{\pi} (1-a^2) \tan^{-1} a - \frac{2a}{1+\epsilon^2} \left(\epsilon + \frac{1-\epsilon^2}{\pi} \log \epsilon \right) \quad (5)$$

Although a is a dummy parameter used for calculation, we can relate it to physical parameters through $a = 2c_n H / d_n$. In the equations above, we can see that the jet leaves the flapper almost parallel to the flapper surface when $\tan \delta < 0.01$. This condition is satisfied when B is larger than the value explained below. The result of the numerical calculation of above equations is shown in Fig. 3. For a practical flapper-nozzle system, we can assume $\delta < 0.01$ and $H/d_n < 0.1$. Then we can neglect the third order of a and the first order of ϵ on Eqs. (4) and (5), and obtain the following simple relation.

$$\frac{2B}{d_n} = 1 + \frac{4H \{ \log(1/\epsilon) - 1 \}}{d_n (\pi + 2)} \quad (6)$$

This gives, for example, $2B/d_n = 1.33$ for $H/d_n = 0.1$ and $\delta = 0.01$. The result of Eq. (6) deviates little from the numerical value shown in Fig. 3. Hence we can say that the fluid leaves the flapper parallel to the surface

Fig. 2 Flow model at flapper

Fig. 3 Flapper width and flow coefficient

when the flapper width is greater than 1.3 times the nozzle diameter. In the following discussion we will neglect the influence of the leaving angle of the flow.

2.2 Fluid force working on flapper

Ohuchi and Ikebe⁽⁵⁾ gave a linearized theory for the force on the flapper. We apply the momentum theory for the control surface illustrated in Fig. 2. The following calculation will be made for the left-side nozzle only, since it is similar for the right-side nozzle.

The rate of increase of the momentum of fluid in the control volume is

$$\rho q_l U. \quad (7)$$

The difference in fluid momentum flowing into and out from the control volume is

$$(Ap_l + \rho q_l u_l) - \left(\int_S p dS + \rho q^* U \right), \quad (8)$$

where

$$A = (\pi/4)d_n^2, \quad q_l = u_l A, \quad U = dx/dt.$$

The pressure integral in the above expression gives the force acting on the flapper:

$$F_l = \int_S p dS. \quad (9)$$

Since the momentum given by Eqs. (7) and (8) is the same,

$$(Ap_l + \rho q_l u_l) - (F_l + \rho q^* U) = \rho q_l U. \quad (10)$$

The equation of continuity gives

$$q^* = q_l - UA = A(u_l - U). \quad (11)$$

Substituting Eq. (11) into Eq. (10), we obtain

$$F_l = A\{p_l + \rho(u_l - U)^2\}. \quad (12)$$

Similarly, for the right side,

$$F_r = A\{p_r + \rho(u_r + U)^2\}. \quad (12.a)$$

We call these forces "fluid jet force" in this paper.

Referring to the fluid conduit connection shown in Fig. 1, and considering the pressure loss in the conduit, we can express the pressure at the back of the spool (backpressure) as

$$p_l = p_{lf} \left[1 + \left\{ \zeta_c \left(\frac{4d_n H c_n}{d_c^2} \right)^2 + \zeta_n \left(\frac{4H c_n}{d_n} \right)^2 \right\} (1 - \xi)^2 \right]. \quad (13)$$

We can give a similar expression for the right-side pressure. Using these expressions for the backpressures,

$$F = F_l - F_r,$$

$$= A(p_{lf} - p_{rf}) + \rho(q_l + q_r) \left\{ \frac{q_l - q_r}{A} - 2U \right\}. \quad (14)$$

We will show in the following that U in Eq. (14) can be neglected.

The nozzle flows are given by

$$q_l = (H - x) c_n \pi d_n \sqrt{2p_{lf}/\rho}, \quad (15)$$

$$q_r = (H + x) c_n \pi d_n \sqrt{2p_{rf}/\rho}. \quad (15.a)$$

Considering that $p_{lf} \sim p_{rf}$,

$$\begin{aligned} \frac{q_l - q_r}{A} &= \frac{(H - x) c_n \pi d_n}{\pi d_n^2/4} \sqrt{\frac{2p_{lf}}{\rho}} \\ &\quad - \frac{(H + x) c_n \pi d_n}{\pi d_n^2/4} \sqrt{\frac{2p_{rf}}{\rho}} \\ &\approx -(8x c_n/d_n) \sqrt{2p_{lf}/\rho}. \end{aligned} \quad (16)$$

When the flapper takes a simple harmonic motion with angular frequency ω , its velocity is

$$U = dx/dt \approx \omega x$$

Therefore, the two terms in the last part of Eq. (16) become equal when

$$(4c_n/d_n) \sqrt{2p_{lf}/\rho} \sim \omega$$

Now let us substitute practical values, e.g., $c_n = 0.6$, $d_n = 3 \times 10^{-3}$ m and $\rho = 1000$ kg/m³, into the last relation. The result is

$$\omega \sim 6.7 \times 10^4 \text{ rad/s} = 1.1 \times 10^4 \text{ Hz}$$

This frequency is well above the servovalve operating frequency. Therefore, U can be eliminated from Eqs. (12) and (12.a).

After eliminating U in Eqs. (12) and (12.a), and using Eqs. (3) and (15) we derive the following non-dimensional expression.

$$f_l = \frac{1 + 2a_d(1 - \xi)^2}{1 + a_q a_c^2(1 - \xi)^2} P_l \quad (17)$$

Similarly, for the right-side of the flapper,

$$f_r = \frac{1 + 2a_d(1 + \xi)^2}{1 + a_q a_c^2(1 + \xi)^2} P_r. \quad (18)$$

The nondimensional resultant force is given by the difference between these expressions, namely,

$$f = f_l - f_r. \quad (19)$$

3. Connection with the Elastic Structure

The flapper moves to the position where the moment on the armature, nozzle jet force and elastic moment due to the tube are balanced. Our next task is to find this position.

From Eq. (2) and Eq. (19),

$$\xi = \alpha_l \phi - \alpha_m (f_l - f_r) \quad (20)$$

This equation is valid when the flapper is not in contact with the nozzle. The state of contact will be

considered later. To estimate f_l and f_r appearing in Eq.(17) and (18), we need to know P_l and P_r . As reported in another paper⁽³⁾, P_l and P_r are determined by a system of equations that includes the hydrostatic bearing pressures P_{lB} and P_{rB} . We cite here the system of equations to explain the calculation procedure.

$$\text{sgn}(1 - P_{lB})\sqrt{1 - P_{lB}} = (\alpha_a + \alpha_b)P_{lB} - \alpha_b P_l, \quad (21)$$

$$\alpha_b(P_{lB} - P_l) = \frac{d\xi}{d\tau} + \frac{\alpha_c(1 - \xi)\sqrt{P_l}}{\sqrt{1 + \alpha_q\alpha_c^2(1 - \xi)^2}}, \quad (22)$$

$$\text{sgn}(1 - P_{rB})\sqrt{1 - P_{rB}} = (\alpha_a + \alpha_b)P_{rB} - \alpha_b P_r, \quad (23)$$

$$\alpha_b(P_{rB} - P_r) = \frac{d\xi}{d\tau} + \frac{\alpha_c(1 + \xi)\sqrt{P_r}}{\sqrt{1 + \alpha_q\alpha_c^2(1 + \xi)^2}}. \quad (24)$$

The above six equations, namely, Eqs.(19) to (24), must be solved simultaneously. The input to the present system is the set of ψ (or T_A) and η ; the output set is ξ and f . If α_m is a small quantity, the problem is greatly simplified since it can be eliminated from Eq.(20), and the influence of f will appear only as an additive load on the armature-flapper structure. In a practical system with actual materials and dimensions of parts, however, the first and the second terms in the right-side of Eq.(20) become the same order of magnitude.

Since a numerical calculation procedure to find P_l , P_r , P_{lB} and P_{rB} from ξ and η is given in another paper⁽³⁾, we use these pressures as parameters. Then the remaining work is, using η and $d\xi/d\tau$ as parameters, to determine the ξ that satisfies

$$\xi + \alpha_m f(\xi, d\xi/d\tau) = \alpha_i \psi, \quad (25)$$

where $f = f(\xi, d\xi/d\tau)$.

The calculation to solve the above equation is equivalent to find the intersection of

$$y = \alpha_m f(\xi, d\xi/d\tau) \text{ and } y = \alpha_i \psi - \xi.$$

The iterative substitution is not suitable for solving this system since the result of iteration diverges when $|d(\alpha_m f)/d\xi| > 1$.

The Newton method is not applicable because f and its derivative are not expressed by explicit functions. Applying the secant method to

$$G(\xi) = \xi - \alpha_i \psi + \alpha_m f(\xi, d\xi/d\tau), \quad (27)$$

we can find the solution of Eq.(25).

Let us consider the state when the flapper contacts the nozzle. This is a case of saturation that may occur when a large input voltage is applied to the armature driving system. A trial simulation reveals that the contact occurs only when a input signal of large amplitude with high frequency is given. When the flapper contacts the nozzle, the magnitude of ξ is unity and its sign matches the sign of ψ . Hence we have

$$\xi = \text{sgn}(\psi). \quad (28)$$

Under this state, the pressures are determined independent of the force acting on the flapper. Conse-

quently, the solution-finding procedure is simplified. For this state, however, we must derive an equation to give a ψ that restricts the flapper end to $\xi = \pm 1$. Note that the armature can move even when the flapper end is fixed. The contact force of the flapper to the nozzle becomes

$$f = \{\alpha_i \psi - \text{sgn}(\psi)\} / \alpha_m. \quad (29)$$

In our servovalve, first an input moment is given to the armature, then ψ is determined as the solution of

Fig. 4 Flowchart to calculate f from ψ and $d\xi/d\tau$ (ξ_A, ξ_B : initial approximations)

Fig. 5 Subroutine to calculate f , ξ and $d\xi/d\tau$

Fig. 6 ξ - f diagram Fig. 7 $d\zeta/d\tau$ - f diagram

the equation of motion of the armature, and finally ξ is determined from ψ and η . The calculation procedure is illustrated in Fig. 4 as a flowchart. The subroutine in Fig. 4 is shown in Fig. 5. Numerical examples are shown in Figs. 6 and 7. The values of $d\zeta/d\tau$ in Fig. 6 were selected assuming normal operating conditions, although they seem small.

The fluid jet force near $d\zeta/d\tau=0$ is approximated by a group of parallel lines taking ξ as the parameter. When the magnitude of $d\zeta/d\tau$ increases, the curve exhibits a break point where the gradient changes abruptly. This corresponds to reverse flow between the nozzle and the hydrostatic bearing. The reverse flow occurs earlier the larger the magnitude of ξ .

In Figs. 6 and 7, we can see that f is approximated by a linear combination of ξ and ϕ , namely,

$$f \approx C_1 \xi + C_2 \eta, \quad (\eta = d\zeta/d\tau). \quad (30)$$

The accuracy of the linear combination is estimated as follows. Using the sum of discrete errors, namely,

$$f_{ij} = f(\xi_i, \eta_j)$$

Table 1 Accuracy of linear approximation of f

Range of η	F_L	C_1	C_1^*	C_2	C_2^*
$ \eta \leq 0.1$	5.796E-4	0.06292	0.06286	-0.09081	-0.09103
$ \eta \leq 0.2$	3.225E-3	0.05936	0.06286	-0.1684	-0.09103

Common parameters: $\alpha_a=0.08732$, $\alpha_b=0.1674$, $\alpha_c=0.08705$, $\alpha_l=512.5$

$\alpha_m=1.463$, $\alpha_q=2.225$, $M=N=20$; $-1 \leq \xi \leq 1$

and

$$I = \sum_{i=1}^M \sum_{j=1}^N (f_{ij} - C_1 \xi_i - C_2 \eta_j)^2, \quad (31)$$

we define flatness as

$$F_L = \sqrt{I / ((\xi_{\max} - \xi_{\min})(\eta_{\max} - \eta_{\min}))}. \quad (32)$$

We can find C_1 and C_2 by applying the least squares method to the calculated set of f_{ij} ($i=1$ to M , $j=1$ to N). Since right side of Eq.(30) is the expression of a plane approximating f , Eq.(31) is a measure of the deviation of f from the plane.

Next we determine the linear approximations of Eqs.(17),(18),(19) and (21) by the method of small perturbation.

$$C_1^* = \frac{\partial f}{\partial \xi} \Big|_0, \quad C_2^* = \frac{\partial f}{\partial \eta} \Big|_0 \quad (33)$$

$$\left. \begin{aligned} C_1^* &= \frac{a_c \sqrt{P_{n0}}}{(1 + a_q a_c^2)^{3/2}} B^*, \quad C_2^* = -B^*, \\ B^* &= \frac{1}{2} + \frac{1}{4 \sqrt{P_{n0}} (1 + a_q a_c^2)} - \frac{a_b \sqrt{1 - P_{B0}}}{1 + 2(a_a + a_b) \sqrt{1 - P_{B0}}} \end{aligned} \right\} \quad (34)$$

Table 1 shows these values. In this table we can see that C_1 and C_2 are close to C_1^* and C_2^* , respectively. Figure 7 shows that when the range of η increases, F_L increases, or in other words, f loses flatness. In the examples shown, errors due to the approximations are less than 1%. The difference of C_1 from C_1^* and the difference of C_2 from C_2^* are around 1% when $|\eta|$ is less than 0.1; these increase considerably when the range of $|\eta|$ is doubled. In the authors' experience, however, $|\eta|$ seldomly exceeds 0.1 in actual valve design.

The method of perturbation here gives a good approximation because the Taylor developments of f_l and f_r lose their odd-order terms of ξ and η as a result of $f = f_l - f_r$. This result is due to the structure of the double nozzle system. Consequently, this flapper-nozzle system shows a quasi-linear characteristic in large part of its operation. Naturally, a suitable parameter selection is important to guarantee this characteristic. An overly large value of H , for example, induces strong nonlinearity between the nozzle pressure and the flapper displacement; then the quasi-linear relationship between f and ξ also vanishes. Figure 8 shows a numerical example with various H . Since a_l is inversely proportional to H , smaller values of a_l indicate larger values of H .

Fig. 8 Influence of nozzle gap on linearity of flow force

Fig. 9 Experimental setup

4. Experiment

For an oil-hydraulic servovalve, the measurement of flapper force has been reported⁽⁶⁾. However, the structure and operating conditions of the present system are different from those already studied. Therefore, the force acting on this flapper-nozzle system must be verified experimentally. It is difficult to measure flapper displacement in water exactly. Instead, we indirectly estimate the relationship between them applying external moment on the armature and by measuring the nozzle pressure.

Figure 9 shows the experimental setup. The supply port p_s in Fig. 9 is connected to a 3.5-MPa-water-pressure source. The measurement of the armature deflection angle is not sufficient to estimate x because it does not include the bending of the flapper. On the other hand the variables T_A , F and x are related by Eqs. (1) and (2). Elimination of ϕ from Eqs. (1) and (2) gives

$$T_A = \frac{K_T x}{l_F - l_t/2} + \left(l_F - \frac{l_t}{2} + \frac{K_T/K_F + K_T/K_M}{l_F - l_t/2} \right) F. \quad (35)$$

Fig. 10 Torque vs control pressure (theoretical and experimental)

Since the relationships between the distance and the nozzle pressures were determined theoretically⁽³⁾, the influence of F appears in the relationships between T_A and pressures p_{nl} and p_{nr} .

In the setup shown in Fig. 9, the moment added to the armature is measured using the load cell. The nozzle pressures are measured simultaneously. Prior to the measurement, the positions of the two nozzles are adjusted such that both nozzle pressures coincide. Measured results of T_A/p_{nl} and T_A/p_{nr} are shown in Fig. 10. The black and white marks in Fig. 10 indicate the values measured during moment increase and decrease, respectively. They clearly show that the hysteresis is small. The bold curves in Fig. 10 are theoretical results. The dotted curves in Fig. 10 are the results calculated neglecting the nozzle jet force. Comparing the bold and dotted curves, we can see influence of the nozzle jet force. In the illustrated example, the change of the actual nozzle pressure is about one-half of the fictitious pressure assuming zero nozzle jet force.

5. Conclusions

The nozzle jet force acting on the flapper and its influence are studied for a water hydraulic servovalve that has a set of hydrostatic bearings supporting the spool. The flow from the hydrostatic bearing is used as the nozzle flow in the valve. The results obtained are summarized as follows.

(1) The jet leaves the flapper almost parallel to the flapper surface when the flapper width exceeds 1.3 times the nozzle diameter.

(2) A nonlinear numerical calculation of the force acting on the flapper shows that the force is well approximated by a linear combination of spool

displacement and velocity when design parameters are properly selected.

(3) The coefficients in the above linear combination can be estimated by the method of small perturbation.

(4) The nozzle jet force reduces the rate of change of the spool driving pressure; it opposes the control action of the armature torque.

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Appendix

Equations (4) and (5) can be derived from the equations given by von Mises⁽⁴⁾, since the jet model for the flapper-nozzle system is a special case of his general model for jets from a container. In our case, however, a definite shape of the flapper-nozzle system enables us to directly derive Eqs. (4) and (5).

Let f_p be a complex potential, and i be the imaginary unit. The two-dimensional potential flow is described as

$$\frac{df_p}{dz} = w = u - iv \quad (\text{A } 1)$$

The physical plane and the velocities are illustrated in Fig. 11(a). The figure shows the right half of the flapper, in which the points C, B and O correspond to the nozzle edge, the flapper end and the origin, respectively. The points A and D are the images of the infinite point. The fluid velocity at A and D are v_0 and v_{inf} , respectively. Denoting w/v_{inf} by q , we can draw the hodograph plane of Fig. 11(b). In this figure, a is given by

$$a = v_0/v_{inf} = h/b \quad (\text{A } 2)$$

The points A and D now become a source of strength m and a sink of strength $-m$, respectively.

Through the conversion $\zeta = q^2$, the quarter-circle drawn in the q -plane is mapped onto the upper half-circle in the ζ plane. Adding the mirror image of the half-plane, we obtain the unit circle shown in Fig. 11(c). With this mapping, the strengths of the sources and sinks are doubled to give flow to the lower half-plane. This circle becomes a streamline when another source of strength $2m$ is placed at the inverse point A with respect to the circle. Then the expression of the complex potential on the ζ plane becomes

$$f_p = 2m[\log(\zeta + a^2) + \log(\zeta + 1/a^2) - \log(\zeta - a^2) - \log(\zeta - 1/a^2)], \quad (\text{A } 3)$$

where

$$a = \exp(i\delta), \quad m = bv_0/(2\pi).$$

Combining Eqs. (A 1) and (A 3) gives

Fig. 11

$$\begin{aligned} dz &= \frac{df_p}{w} = \frac{1}{w} \cdot \frac{df_p}{d\zeta} \cdot \frac{d\zeta}{dq} \cdot dq \\ &= \frac{2ab}{\pi} \left[\frac{1}{q^2 + a^2} + \frac{1}{q^2 + 1/a^2} - \frac{1}{q^2 - a^2} - \frac{1}{q^2 - 1/a^2} \right] dq. \end{aligned} \quad (\text{A } 4)$$

Integrating this equation, and considering that $z=0$ at $q=0$,

$$\begin{aligned} z &= \frac{ab}{\pi} \left[\frac{1}{ia} \log\left(\frac{ia-q}{ia+q}\right) + \frac{a}{i} \log\left(\frac{i/a-q}{i/a+q}\right) \right. \\ &\quad \left. - \frac{1}{a} \log\left(\frac{a-q}{a+q}\right) - a \log\left(\frac{1/a-q}{1/a+q}\right) \right]. \end{aligned} \quad (\text{A } 5)$$

The point C on Fig. 11(a) gives $z=b+iH$, and the velocity at that point is shown in Fig. 11(b) as $q=i$. Substituting these values into Eq. (A 5), and rearranging, we obtain

$$\begin{aligned} \frac{H}{b} &= \frac{1+a^2}{\pi} \log\left(\frac{1+a}{1-a}\right) \\ &\quad + a \left[\cos \delta - \frac{2 \sin \delta}{\pi} \log\left(\frac{1+\tan(\delta/2)}{1-\tan(\delta/2)}\right) \right]. \end{aligned} \quad (\text{A } 6)$$

Equation (4) is obtained when we substitute $\varepsilon = \tan(\delta/2)$, $\sin \delta = 2/(1+\varepsilon^2)$, etc. Similarly, since $z=B$ and $q=1$ at the point B, we substitute these values into Eq. (A 5), which gives

$$\begin{aligned} \frac{B}{b} &= \frac{2}{\pi} \left(\tan^{-1} \frac{1}{a} + a^2 \tan^{-1} a \right) - a \left[\sin \delta \right. \\ &\quad \left. + \frac{2}{\pi} \cos \delta \cdot \log\left(\tan \frac{\delta}{2}\right) \right]. \end{aligned} \quad (\text{A } 7)$$

Applying the following transformation of variables to Eq. (A 7),

$$\varepsilon = \tan(\delta/2), \quad \tan^{-1}(1/a) = \pi/2 - \tan^{-1} a,$$

we obtain Eq. (5).

Note that the algebraic calculation to transform $\log[(a-1)/(a+1)]$ to the standard form of a complex number is greatly simplified when the geometric

relation illustrated in Fig. 11(d) is considered. In this figure we see that

$$\overline{PQ} = \overline{OP} = \overline{PR} = 1.$$

Hence

$$\angle ROQ = \pi/2 \text{ and } \angle POQ = \delta/2$$

$$\therefore \log[(\alpha-1)/(\alpha+1)] = \tan(\delta/2) + i\pi/2$$

The same procedure can be adopted for other cases.

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