High-Order Finite-Difference Method for Incompressible Flows Using Collocated Grid System*

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To apply the direct numerical simulation (DNS) and the large-eddy simulation (LES) of turbulence to flow fields of complicated geometry, a higher-order finite-difference method (FDM) has been developed for the body-fitted coordinate system. The consistency and the conservation property of FDMs are discussed for the collocated grid. As numerical examples, DNS results of decaying isotropic turbulence and DNS/LES results for plane channel flow are shown and the influence of variable arrangement is examined. The results by the consistent 'interpolation' method for gradient form on the collocated grid agree well with those by other proper FDMs and the spectral method.

Key Words: Finite Difference Method, Collocated Grid System, Turbulent Flow, Direct Numerical Simulation, Large-Eddy Simulation, Incompressible Flow, Numerical Analysis, Computational Fluid Dynamics

1. Introduction

The purpose of this study is to apply a finite-difference method (FDM) of high-order accuracy to the Direct Numerical Simulation (DNS) or the Large-Eddy Simulation (LES) of turbulent flows of engineering interest. The spectral method⁽¹⁾ has been used as a standard tool for the DNS of turbulent flows, although its accuracy and efficiency are kept in very simple flow fields. The finite-element method with unstructured grid system, on the contrary, has been developed intensively for the simulation of engineering application, but the accuracy of it is not enough for the basic research of turbulent phenomena by DNS. The FDM of high-order accuracy on the boundary-fitted coordinate system is therefore expected to be most suitable to expand the applicability of DNS.

It seems that the FDM has not been relied as a

numerical tool for the DNS of turbulent flows. We offer here some reasons. Many simulations used an upstream finite-difference which contains numerical diffusion. In usual upstream methods, the numerical diffusion was added to an inappropriate form of central FDM as pointed out by one of the authors⁽²⁾. The DNS using the 5th-order upstream FDM by Rai and Moin⁽³⁾ provides an example. At the same time, the reliability of the FDM for DNS has been improved by the proper form of central FDM. One of the authors applied a high-order FDM^{(4),(5)} to the DNS of turbulent flows and obtained reasonable agreement with that by the spectral method. The conservation properties of the high-order scheme was recently modified by Morinishi⁽⁶⁾.

The application of higher-order FDM to DNS of turbulent flows has been limited in the orthogonal coordinate systems, while the merit of FDM in comparison with the spectral method is its wider applicability. In this study, we develop a consistent FDM, namely the interpolation method for convection term^{(4),(7)}, to the generalized curvilinear coordinate system.

There are some possibilities in choosing the basic

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variables in the curvilinear coordinate system⁽⁸⁾. Contravariant components of velocity are suitable for the calculation using a staggered arrangement of variables. But such a formulation contains complex terms due to the coordinate curvature. On the other hand, Cartesian components of velocity are usually used for collocated arrangement of variables^{(9)–(11)}. The so-called strong conservative form is available for the collocated grid proposed by Rhie and Chow⁽⁹⁾. We use here the collocated grid for such a reason.

Some DNS and LES results of turbulent flows using collocated grids have been reported. Satake et al.⁽¹²⁾ conducted a DNS of turbulent flow in a plane channel by the central difference method of the 4th-order accuracy with enough spatial resolution. They reported good agreement between their results by staggered and collocated arrangements. Morinishi⁽⁶⁾, on the other hand, pointed out that the energy conservation properties can not be obtained by the collocated grid due to the pressure gradient term. An LES result by Ooka et al.⁽¹³⁾ for a plane channel flow using the 2nd-order FDM and a relatively coarse grid seems to manifest this problem. We can not decide the reason of these opposite results about the variable arrangement because there are some varieties in the allocations of variables and metrics for coordinate transform as well as the interpolation scheme^{(8)–(11)}.

In the present paper, a convective-interpolation method is introduced to improve the accuracy of FDM on the generalized curvilinear coordinate system. First, the consistency, the compatibility and the conservation properties of our method are discussed. To examine the reliability of our method, DNS and LES results for turbulent channel flows using the rectangular coordinate system are compared to reliable data.

2. Numerical Method

2.1 Basic equations

A numerical simulation of incompressible fluid flow using a collocated grid system on the curvilinear coordinate system as shown in Fig. 1 is considered. The basic variables for velocity field are the components u_i ($u_1=u$, $u_2=v$, $u_3=w$) in the Cartesian coordinate system x_i ($x_1=x$, $x_2=y$, $x_3=z$). The contravariant components U^j ($U^1=U$, $U^2=V$, $U^3=W$) in the curvilinear coordinate system ξ^j ($\xi^1=\xi$, $\xi^2=\eta$, $\xi^3=\zeta$) are used in evaluating convective flux at the cell interface. These components, u_i and U^j , are transformed to each other by

$$U^j = \frac{\partial \xi^j}{\partial x_i} u_i, \quad u_i = \frac{\partial x_i}{\partial \xi^j} U^j \quad (1)$$

where the summation rule is applied if we do not comment about exception. The Jacobian of the coordinate transform is represented as

Fig. 1 Collocated variable arrangement

$$J = \frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)} \quad (2)$$

The mass continuity equation, $\nabla \cdot \mathbf{u} = 0$, is represented in ξ^j coordinate as

$$\frac{1}{J} \frac{\partial J U^j}{\partial \xi^j} = 0 \quad (3)$$

The Navier-Stokes equation of motion is

$$\frac{\partial u_i}{\partial t} + U^k \frac{\partial u_i}{\partial \xi^k} + \frac{\partial \xi^k}{\partial x_i} \frac{\partial P}{\partial \xi^k} - \nu \frac{\partial \xi^k}{\partial x_m} \frac{\partial}{\partial \xi^k} \left(\frac{\partial \xi^l}{\partial x_m} \frac{\partial u_i}{\partial \xi^l} \right) = 0 \quad (4)$$

in the gradient (or non-conservative) form and

$$\frac{\partial u_i}{\partial t} + \frac{1}{J} \frac{\partial}{\partial \xi^k} \left\{ J U^k u_i + J \frac{\partial \xi^k}{\partial x_i} P \right. \\ \left. - \nu J \frac{\partial \xi^k}{\partial x_m} \left(\frac{\partial \xi^l}{\partial x_m} \frac{\partial u_i}{\partial \xi^l} + \frac{\partial \xi^l}{\partial x_i} \frac{\partial u_m}{\partial \xi^l} \right) \right\} = 0 \quad (5)$$

in the divergence (or conservative) form, where $P = p/\rho$. The formula (5) is sometimes referred to as the strong conservation form. In the above equations, p denotes the static pressure, ρ the fluid density and ν the kinematic viscosity of the fluid. When the equation of motion is written only by the contravariant components, it is represented as

$$\frac{\partial U^j}{\partial t} + U^k U^j|_k + \frac{\partial \xi^j}{\partial x_i} \frac{\partial \xi^k}{\partial x_i} \frac{\partial P}{\partial \xi^k} - \nu \frac{1}{J} \frac{\partial}{\partial \xi^l} \left(J \frac{\partial \xi^k}{\partial x_m} \frac{\partial \xi^l}{\partial x_m} \frac{\partial U^j}{\partial \xi^k} \right) = 0, \quad (6)$$

where $U^j|_k$ means the covariant difference. We do not use this form for momentum conservation, but the pressure gradient term is used to couple the pressure and velocity fields through Eq. (3).

2.2 Finite-difference method for convection term

As the two formulations (4) and (5) for the equation of motion are compatible with Eq. (3), the momentum should be conserved even by the 'non-conservative form'. In addition, the energy conservation law which obeys the mass and momentum conservation laws is not indispensable in the solution algorithm for incompressible fluid flows, but it should be satisfied implicitly. These properties are required

even for the discretized form of basic equations. The reasons are:

1. The compatibility, which means here that the numerical solution is independent of the form used as a basic equation, is essential for reliability.

2. The conservation of energy, which means the numerical conservation of quadratic quantities⁽¹⁴⁾, is a condition for numerical stability.

3. The consistency, which means that the relationships $\partial(ab)/\partial x = a\partial b/\partial x + b\partial a/\partial x$ and $\partial^2 a/\partial x^2 = \partial(\partial a/\partial x)/\partial x$ are numerically satisfied, is required when the budget of high-order statistics such as Reynolds stress components is evaluated without any numerical error.

One of the authors proposed an interpolation scheme for the convection term and then extended to higher-order version on the staggered grid in the Cartesian coordinate system^{(4),(7)}. The scheme has been applied to a decaying isotropic turbulence⁽⁴⁾ and a turbulent flow in a plane channel⁽⁵⁾. The numerical results agreed satisfactorily with those by the spectral method and the budget of energy as well as Reynolds stress tensor were accurately evaluated. For the 2nd-order accuracy form, our method proves the compatibility among divergence, gradient and mixed forms⁽¹⁴⁾ of convective terms and the energy conservation in the condition that the mass continuity is numerically satisfied. Suzuki and Kawamura⁽¹⁵⁾ used the same scheme to derive the Reynolds stress equation and they named it as 'Consistent scheme'. Although our original version breaks the local conservation of energy in higher than the 4th-order scheme, the numerical result was improved by increasing the order of accuracy. This defect is modified recently by Morinishi⁽⁶⁾. He pointed out the inadequacy of the 4th-order scheme based on the divergence form⁽¹⁶⁾ and the gradient form⁽⁴⁾ for the case of poor resolution of grid showing an improvement by his method.

In the present paper, we extend the interpolation method of convective term for the collocated grid on the generalized curvilinear coordinate system. The discretized forms of convective terms, $U^k \partial u_i / \partial \xi^k$ in the gradient form (4) and $(1/J) \partial (JU^k u_i) / \partial \xi^k$ in the divergence form (5), should be compatible to each other. We will begin by considering the 2nd-order discretization. The central finite-difference and interpolation are written as

$$[\delta_\xi f]_i = -f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}, \quad (7)$$

$$[\bar{f}^\xi]_i = \frac{1}{2}(f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}), \quad (8)$$

$$[\delta_\xi^2 f]_i = \frac{1}{2}(-f_{i-1} + f_{i+1}) \quad (9)$$

in the ξ direction, for example. Note that the form (9) is not compact in comparison with Eq. (7). The

width of the computational grid is fixed as $\Delta_\xi = 1$ as usually used in the computational coordinate. The subscripts J and K for the other directions η and ζ are omitted for simplicity. At the cell center of the collocated grid, the gradient form may be discretized intuitively as

$$\overline{U}^\xi \delta_\xi u + \overline{V}^\eta \delta_\eta u + \overline{W}^\zeta \delta_\zeta u \quad (10)$$

or

$$\bar{U}^\xi \delta_\xi u + \bar{V}^\eta \delta_\eta u + \bar{W}^\zeta \delta_\zeta u. \quad (11)$$

because u, v, w are allocated at the cell center while U, V, W are at the cell interface. On the other hand, the divergence form may be

$$\frac{1}{J} [\delta_\xi (JU \bar{u}^\xi) + \delta_\eta (JV \bar{u}^\eta) + \delta_\zeta (JW \bar{u}^\zeta)], \quad (12)$$

or

$$\frac{1}{J} [\delta_\xi (\bar{J} U^\xi u) + \delta_\eta (\bar{J} V^\eta u) + \delta_\zeta (\bar{J} W^\zeta u)] \quad (13)$$

corresponding to Eqs. (10), (11), respectively. The compact forms such as Eqs. (10), (12) are generally recommended rather than Eqs. (11), (13). However, all of finite-difference forms (10)-(13) provide different numerical results. Then, we exchange the variable located at the cell interface from U^k to $(JU)^k$ and rewrite the gradient form as

$$\frac{1}{J} [\bar{J} U^\xi \delta_\xi u + \bar{J} V^\eta \delta_\eta u + \bar{J} W^\zeta \delta_\zeta u]. \quad (14)$$

The difference between Eq. (12) and Eq. (14) is $u[\delta_\xi (JU) + \delta_\eta (JV) + \delta_\zeta (JW)]/J$. If the discretized equation of mass conservation

$$\frac{1}{J} [\delta_\xi (JU) + \delta_\eta (JV) + \delta_\zeta (JW)] = 0 \quad (15)$$

is satisfied, in other words, if the Poisson equation of pressure is accurately solved in consistent with Eq. (15), Eq. (12) and Eq. (14) are identical. There is no other couple in Eqs. (10)-(14).

Next, we discuss about the energy conservation property. Multiplying u to Eq. (12) and Eq. (14) derives respectively

$$\begin{aligned} & \frac{1}{J} [\delta_\xi (JU \bar{u}^{2\xi}) + \delta_\eta (JV \bar{u}^{2\eta}) + \delta_\zeta (JW \bar{u}^{2\zeta})] \\ & \mp \frac{u^2}{J} [\delta_\xi (JU) + \delta_\eta (JV) + \delta_\zeta (JW)] \end{aligned} \quad (16)$$

where $\bar{u}^{2\xi}$ is a quadratic value⁽¹⁴⁾ defined by

$$\bar{u}^{2\xi}_{I+\frac{1}{2},J,K} = u_{I,J,K} \times u_{I+1,J,K} \quad (17)$$

(same in η, ζ directions). The second term in Eq. (16) is again negligible corresponding to the numerical accuracy of mass continuity equation (15). Thus the convective term in energy budget is written as $(1/J) \delta_\xi (JU \bar{u}^{2\xi})$, where summation for i is not indispensable. Equations (12) and (14) have energy conservation property in the sense that the energy in the computational domain is determined only by fluxes through the domain. In other words, there is not any unphysical source or sink. We may note, in passing,

that the average of Eqs. (12) and (14) conserves energy completely because the error due to the second terms in Eq. (16) are eliminated.

It is concluded that, from the above discussion, that we must use Eq. (14) for the gradient form, Eq. (12) for the divergence form or an average of them for the mixed form to obtain compatibility and energy conservation properties. In addition, the consistent scheme⁽¹⁵⁾ for the Reynolds stress budget should be derived using these proper formulations in the curvilinear coordinate system.

2.3 Higher order finite difference

The discussion in the former section can be extended to the 4th-order central scheme if Eqs. (7)–(9) are replaced by

$$[\delta_\varepsilon f]_i = \frac{1}{24}(f_{i-\frac{3}{2}} - 27f_{i-\frac{1}{2}} + 27f_{i+\frac{1}{2}} - f_{i+\frac{3}{2}}), \quad (18)$$

$$[\bar{f}^\varepsilon]_i = \frac{1}{16}(-f_{i-\frac{3}{2}} + 9f_{i-\frac{1}{2}} + 9f_{i+\frac{1}{2}} - f_{i+\frac{3}{2}}), \quad (19)$$

$$[\delta_\varepsilon^2 f]_i = \frac{1}{12}(f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2}). \quad (20)$$

The same is said of the central FDM of higher than the 6th-order accuracy. However, the errors in compatibility as well as energy conservation are not proportional to the mass conservation error for the higher-order forms of Eqs. (12) and (14). The defect is supplemented by high-order accuracy to some extent, but one must introduce Morinishi's method⁽⁶⁾ to attain a complete compatibility and energy conservation. The drawback of his method, on the other hand, is the increase of computation and required memory for high-order finite-difference because it uses M kinds of the 2-point finite-differences and 2-point interpolations at each point for the $2M$ -th order accuracy. Considering these matters, we use the interpolation method for convective term of our own.

The pressure gradient term as well as the convective term affect the energy conservation properties. The pressure gradient term in Eq. (4) is used for the equation of motion, while that in Eq. (6) is used to be coupled with mass conservation equation (3). The discretized form of them are not compatible to each other. Morinishi⁽⁶⁾ pointed out that the energy conservation scheme is impossible due to this inconsistency in pressure gradient term. We will discuss this problem qualitatively on the basis of our numerical result.

2.4 Time-marching method

We use a fractional-step method based on the 2nd-order Adams-Bashforth method for the convective and viscous terms in the momentum equation. The pressure gradient term and the mass conservation equation are treated implicitly. The scheme is as follows:

1. The Cartesian components of fractional-step velocity are calculated at the cell center:

$$u_i^* = u_i^{(n)} + \frac{\Delta t}{2}[3F_i^{(n)} - F_i^{(n-1)}]. \quad (21)$$

F_i contains convective and viscous terms as

$$F_i = -\frac{1}{J}(\overline{JU})^k \delta_k u_i^* + \frac{\nu}{J} \delta_k (\overline{G^{kl}} \delta_l u_i) \quad (22)$$

where $\delta_k f$ and $\bar{f}^k$ denote $\delta_{\varepsilon^k} f$ and $\bar{f}^{\varepsilon^k}$, respectively. The summation rule is not applied for superscript in the interpolation $\bar{f}^k$. The tensor G^{kl} is defined as

$$G^{kl} = J \frac{\partial \xi^k}{\partial x_m} \frac{\partial \xi^l}{\partial x_m}.$$

The discrete time step is denoted by $t_n (= n\Delta t)$, where n is a time-step count and Δt is a time increment.

2. Transforming the result of Eq. (21) to the contravariant components in the curvilinear coordinate and then interpolated at the cell interface:

$$(JU)^{j*} = J \frac{\partial \xi^j}{\partial x_i} u_i^*. \quad (23)$$

3. Poisson equation for a potential ϕ

$$\delta_j (\overline{G^{jk}} \delta_k \phi) = \delta_j (JU)^{j*} \quad (24)$$

is solved.

4. Adding the contribution of ϕ , the flow field at the new time step is obtained:

$$(JU)^{j(n+1)} = (JU)^{j*} - \Delta t \overline{G^{jk}} \delta_k \phi, \quad (25)$$

$$u_i^{(n+1)} = u_i^* - \Delta t \frac{\partial \xi^k}{\partial x_i} \delta_k \phi. \quad (26)$$

The parameters such as Jacobian J and the components of matrix $\partial \xi^k / \partial x_i$ are all allocated at the cell center.

Some alternatives are possible for the finite-difference of the gradient of ϕ in Eq. (26) such as $(\partial \xi^k / \partial x_i) \delta_k \bar{\phi}^k$ or $(\partial \xi^k / \partial x_i) \delta_k \phi^k$. They are compatible in the 2nd-order form but different from each other in the higher order forms. However, it must be noted that the inverse transforms from the contravariant components at a new time step obtained by Eq. (25)

$$u_i^{(n+1)} = \frac{\partial x_i}{\partial \xi^j} \frac{1}{J} (JU)^{j(n+1)*} \quad (27)$$

should not be used in place of Eq. (26), especially for the DNS/LES of turbulent flows. The reason is because a filter for smoothing is introduced by the twice interpolations on the convective and viscous terms, $\bar{F}_i$.

When a very fine grid in the limited region such as the vicinity of solid walls is needed, the implicit scheme^{(8),(18)} for the viscous term is recommended to avoid too small time increment. But the grid clustering to the wall can usually be milder in FDM than that in spectral method which requires special mapping such as the Gauss-Lobatto distribution for the Chebyshev spectral method⁽¹⁾.

3. Numerical Results

The feasibility of FDM for DNS/LES should be

evaluated step-by-step such as: the difference between staggered and collocated arrangements; a non-equidistant finite-difference in the physical space for $\partial f/\partial x$ or a transformation from the equidistant finite-difference in a computational space, $(\partial \xi/\partial x)(\partial f/\partial \xi)$; the treatment of non-diagnostic component in the non-orthogonal coordinate system; the numerical method for the boundary conditions, and so on. We will begin with a comparison between the staggered and collocated arrangements in the Cartesian coordinate system. The central FDM is used to avoid the artificial diffusion in our simulation.

3.1 DNS of decaying isotropic turbulence

In order to compare the staggered and collocated arrangements in cubic grid, a DNS of decaying isotropic turbulence was carried out. The computational condition is same as given in the JSME workshop held in Computational Dynamics Conference, Sendai (1993). The computational domain 2π was divided into 32 cells in each direction. The non-dimensional eddy viscosity was $\nu=1/41.1$. The given initial condition satisfied the mass conservation law in Fourier space and the initial Reynolds number was $R(=k^2/\varepsilon\nu)=465$. A result by the Fourier spectral method⁽¹⁹⁾ was distributed in the Workshop for comparison. As the spectral method utilized the Crank-Nicholson method for the viscous term⁽¹⁸⁾, the same time-marching scheme was applied in our computation. The spatial derivatives are approximated by a central FDM of the 4th-order accuracy.

Using the 4th-order scheme on a collocated grid, three cases for the convective term were tested: Case A, the divergence form corresponding to Eq. (12); Case B, the gradient form corresponding to Eq. (14), same as Eq. (10) in this particular case; and Case C, the alternative gradient form corresponding to Eq. (11). Numerical results are shown in Fig. 2 comparing with the result by the Fourier spectral method⁽¹⁹⁾. Figure 2(a) compares the time evolution of turbulence kinetic energy k . Figure 2(b) shows each term in the energy budget equation $\partial k/\partial t = -\varepsilon$. Figure 2(c) shows each term in the dissipation rate equation, $\partial \varepsilon/\partial t = P_\varepsilon - D_\varepsilon$, where P_ε and D_ε represent respectively the production and destruction of the energy dissipation rate

$$P_\varepsilon = -2\nu \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}, \quad (28)$$

$$D_\varepsilon = 2\nu^2 \frac{\partial^2 u'_i}{\partial x_j \partial x_k} \frac{\partial^2 u'_i}{\partial x_j \partial x_k}.$$

The numerical error in budget due to the discretization is denoted by Residual in Figs. 2(b), (c).

The difference between the results by Case A and Case B is not observed in Fig. 2. In addition, they agree well with the Fourier spectral method⁽¹⁹⁾. The

residuals in energy and dissipation rate budgets are very small in comparison with other main terms. From these results, the compatibility and conservation properties are proved by our computational result.

The result by Case C differs from others. There observed considerable residual in the energy and dissipation rate budgets due to the inconsistency in finite-difference. Case C shows closer agreement with the spectral method for some quantities in Fig. 2 than Cases A and B, but this seems to be an occasion due to the balance between numerical error and increase

(a) Kinetic energy

(b) Energy budget

(c) Dissipation rate budget

Fig. 2 Decay of 3-dimensional isotropic turbulence

in dissipation rate as shown in Fig. 2(b).

The results by the collocated grid agree with those by the staggered grid⁽⁴⁾ for all quantities shown in Fig. 2. Thus it can be considered that the consistent schemes, Cases A and B are not influenced very much by the variable arrangement. On the other hand, there observed some discrepancy between the results by different arrangement for the inconsistent scheme, Case C.

Contravariant components of velocity allocated at the cell interface satisfy the discretized mass conservation equation (15) in the collocated grid. Cartesian components, on the contrary, do not necessarily satisfy it. Morinishi⁽⁶⁾ pointed out this numerical error breaks the energy conservation property and Ooka et al.⁽¹³⁾ reported this problem showing their results of LES with relatively coarse grid. Figure 3 evaluates the error in mass continuity in our simulation, where $\|a\|$ denotes the root-mean-square (rms) value of a . Error in $\text{div } \mathbf{u}$ is plotted as a function of the relative residual in the Poisson equation, which was changed by limiting the iteration number of the successive over-relaxation (SOR) method. The vertical axis is the error normalized by $\sqrt{k}/\Delta_x$ at the instantaneous flow field, $t=0.1$. The error for the contravariant components, $\|\delta_j U^j\|$, decreases with the convergence of Poisson equation because these equations are consistent to each other. The error for the Cartesian component, $\|\delta'_i u_i\|$, which is not consistent with Poisson equation follow the decrease of $\|\delta_j U^j\|$ to some extent but it can not be smaller than 2×10^{-4} . This is a defect of collocated grid system already pointed out, but the problem is how this error affects the flow field. Actually, the discrepancy in flow field data (u, v, w, p) was less than 10^{-5} between two cases, $\|\nabla^2 P - Q\|/\|Q\| \approx 10^{-1}$ and greater accuracy 10^{-3} . The relative error $\|\nabla^2 P - Q\|/\|Q\| \approx 10^{-1}$ is the limitation that the decrease in $\|\delta'_i u_i\|$ follows the convergence of Poisson

Fig. 3 Numerical error for mass continuity

equation. Thus, the error in $\|\delta'_i u_i\|$ is not fatal to the reliability of flow field if enough convergence in Poisson equation is attained.

From the above results for DNS of isotropic turbulence with equidistant grid of relatively poor resolution, we suppose that an enough accuracy is obtained by the FDM on the collocated grid system in comparison with that on the staggered grid system as well as the spectral method. The discrepancy between collocated and staggered grids are significant only in the case that the inconsistent finite-difference is used. The influences on flow field and energy conservation due to the pressure gradient term are not significant even in the collocated grid.

3.2 DNS of turbulent flow in a plane channel

Fully-developed turbulent flow in a two-dimensional channel driven by a constant pressure gradient is considered as an example of wall turbulence. The mainstream direction, the direction normal to walls and the spanwise direction are denoted by x, y and z , respectively. The velocity component in each direction is u, v and w . Quantities are non-dimensionalized by the channel width H , the mean friction velocity at the walls $u_{\tau w}$, and the fluid density ρ . The non-dimensional mean pressure gradient is $\partial \bar{P} / \partial x = -2$ and, therefore, the Reynolds number $R_\tau = H u_{\tau w} / \nu$ is a single parameter to characterize the flow field. The computational domain is limited in x and z directions, in which the periodic boundary condition was applied for the velocity and the pressure fluctuation. The width of computational grid is uniform in those homogeneous directions. In y direction, on the other hand, the computational grid is clustered to the walls by tanh function to apply the no-slip condition for velocity. Table 1 gives the computational setup. The condition in DNS using the spectral method by Kuroda and Kasagi⁽²⁰⁾ is also shown in Table 1 for comparison. In our computation with collocated grid system, the finite-difference approximation $\delta_h f$ and interpolation $\tilde{f}^h$ have the 4th-order accuracy in space. The time increment is fixed as $\Delta_t = 2 \times 10^{-4} H / u_{\tau w}$.

As the velocity components u_i are not allocated at the wall for the collocated grid system, we used additional points at the wall to apply the boundary condition $u_i = 0$. For the potential ϕ in Eq. (24), on the other hand, an additional point is given inside the wall to apply the Neumann boundary condition, $\delta_y \phi = 0$ for $v^* = 0$ at the wall. However, no additional grids are not given outside the flow field even for the wide stencil of higher-order FDM.

The initial flow field is given by interpolating the data obtained by the staggered grid⁽⁵⁾. Observing the time evolution of the instantaneous average in x - z plane, the time average was obtained after the

Table 1 Numerical setups for DNS and LES of plane channel turbulence

	DNS(present)	DNS(Kuroda)	LES(present)	DNS(Kim)
Method	4th FDM	Spectral	2nd/4th FDM	Spectral
$R_\tau (= H u_{\tau w} / \nu)$	300	300	360	360
Number of Grid Points				
N_x	64	192	32	192
N_y	64	144	64	129
N_z	64	192	32	160
Domain($\times H$)				
H_x	3.84	7.85	3.2	6.28
H_y	1.0	1.0	1.0	1.0
H_z	1.92	3.14	1.6	3.14
Grid resolution($\times \nu / u_\tau$)				
h_x^+	18.0	12.3	36.0	11.8
h_y^+	0.93~9.03	0.036~3.28	1.12~10.8	0.05~4.4
h_z^+	9.0	4.91	18.0	7.1

(a) Mean velocity

(a) Energy budget

(b) Turbulence intensity

(b) Pressure-strain correlation

Fig. 4 Time averaged flow field in a plane channel by DNS at $R_\tau=300$ Fig. 5 Reynolds stress budget in a plane channel by DNS at $R_\tau=300$

statistically steady state was reached. The Reynolds numbers based on the bulk velocity u_m and centerline velocity u_c , represented respectively as $R_m = H u_m / \nu$ and $R_c = H u_c / \nu$, were $R_m = 4582$ and $R_c = 5379$ by the collocated grid and $R_m = 4544$ and $R_c = 5340$ by the staggered grid. Their differences from the result by the spectral method⁽²⁰⁾, $R_m = 4560$ and $R_c = 5356$, were less than 0.5%.

Figures 4 and 5 compare turbulence statistics. The results by the FDM with the collocated grid as

well as those with the staggered grid agree well with those by the spectral method. The turbulence intensity by our FDM is smaller than that by the spectral method and it may be due to the coarser resolution of FDM. The energy budget is satisfactorily obtained by our scheme as shown in Fig. 5(a). The residual in the budget is within 1% of the energy dissipation rate except for the first grid point from the wall where the finite-difference stencil inside the wall is omitted. The maximum error in the budget at that point is 1.2%,

but we consider it might be acceptable for the DNS database.

Figure 5(b) shows orthogonal components of pressure-strain correlation tensor in the Reynolds stress budget, $T_{ij} = \overline{p'(\partial u_i'/\partial x_j + \partial u_j'/\partial x_i)}$. The discrepancy among the methods is very small considering the magnified scale of vertical axis in comparison with Fig. 5(a). The negative value of T_{22} in the vicinity of the wall means the transfer of energy from v component to other components due to the resistance of the wall, which is often referred to as the splatting. Concerning to the energy transfer from v to w component, the collocated grid reproduced closer agreement with the spectral method than the staggered grid, while there is no wide difference. The trace of three orthogonal component T_{kk} should be 0 because of the mass conservation constraint. The error in this summation is always zero for the staggered grid because of $\delta_i u_i = 0$ everywhere. As the divergence of finite-difference form, $\delta_i u_i$, is not exactly zero for the collocated grid, on the other hand, we plotted T_{kk} only for the collocated grid in Fig. 5(b). The maximum of T_{kk} is 0.0004, which is the order of 10^{-2} to each component of T_{ij} . Thus its influence to the energy budget is

negligibly small because the magnitude of T_{ij} is much smaller than principal terms in the energy budget.

It is concluded that the difference among the grid arrangements and the defect of the collocated grid do not manifest themselves even for the wall-bounded turbulence in which the non-equidistant grid is used, if the grid resolution, the number of grid as well as the degree of accuracy, is enough for the phenomenon to be analyzed.

3.3 LES of turbulent flow in a plane channel

The influence of variable arrangement and the degree of accuracy are evaluated for LES as an example of engineering application. The Reynolds number $Re_\tau = 360$ in this case is same as used in the DNS by the spectral method conducted by Kim, Moin and Moser⁽²¹⁾. The Smagorinsky model was used for subgrid scale model, which was damped by the van Driest function to apply the no-slip boundary condition. The filter width is $\Delta = (\Delta_x \Delta_y \Delta_z)^{1/3}$ for both staggered and collocated grids. For discretization of the diffusion term, the divergence form as shown in Eq. (5) was used where ν is replaced by the total of molecular viscosity and SGS eddy viscosity, $\nu + \nu_{SGS}$. In addition to the 4th-order scheme, the 2nd-order one

(a) Mean velocity

(b) Turbulence intensity

Fig. 6 Time averaged flow field in a plane channel by LES with 2nd-order finite-difference method at $Re_\tau = 360$

(a) Mean velocity

(b) Turbulence intensity

Fig. 7 Time averaged flow field in a plane channel by LES with 4th-order finite-difference method at $Re_\tau = 360$

was also tested because it is commonly used for practical application of LES.

The right half of Table 1 shows computational condition comparing to those used in spectral DNS⁽²¹⁾. The time increment is fixed as $\Delta t = 2.5 \times 10^{-4} H/u_{\tau w}$ for our LES.

Figures 6 and 7 compare main flow profiles and the intensities of grid-scale portion of velocity fluctuations to the DNS results by Kim, Moin and Moser⁽²¹⁾. For both the 2nd-order accuracy shown in Fig. 6 and the 4th-order accuracy shown in Fig. 7, the staggered grid gives better agreement with the spectral method than the collocated grid. The improvement by the higher-order accuracy is more significant in the collocated grid than in the staggered grid. In the result by the 4th-order accuracy, the difference caused by the grid arrangement is much smaller than in the 2nd-order finite-difference.

In our LES results with relatively sparse grid, the collocated grid received greater influence by grid resolution, which means the grid width as well as the order of accuracy, in comparison with the staggered grid. The filter width to divide the flow field into grid scale and subgrid scale might depend on the variable arrangement of the grid system⁽²²⁾. As we had not taken this into account, the comparison in this section can not conclude that the collocated grid is always worse than the staggered grid. With all some quantitative difference, what is important may be that the result by the two grid systems are not qualitatively different to each other.

4. Conclusion

To apply the DNS or LES of turbulent flows to wider range of flow field, properties of central FDM of the high-order accuracy have been discussed. First, we proposed a proper FDM on the collocated grid suited for the generalized curvilinear coordinate system. Next, the compatibility, the consistency and the conservation properties were surveyed. Then the DNS of isotropic turbulence and the DNS/LES of turbulent flows in a plane channel were carried out. In conclusion,

1. The interpolation method for convective form which is compatible with the divergence form can be constructed on the collocated grid as well as the staggered grid.

2. By a proper discretization with high-order accuracy, a DNS by FDM on the collocated grid can be comparable to that by the spectral method.

3. In the LES with relatively poor resolution, the collocated and staggered grids give qualitatively same result, but the staggered grid allows closer agreement with the spectral method than the collocated grid for

the same resolution and accuracy.

4. A greater improvement is expected by the grid refinement or higher-order method in the collocated grid than in the staggered grid.

We consider that a discussion here gives some basis about the reliability of central FDM as a scheme for DNS/LES for flow fields of complicated geometry.

Most of calculations shown in this paper adopted the 4th-order accuracy. But we do not conclude that it is the best selection for LES and DNS of turbulent flows. If the dynamics in the scale of grid width is important, one must improve the order of accuracy.

As a benchmark case is not established for the generalized coordinate system, especially for the non-orthogonal curvilinear grid, we limited the discussion here into the Cartesian coordinate system. At the same time, we have successively applied our 4th-order scheme to a turbulent flow in a wavy channel in which the channel width changes sinusoidally in the main-stream direction. We may leave the details to another paper⁽²³⁾.

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