A Theoretical Evaluation of Fractal Dimension of Turbulent Flame*

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The fractal dimension on turbulent premixed flame based on HTA (Hyperbolic Tangent Approximation) model for fully developed turbulent field was evaluated. Main results are as follows: (1) The fractal dimension evaluated without flame stretch effect is 2.67. (2) The fractal dimension evaluated with flame stretch effect is 2.33. This value is in good agreement with experimental data.

Key Words: Turbulent Flame, Flame Structure, Fractal Dimension, Flame Speed

1. Introduction

Recently, a concept of fractal has been the subject of intense research as a new evaluation technique of turbulent premixed combustion. The concept was suggested by Mandelbrot[1]. For the complicated structures which have self-similarity, the fractal estimates the structures using of non integer dimension. For the turbulent premixed flames, the fractal was discussed by Peters[9] first, and was applied to evaluate the turbulent flame speed by Gouldin et al[10]. The fractal dimension has been determined by experiments[11][12], and was about 2.3.

On the field of uniform turbulence, the fractal dimension of distribution of dissipation rate of turbulence energy was conformed and estimated as 2.5 – 2.7[13]. So, the fractal dimension of the premixed flame is less than the case of turbulence field. For the fractal of turbulence, some theoretical approach[6] was performed to determined the dimension. However, for the turbulent flames, the theoretical approach is very few and the relationship with the fractal of turbulence field is not clear yet.

The authors developed the HTA (Hyperbolic Tangent Approximation) theory that is a new theoretical approach of turbulent premixed combustion and applied to the modeling to simulate laminar and turbulent premixed combustion[9][10]. In the present work, to verify the combustion model a three dimensional numerical simulation of turbulent flame around a bluff body was performed with LES (Large Eddy Simulation) turbulent model[11][12]. The results were in good agreement with experimental results. However, if the model is including the correct physics of turbulent combustion, the structure of flames has to been explained through it. Then, by using the concept of LES turbulent model for HTA combustion model, the fractal dimension of wrinkled laminar flame was estimated through the turbulent flame speed in mesh scale Δ. Further, the theoretical fractal dimension of wrinkled laminar flame was compared with experimental results qualitatively.

2. Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>c</td>
<td>reactive progress variable</td>
</tr>
<tr>
<td>C</td>
<td>time averaged reaction progress variable</td>
</tr>
<tr>
<td>( C_0 )</td>
<td>capacity of specific heat, J/kg\cdot K</td>
</tr>
<tr>
<td>D</td>
<td>fractal dimension of wrinkled laminar flame</td>
</tr>
<tr>
<td>( D_t )</td>
<td>fractal dimension of turbulent field</td>
</tr>
<tr>
<td>k</td>
<td>turbulent energy, ( m^2/s^2 )</td>
</tr>
<tr>
<td>L</td>
<td>macro scale of turbulent field, m</td>
</tr>
<tr>
<td>( Pr_t )</td>
<td>turbulent Prandtl number</td>
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<tr>
<td>R</td>
<td>gas constant, J/mol\cdot K</td>
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</tbody>
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$Su$ : laminar flame speed, m/s  
$St$ : turbulent flame speed, m/s  
$x$ : distance, m  
$\Delta$ : mesh scale, m  
$\delta$ : thickness of laminar flame (=$(1/(dc/dx)_{max})$, m  
$\delta_t$ : thickness of turbulent flame (=$(1/(dC/dx)_{max})$, m  
$\varepsilon$ : turbulent dissipation rate, m$^2$/s$^3$  
$\lambda$ : thermal conductivity, W/m·K  
$\lambda_t$ : eddy thermal conductivity, W/m·K  
$\rho$ : density, kg/m$^3$  
$\omega$ : reaction rate, kg/s·m$^3$  

Subscripts  
$b$ : burned state  
$u$ : unburned state  

3. Turbulent Premixed Combustion Model

3.1 Turbulent premixed combustion model

Details of the modeling were described previously$^{[10,11]}$. Here, only the features of the model is described. HTA model gives the chemical reaction rate for the equation of reactive progress variable $c$ as follows,

$$
\omega = 8\rho_u \frac{S_t}{\delta_t} c^2 (1 - c)
$$

(1)

where, $S_t$ : turbulent flame speed, $\delta_t$ : thickness of turbulent flame, $(S_t/\delta_t)$ is the mean chemical reaction rate through the turbulent flame. And, $S_t$ and $\delta_t$ satisfies the following relationship.

$$
S_t = 4\lambda_t C_{p_u} \rho_u \frac{k^2}{\varepsilon}
$$

(2)

where, $\lambda_t$ : turbulent thermal conductivity. Subscript $i$ means the physical value under the mean temperature of unburned and burned temperature. The combustion model of Eq. (1) is applicable with the time averaged turbulent model like the $k$-$\varepsilon$ model, the LES turbulent model and the direct simulation of turbulent through the each turbulent thermal conductivity $\lambda_t$. $S_t$ and $\delta_t$ are defined for each turbulent models. For example, on the case of LES, $S_t$ and $\delta_t$ are defined in each meshes and are functions of mesh scale $\Delta$.

Further, for wrinkled laminar flame the following relationship is given$^{[9,10]}$.

$$
\frac{S_{tu}}{\delta_t} = \frac{S_{tu}}{\delta_t} I^2 (I > 0.7)
$$

(3)

$$
0 (I \leq 0.7)
$$

where, $S_{tu}$ : laminar flame speed, $\delta$ : thickness of laminar flame, and script 0 means the physical value without the flame stretch effect. On Eq. (3), $I$ is a factor of flame stretch that is change from 0 to unity due to the magnitude of the flame stretch. $I$ is estimated by the following equation.

$$
\frac{S_{tu}}{\delta_t} = \frac{1}{\rho_u} \frac{1}{16} \ln (1 - I)
$$

(4)

where, $\rho_u$ : density of unburned gas under room temperature and atmosphere pressure conditions, $\rho_u$ : density of unburned gas, $g$ : velocity gradient. The laminar flame speed and thickness of laminar flame have the following relationship like Eq. (2).

$$
S_{tu} = \frac{4\lambda_t}{C_{p_u} \rho_u \delta_t}
$$

(5)

The effects of equivalence ratio, temperature of unburned gas and pressure are considered trough the laminar flame speed $S_{tu}$.

3.2 Evaluation of turbulent flame speed

Using of Eqs. (1)-(5) and time averaged turbulent model, the turbulent flame speed is determined. When, as a time averaged turbulent model, the $k$-$\varepsilon$ model is considered, $\lambda_t$ is estimated as,

$$
\lambda_t = c_k \rho_u C_{p_u} k^2
$$

(6)

where, $c_k$ : model constant. From Eqs. (2)-(4) and (6), on the case of $I > 0.7$, the following relationship is given,

$$
\frac{S_{tu}}{\delta_t} = I \left(1 + c_k \rho_u C_{p_u} k^2 \frac{\varepsilon}{\delta_t}ight)^{1/2}
$$

(7)

And by using

$$
k = \frac{3}{2} \frac{u^2}{l^2}, \quad \varepsilon = \frac{15}{40.4} \frac{u^3}{l}
$$

(8)

where, $L$ : macro scale of turbulence, $l$ : micro scale of turbulence. Then, the following turbulent flame speed is estimated:

$$
\frac{S_{tu}}{\delta_t} = I \left(1 + c_k \rho_u \frac{30.3}{5} R_l \right)^{1/2}
$$

(9)

where, $R_l$ : turbulent Reynolds number (= $u' L / \nu_l$) defined by $L$. On the case of strong turbulence,

$$
\frac{S_{tu}}{\delta_t} = I \sqrt{c_k \rho_u \frac{30.3}{5} R_l}
$$

(9')

3.3 Verification of the combustion model by a numerical simulation

Details of the simulation were described previously$^{[12,14]}$. To verify the turbulent combustion model including the quenching effect, a numerical simulation of a turbulent premixed flame around a bluff body with a pilot flame is performed with the LES turbulent model. The calculation domain consists of a straight chamber with a rectangular cross

![Experimental and calculation domain](image-url)

Fig. 1 Experimental and calculation domain
section. The experimental domain by Kobayashi et al.\textsuperscript{(15)} is shown in Fig. 1. Air/propane premixed gas of 300°C is supplied at the upper inlet. The equivalence ratio is 0.6 (–) and the inlet velocity is 21.5 m/s. At the lowest inlet, burned high temperature gas of 1250°C is supplied. This hot gas is the burned air/propane premixed gas whose equivalence ratio is 0.6 (–). The inlet velocity is 53.0 m/s. Downstream from the upper inlet, a bluff body is placed for the flame stabilization. The turbulent premixed flame is stabilized by a recirculation zone behind the bluff body and the pilot flame from the lowest inlet. The chemical reaction is assumed to be a single-step reaction like:

\[ \text{C}_4\text{H}_10 + 5 \text{O}_2 \rightarrow 3 \text{CO}_2 + 4 \text{H}_2\text{O} \]

The calculation domain is divided into \( z \times y \times z = 60 \times 100 \times 10 = 60,000 \) meshes. As only 10 meshes are used in the direction, to simulate the three-dimensional domain, periodic condition is enforced in the \( z \)-direction. The transport equations of mass, momentum, energy and the reaction progress variable with Favre averaged LES are solved with the following boundary conditions. As a wall condition, the gas velocity is estimated using the wall function. In the experimental apparatus, the wall of the chamber is cooled by water. However, the wall is treated as adiabatic, as the experimental heat loss is unknown. As the boundary condition of reactive progress variable \( c, c = 1.0 \) at the lowest inlet, and the other, at the upper inlet \( c = 0 \) is given, respectively. The calculation procedure is based on a second order finite-volume scheme for the momentum equation using the second order Adams–Bashforth method. As the flow solver, the SMAC (Simplified Marker And Cell) method is used. To solve the Poisson equation of the system pressure, the ICCG (Incomplete Cholesky Conjugate Gradient) method is used. To estimate the flame stretching effect, the velocity gradient of each mesh is necessary. In this calculation, the maximum velocity gradient of \( (du/dx, dv/dy, dw/dz) \) is chosen as a representative velocity gradient for each mesh. The time averaged temperature and streamlines distributions are shown in Fig. 2.

The flame between the burned and unburned gas is stabilized at the downstream from the edge of the separator. This is caused by the quenching effect due to flame stretch trough the strong shear layer. As ordinary combustion models omit the flame stretch effect, the numerical results were stabilized at the edge of separator\textsuperscript{(15)}. The present model can modify this problem. Time averaged temperature distribution of \( y = 63 \text{(mm)} \) from the edge of the separator is shown in Fig. 3. The low temperature region around the separator is seen like the calculation result. Tough there are some discrepancy with the experimental data around \( x = 20 ~ 30 \text{(mm)} \), the overall distribution is in good agreement with the experimental results. So, the present model is applicable to the three-dimensional turbulent combustion field and has a enough accuracy.

4. Application of the Fractal Dimension to the Turbulent Flame Speed

A cubic with a side \( L \) long that is equal to thickness \( \delta_r \) of time averaged turbulent flame is considered. In this cubic, the premixed gas change from the unburned state to the burned state through a flame. Here, the cubic with a side \( L \) long is divided to small cubes with a side \( \Delta_n \) (where, \( L > \Delta_n, n = 1, 2, 3, \ldots \)). For each cubes with a side \( \Delta_n \), a number of the cubes including the reactive zone and the total number \((L/\Delta_n)^3\) of cubes gives the probability \( \phi \) of reactive region in \( L^3 \) cubic. On the case of plain flame like a laminar flame, probability \( \phi \) is independent of \( \Delta_n \) and constant. And the other, when the turbulent flame has fluctuations due to turbulence, by using smaller \( \Delta_n \), more fine structure can be estimated. Then, the probability \( \phi \) depends on \( \Delta_n \). Especially, when the turbulent flame has a self-similarity, the number of cubes including reactive zone is given as follows,

\[ n_s = \left( \frac{L}{\Delta_n} \right)^D \]

where, \( n_s \) : the number of cubes including reactive zone, \( \Delta_n \) : scale of cubes, \( D \) : fractal dimension.
Then, the probability $\phi$ estimated by scale $\Delta_n$ is:

$$
\phi = \left( \frac{L}{\Delta_n} \right)^d \left( \frac{\Delta_n}{\Delta} \right)^{-3} = \left( \frac{L}{\Delta_n} \right)^{-3} \tag{11}
$$

The small cubic with a side $\Delta_n$ long includes more fine structures of flames. So, a local turbulent flame speed $S_t(\Delta_n)$ in the small cubic is determined for each cubic. The mean chemical reaction rate in the small cubic $\bar{\omega}(\Delta_n)$ has a relationship with the local turbulent flame speed $S_t(\Delta_n)$ as follows:

$$
\bar{\omega}(\Delta_n) = \rho_s S_t(\Delta_n) \frac{L}{\Delta_n} \tag{12}
$$

The mean chemical reaction rate $\omega_t$ through the cubic with a side $L$ long is given by the product of Eqs. (11) and (12). As the product of the turbulent flame speed and the density of unburned gas is equal the product of the thickness $\delta_c (= L)$ of the turbulent flame and mean chemical reaction rate $\omega_t$, the following relationship is satisfied:

$$
\rho_s S_t = L \bar{\omega}(\Delta_n) \left( \frac{L}{\Delta_n} \right)^{-3} = L \rho_s S_t(\Delta_n) \left( \frac{L}{\Delta_n} \right)^{-3} \tag{13}
$$

Gouldin et al. also related the turbulent flame speed to the increase of the flame area by the wrinkled laminar flame. And, by using the fractal dimension, the ratio of $S_t$ and $S_u$ is estimated by them\(^3\) as follows:

$$
\frac{S_t}{S_t} = \left( \frac{\Delta_{\text{max}}}{\Delta_{\text{min}}} \right)^{d-2} \tag{14}
$$

where, $\Delta_{\text{max}}$: the maximum scale in which satisfy the fractal structure, $\Delta_{\text{min}}$: the minimum scale in which satisfy the fractal structure. On Eq. (13), when $\Delta_n$ is equal to $\Delta_{\text{min}}$, $S_t(\Delta)$ is quite equal to $S_t$. Then, Eq. (13) is quite equal to Gouldin's formula (14).

On the next section, the local turbulent flame speed $S_t(\Delta_n)$ is estimated by using the concept of LES turbulent model.

5. Estimation of the Fractal Dimension

5.1 Fractal dimension without the flame stretch effect

On the case without the flame stretch effect, the factor of the stretch $I$ is treated as unity on Eqs. (1) – (9'). First, the local turbulent flame speed in the cubic with a side $\Delta_n$ (written as $\Delta$ on the following, briefly) long is estimated. If there are wrinkled laminar flame structures in the smaller cubics, the local turbulent flame speed satisfies the following relationship by Eqs. (3) and (5):

$$
\frac{S_t(\Delta)}{S_t} = \frac{\delta_c(\Delta)}{\delta} = \frac{4 C_p \rho_s S_t(\lambda_1 + \lambda_2)}{4 C_p \rho_s \lambda_1 S_t(\Delta)} = \frac{S_t(\lambda_1 + \lambda_2)}{\lambda_1 S_t(\Delta)} \tag{15}
$$

where, $\delta_c(\Delta)$ is the thickness of the turbulent flame defined in the cubics. On Eq. (15), the turbulent thermal conductivity $\lambda_t$ is given by LES as follows:

$$
\lambda_t = \rho_s C_p C_t \Delta^{d-2} \tag{16}
$$

where, $C_t$: model constant, $\Delta$: mesh scale, $Pr_t$: turbulent Prandtl number. Then, from Eqs. (15) and (16),

$$
\frac{S_t(\Delta)}{S_t} = \left( 1 + \frac{\rho_s C_p C_t}{Pr_t} \frac{A^{d-2}}{\lambda_t} \right)^{1/2} \tag{17}
$$

By using Eq. (8) for dissipation rate of turbulence energy $\varepsilon$, Eq. (8) is:

$$
\frac{S_t(\Delta)}{S_t} = \left( 1 + \frac{\rho_s C_p C_t}{Pr_t} \frac{A^{d-2}}{\lambda_t} \right)^{1/2} \frac{\Delta}{\lambda_1} \tag{18}
$$

Under the strong turbulence fields, the first term of Eq. (18) is far smaller than the second term. Then,

$$
\frac{S_t(\Delta)}{S_t} = \sqrt{C_t \left( \frac{\Delta}{\lambda_1} \right)^{1/3} \left( \frac{A}{\lambda_1} \right)^{2/3}} \tag{19}
$$

where,

$$
C_t = \frac{\rho_s C_p C_t}{Pr_t} \left( \frac{A}{\lambda_1} \right)^{1/3} \tag{20}
$$

From Eq. (19) and the global turbulent flame speed (13) defined through the cubic with a side $L$ long, the following relationship is given:

$$
\frac{S_t}{S_t} = \sqrt{C_t R_t \left( \frac{A}{\lambda_1} \right)^{1/3}} \tag{21}
$$

As Eq. (21) is the global turbulent flame speed, it has to be equal to Eq. (9') that do not depends on $\Delta$. When the effect of the flame stretch is omitted, $I$ is equal to unity on Eq. (9'). Then, the following relationship is given,

$$
D - 2 \times \frac{2}{3} = 0 \tag{22}
$$

So, the fractal dimension of turbulent flame is $8/3 = 2.6666$. This value is quite equal to the fractal dimension of uniform turbulence and is larger than the experimental data.

5.2 Fractal dimension with the flame stretch effect

When the flame stretch effect is considered, from Eqs. (9') and (14) the ratio of the turbulent and laminar flame speeds satisfies the following relationship,

$$
\left( \frac{\Delta_{\text{max}}}{\Delta_{\text{min}}} \right)^{d-2} = \left( \frac{S_t}{S_t} \right)^{I} \tag{23}
$$

where, script 0 means the state without the flame stretch effect. As the fractal dimension of $(S_t/S_t)_0$ is $8/3 (= 2.6666)$,

$$
D = \frac{8}{3} + \frac{\ln(I)}{\ln(\Delta_{\text{max}}/\Delta_{\text{min}})} \tag{24}
$$

On Eq. (24), as $I$ is change to $0.7 - 1.0$, then $\ln(I)$ is change from $-0.35$ to $0.0$. So, the flame stretch always has a effect to reduce the fractal dimension. As $I$ is given as a solution of Eq. (4), the value of $I$ depends on the velocity gradient $g$ strongly. As the distribution of velocity gradient has a fractal structure\(^4\), the distribution of the flame stretch factor for each scales is considered to have a fractal dimension $D_t$. The global factor $I$ is related to the local
flame stretch factor \( i \) for each scales through \( D_i \) as follows:

\[
I \left( \frac{L}{\Delta} \right)^3 = i \left( \frac{L}{\Delta} \right)^{D_i},
\]

(25)

So,

\[
i = I \left( \frac{L}{\Delta} \right)^{3-D_i}
\]

(26)

On the case with the flame stretch effect, Eq. (20) is modified as follows,

\[
\frac{S}{S_{tu}} = \sqrt{C_i R_e} \left( \frac{A}{L} \right)^{2.3} \left( \frac{L}{\Delta} \right)^{3-D_i}
\]

(27)

Then, the fractal dimension of premixed flame is estimated by the following equation.

\[
D = D_i - \frac{1}{3}
\]

(28)

where, \( D_i \): fractal dimension of turbulence. The fractal dimension is estimated 2.5 - 2.7 by experiment, and 8/3 = 2.666 by theoretical approach. By using \( D_i = 8/3 \), the fractal dimension of premixed flame is given \( D = 7/3 = 2.333 \). At the region of strong velocity gradient, the local quenching of flame is caused due to flame stretch. The quenching causes local reductions of flame area in the turbulent combustion field. This is the main reason of the reduction of the fractal dimension.

6. Comparison with Experimental Results

The comparison of the fractal dimension related to \((u'/Su)\) with the experimental results by Yoshida et al. (4,5), Bracco (7) is shown in Fig. 4. Figure 4 shows that the measured fractal dimension increases with \((u'/Su)\) from 2.0 that means the two dimensional flat flame to the limiting value 2.3. On the case with flame stretch effect, the theoretical fractal dimension is 2.33 and is close to the measured value.

7. Conclusions

The fractal dimension on turbulent premixed flame based on HTA (Hyperbolic Tangent Approximation) model for fully developed turbulent field is estimated. Main results are as follows:

1. The fractal dimension evaluated without an effect of flame stretch is 2.67.
2. The effect of flame stretch reduces the fractal dimension. The fractal dimension evaluated with the effect of flame stretch is 2.33. This value is in good agreement with experimental data. This is caused by the reduction of flame area due to the local quenching in the strong stretch region.

References