Thermal Boundary Condition Effects on Forced Convection Heat Transfer*
((application of numerical solution of adjoint problem)

Kazunari MOMOSE**, Kiyoshi SASOH***
and Hideo KIMOTO**

A numerical approach based on the adjoint formulation of forced convection heat transfer is proposed, to evaluate the total or mean heat transfer rate under arbitrary thermal boundary conditions. Using the numerical solution of the adjoint problem under the uniform Dirichlet condition, which can be computed by a slightly modified CFD code, an influence function of the local surface temperature can be obtained. As a result, the total heat transfer for arbitrary surface temperature distributions can be calculated by the influence function. In a similar fashion, using the numerical solution of the adjoint problem under the Neumann condition, the mean surface temperature can also be expressed as an influence function of the local heat flux distributions. The influence functions for a circular cylinder and for an in-line square rod array are presented to illustrate the present method.

Key Words: Forced Convection, Heat Transfer, Thermal Boundary Condition, Integral Equation, Adjoint Problem, Numerical Analysis

1. Introduction

When both, the flow and thermal similarities are satisfied, the Nusselt number can be expressed as a function of the Prandtl and Reynolds numbers for forced convection heat transfer. Using this similarity relationship, many correlation equations have been proposed for various geometric configurations. However, most of the thermal boundary conditions adopted in the previous studies were limited to an isothermal or a uniform heat flux condition due to the requirement of thermal similarity, although the thermal boundary condition effects can generally be evaluated for special cases such as for a flat plate or for a blade by employing the boundary layer approximation(1)-(3). In practical applications, of course, not all the thermal boundary conditions take the isothermal or the uniform heat flux condition. In other words, the heat transfer characteristics evaluated under such uniform boundary conditions are no longer meaningful if the uniform condition is violated. Further, even under the uniform thermal boundary conditions, the heat transfer characteristic under the uniform heat flux condition is different from that under the isothermal condition(4)-(10).

With the recent progress in the numerical simulation technique, several numerical results have been reported under nonuniform thermal boundary conditions as well as under the uniform ones. Although the conventional numerical method enables us to predict the heat transfer characteristics even under the nonuniform thermal boundary conditions, each result gives only a particular solution under a specific boundary condition. Thus, for general evaluation of the thermal boundary condition effects, we need to not only compute the heat transfer rates under various thermal boundary conditions, but to also develop a general expression of the heat transfer under arbitrary thermal boundary conditions.

From this reason, we have proposed a Fredholm-type boundary integral expression of forced

** Faculty of Engineering Science, Osaka University, 1-3 Machikaneyama, Toyonaka, Osaka 560-8531, Japan. E-mail: momose@me.es.osaka-u.ac.jp
*** Graduate School of Osaka University
convection heat transfer\(^{27,19}\), in which the local heat transfer rate is expressed as a function of the surface temperature distribution. This expression yields the local heat transfer rate under arbitrary thermal boundary conditions and gives the detailed mechanism of forced convection heat transfer. However, a large number of numerical simulations of forced convection fields are required to construct Fredholm’s kernel.

In this paper, we focus attention on the total or mean heat transfer rate instead of the local one. To evaluate the total heat transfer rate under arbitrary thermal boundary conditions, we propose the use of numerical solution of an adjoint problem, which can be calculated by a slightly modified heat transfer code. Using the solution of the adjoint problem, we can construct an influence function of the thermal boundary condition on the total heat transfer rate; thus the total heat transfer rate can be obtained under arbitrary surface temperature distributions. The influence functions for a circular cylinder and for an in-line square rod array are presented to illustrate the present method. Moreover, as an application of the influence function, we present an optimal arrangement of in-line square rods with different temperatures. The present influence function can immediately provide the optimal arrangement.

2. Integral Equation for Thermal Field

Consider a forced convection heat transfer problem as shown in Fig. 1. When the fluid is incompressible and has thermally independent properties, the governing equations can be written as

\[
\begin{align*}
div \mathbf{u} &= 0, \\
(\mathbf{u} \cdot \text{grad}) \mathbf{u} &= -\frac{1}{\rho} \text{grad} \rho + \nu \Delta \mathbf{u} \\
(\mathbf{u} \cdot \text{grad}) T &= a \Delta T
\end{align*}
\]  

(1) (2)

where \( \mathbf{u} \) is the velocity vector, \( T \) is the temperature, \( \rho \) is the density, \( \nu \) is the kinematic viscosity, and \( a \) is the thermal diffusivity. As can be seen from these equations, the flow field is obviously governed by nonlinear equations, but the energy field is governed by a linear equation with space-dependent coefficient \( \mathbf{u} \), which can be solved by a certain numerical method. Further, by defining the temperature as the difference from the main flow temperature, such as

\[
\theta = T - T_m
\]  

(3)

the nonhomogeneous boundary condition for Eq.(2) becomes only the surface temperature of the object.

Here, using a linear differential operator \( A \), for notational convenience, we rewrite Eq.(2) as

\[
A \theta = \left( \frac{\mathbf{u}}{a} \cdot \text{grad} - \Delta \right) \theta = 0
\]  

(4)

Then the weak solution of Eq.(4) can be expressed as

\[
\int_\Omega (A \theta) \theta^* d\Omega = 0
\]  

where \( \theta^* \) is a test function defined in \( \Omega \). Applying the divergence theorem to Eq.(5), we obtain the following integral equation.

\[
\int_\Omega (A^* \theta^*) \theta d\Omega = \int_\Gamma (\frac{\partial \theta^*}{\partial n} \theta - \frac{\partial \theta}{\partial n} \theta^*) d\Gamma
\]  

(6)

where \( A^* \) is the adjoint operator for \( A \). Since the differential operator \( A \) is not self-adjoint, the adjoint operator is given as

\[
A^* = \frac{\mathbf{u}}{a} \cdot \text{grad} + \Delta
\]  

(7)

With \( x \) denoting the thermal conductivity of the fluid, we define fluxes \( q \) and \( q^* \) as

\[
q = -x \frac{\partial \theta}{\partial n}, \quad q^* = -x \frac{\partial \theta^*}{\partial n}
\]  

(8)

Then Eq.(6) can be written in terms of the fluxes, as

\[
\int_\Omega (A^* \theta^*) \theta d\Omega = \frac{1}{x} \int_\Gamma (q^* - q \theta^*) d\Gamma
\]  

(9)

In this paper, we term \( \theta^* \) and \( q^* \) as “adjoint temperature” and “adjoint heat flux”, respectively.

3. Influence Functions of Thermal Boundary Conditions

As derived above, the mathematical formulation of the thermal field in the forced convection problem can also be expressed in an integral form. Thus, if we could construct Green’s function for Eq.(9), the temperature at an arbitrary location would be determined only by the thermal boundary condition. Unfortunately, it is impossible to construct Green’s function for the present problem.

In the present work, we adopt the following adjoint problem for Eq.(4):

\[
A^* \theta^* = \left( \frac{\mathbf{u}}{a} \cdot \text{grad} + \Delta \right) \theta^* = 0
\]  

(10)

As a result, Eq.(9) can be simplified to

\[
\int_\Gamma q^* \theta d\Gamma = \int_\Gamma q \theta d\Gamma
\]  

(11)

Further, we choose the uniform Dirichlet boundary condition for the adjoint problem, such as

\[
\theta^* = 1 \text{ on } \Gamma
\]  

(12)

Then the total heat transfer rate \( Q \) can be obtained as an inner product of arbitrary surface temperature

\[
Q = \int_\Gamma q \theta d\Gamma
\]  

(13)
distribution $\theta$ and the adjoint heat flux distribution $q^*$, namely
\[ Q = \int_\Gamma q d\Gamma = \int_\Gamma q^* \theta d\Gamma \]  
(13)

Equation (13) indicates that if we numerically calculate the adjoint Eq.(10) under the Dirichlet condition (12), instead of solving the original Eq.(4) under a particular thermal boundary condition, we can predict the total heat transfer rate under arbitrary temperature distributions using the adjoint heat flux. Thus, the adjoint heat flux distribution obtained from the numerical solution of the adjoint problem can be regarded as an influence function of the surface temperature distribution on the total heat transfer.

In a similar fashion, solving Eq.(10) under the following Neumann condition
\[ q^* = 1 \text{ on } \Gamma \]  
(14)
we obtain an influence function of the surface heat flux distribution on the mean surface temperature $\bar{\theta}$, such that
\[ \bar{\theta} = \frac{1}{S} \int_\Gamma \theta d\Gamma \]  
(15)
where $S$ is the surface area of the object.

4. Numerical Examples

4.1 Numerical method

In the present study, we computed the influence functions, i.e., the adjoint heat flux distributions or the adjoint temperature distributions, for a circular cylinder and for an in-line square rod array. In the computations, a standard flow and temperature calculation code based on the finite difference method was employed by reversing the direction of velocity vector to compute Eq.(10) instead of Eq.(4). It should be noted that Eq.(1) and Eq.(10) were computed as an unsteady problem and the time-averaged values were used as a solution, because the convection fields around a cylinder and around an in-line square rod array show the periodic behavior with the generation of Karman vortex street. In addition, the computations were based on the nondimensional form; then the nondimensional heat transfer coefficient $N\nu$ (Nusselt number) and the nondimensional adjoint heat flux $N^*$ are defined as follows:
\[ N\nu = \frac{hL}{k} \quad N^* = \frac{Lq^*}{\kappa \theta^*} \]  
(16)
where $h$ is the conventional heat transfer coefficient and $L$ is the characteristic length, i.e., the cylinder diameter or the side length of a rod in the present examples. Likewise, nondimensional temperature $\Theta$ and nondimensional adjoint temperature $\Theta^*$ are defined as follows:
\[ \Theta = \frac{x \theta}{Lq} \quad \Theta^* = \frac{x \theta^*}{Lq^*} \]  
(17)

4.2 Results for a circular cylinder

As a first example, let us consider the forced convection heat transfer around a circular cylinder, as shown in Fig. 2. In the simulation, the Prandtl number ($Pr$) is 0.71 and the Reynolds number ($Re$) based on the cylinder diameter is 200, since the convection field at this Reynolds number can be reasonably simulated by the 2-dimensional laminar code.

The flow field, the temperature field around an isothermal cylinder and the adjoint temperature field obtained as a solution of the adjoint problem (Eq. (10)) under the uniform Dirichlet boundary condition ($\theta^*=1$) are shown in Figs. 3(a), 3(b) and 3(c), respectively. As can be seen from Figs. 3(b) and 3(c), the adjoint temperature field is rather different from the original temperature field, because the adjoint operator for the energy equation is not self-adjoint.

![Fig. 2 Configuration of forced convection heat transfer from a circular cylinder](image)

![Fig. 3 Forced convection field around a circular cylinder](image)
Fig. 4 Influence function of surface temperature on total heat transfer \((Re=200, Pr=0.71)\)

From the time series of the adjoint temperature fields, the time-averaged adjoint heat flux, i.e., the influence function of the local surface temperature on the total heat transfer rate, was calculated. Figure 4 shows the influence function as well as the conventional Nusselt number distribution around an isothermal cylinder. It is well known that the conventional Nusselt number takes a peak at the forward stagnation point and decreases toward the separation point. Thus, if we employ the expression of the total heat transfer rate based on Newton's cooling law, such as

\[
Q = \int_{\Gamma} h dT
\]

(18)

and if we regard the heat transfer coefficient as a weighting function of the local surface temperature, it seems as if the forward stagnation temperature would largely affect the total heat transfer rate. However, as mentioned earlier, the heat transfer coefficient defined in Newton's cooling law is meaningful only under a similar thermal boundary condition (under the isothermal condition in this case). In other words, the heat transfer coefficient is no more meaningful when the similarity of the thermal boundary condition is violated. On the other hand, the present influence function defined in Eq. (13) does not require the similarity of the thermal boundary condition and provides the influence of the local surface temperature on the total heat transfer rate. Thus, the temperatures near the forward and backward stagnation points do not significantly contribute to the total heat transfer rate, but in reality the temperature on the side surface largely affects the total heat transfer rate. This can be interpreted by the fact that high temperature on the upstream surface makes its subsequent thermal boundary layer thick; thus the thick thermal boundary layer prevents the heat transfer efficiency. The present influence function shown in Fig. 4 suggests that the high heat transfer capability can be maintained by increasing the surface temperature with an increase of the thermal boundary layer thickness.

Similarly, the influence function of the surface heat flux on the mean temperature was calculated by the time-averaged solution of the adjoint problem under the uniform Neumann condition, and is shown in Fig. 5 along with the temperature distribution of the uniform heat flux cylinder. Figure 5 implies that high heat fluxes near the forward and backward stagnation points increase the mean temperature.

As indicated above, using the present influence functions instead of the conventional heat transfer coefficient, we can quantitatively predict the arbitrary thermal boundary condition effects on the total heat transfer rate or on the mean surface temperature.

4.3 Results for an in-line square rod array

For the application of the present influence function to multiple objects, we will now consider the forced convection heat transfer from an in-line square rod array shown in Fig. 6. The Reynolds and Prandtl numbers in this problem are the same as those in the previous single cylinder case, namely, \(Re=200\) and \(Pr=0.71\), and all the intervals between the rods are equal to the side length of the square rod. For convenience, the square rods are denoted as \(R_1, R_2, \ldots, R_n\) (where \(n\) is the number of square rods), and the side surfaces on each rod are indicated by \(S_1, S_2, S_3, S_4\) as shown in Fig. 6.
In the case of a five-rod array, we numerically obtain the time-averaged solution of the adjoint problem under the uniform Dirichlet condition, namely $\theta^* = 1$ on all rod surfaces. From this solution, the adjoint heat flux distributions on all the rods are shown in Fig. 7, in which the local Nusselt number distributions under isothermal condition are also indicated by dashed lines.

As shown in Fig. 7, the present influence function is obviously different from the Nusselt number distribution. For example, the temperature effect of the front face of the first rod on the total heat transfer rate is not so large, while the heat transfer coefficient on this face is larger than those on the other faces. On the contrary, the temperature effect of the side faces of the end rod is rather large, while the heat transfer coefficient on these faces is small. These results indicate that the effect of the temperature of each rod on the total heat transfer rate cannot be evaluated by the conventional heat transfer coefficient.

To clarify the contribution of the temperature of each rod on the total heat transfer rate, the integrals of the influence function over the surface of each rod are calculated for $n = 1, 2, \ldots, 5$ and are shown in Fig. 8, in which the corresponding integrals of the Nusselt number are also indicated. As shown in Fig. 8, the total heat transfer rate of each isothermal rod decreases toward the downstream direction as expected. On the contrary, the integral of the influence function over the surface of each rod, which means the contribution of the temperature of each rod on the total heat transfer, does not decrease monotonically; for example, the contribution of the end rod temperature is largest for $n = 5$. This suggests that if there are five rods with different temperatures, the end rod temperature should be the highest to efficiently increase the total heat transfer. To demonstrate this fact, let us consider five rods with different surface temperatures from 0.6 to 1.4 at intervals of 0.2. The maximum and minimum heat transfer arrangements obtained from the present influence function are shown in Table 1, in which the symbols $\Theta_1, \Theta_2, \ldots, \Theta_n$ denote the surface temperatures of rods $R_1, R_2, \ldots, R_n$, respectively. For this example, the maximum or minimum heat transfer arrangement gives 11% higher or 9% lower heat transfer rate than the isothermal arrangement; thus, the total heat transfer rate in the maximum heat transfer arrangement is 20% larger than that in the minimum heat transfer arrangement. This result can be explained by the corresponding temperature fields shown in Fig. 9.
difference between the surface of each rod and the surrounding fluid, whose temperature rises toward the downstream direction, can be maintained by increasing the temperature of each rod toward the downstream direction; the first rod is special, because the heat transfer efficiency of this rod is affected by the absence of a front rod. On the contrary, by decreasing the temperature of each rod toward the downstream direction, the temperature difference also decreases; this lowers the heat transfer efficiency.

It should be noted that although these optimization results indicated in Table 1 may be obtained by direct numerical simulations of the forced convection fields, the number of simulations required in the direct approach is much larger than that required in the present method; for this problem, at least five numerical simulations are required in the direct approach by employing the superposition principle. On the other hand, the present influence function can be obtained by only one numerical computation of the adjoint problem even for a large number of objects. Moreover, the present influence function provides the information of local temperature effect on the total heat transfer rate, as shown in Fig. 7. The present method demonstrates very attractive features when compared with the direct approach method.

Table 1 Optimization results

<table>
<thead>
<tr>
<th></th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
<th>Total Heat Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isothermal Case</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>56.2 (1.00)</td>
</tr>
<tr>
<td>Max. Heat Transfer Case</td>
<td>1.2</td>
<td>0.6</td>
<td>0.8</td>
<td>1.0</td>
<td>1.4</td>
<td>62.1 (1.11)</td>
</tr>
<tr>
<td>Min. Heat Transfer Case</td>
<td>0.8</td>
<td>1.4</td>
<td>1.2</td>
<td>1.0</td>
<td>0.6</td>
<td>51.2 (0.91)</td>
</tr>
</tbody>
</table>

5. Conclusions

In this paper, we propose the use of numerical solution of an adjoint problem of forced convection heat transfer to evaluate the total or mean heat transfer characteristic under arbitrary thermal boundary conditions. The results can be summarized as follows:

1. By introducing an adjoint formulation for the forced convection heat transfer, the total heat transfer rate can be expressed as an influence function of the surface temperature distribution. As a result, the total heat transfer rate can be calculated by the influence function under arbitrary surface temperature distributions.

2. In a similar fashion, the mean surface temperature can also be expressed as an influence function of the surface heat flux distribution, and can be calculated under arbitrary surface heat flux distributions by the influence function.

3. Each influence function stated above can be obtained by a conventional numerical simulation technique. The computation time for the influence function is equal to that for the direct simulation of forced convection heat transfer under a specific thermal boundary condition.

References

6. Baughn, J.W., Hechanova, A.E. and Yan, X.,


