Vibration Analysis of a Multi-Stage Gear System Including Drive Mechanism Elements
(Proposal of Three-Dimensional Vibration Model and Eigenvalue Analysis of Helical Gear System)

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We have developed a new method for building a vibration model of an actual gear-drive system. Each gear has six degrees of freedom: three translational and three rotational motions. Stiffness matrixes representing each gear shaft and tooth meshing were made and overlapped into one stiffness matrix for the whole system. This enabled easy modelling of complex gear-drive systems such as multi-stage gears and branched gears. The results are in good agreement with experimental results with respect to the major resonance frequencies and vibration modes.

Key Words: Gear, Modelling, Coupled Vibration, Vibration of Mechanism, Eigenvalue Analysis, Multi-Stage Gear

1. Introduction

The increasing demand for quiet gear-driven machines, such as vehicles, elevators and generators, has created an increasing demand for precise analysis of gear drive vibration. There has been a lot of research on gear vibration analysis and several vibration models have been proposed. And most of these models deal with a single gear pair. However, it is necessary to consider coupled vibration of multi-stage gears[1]-[3] for the analysis of the gear systems used in industry. Furthermore, an actual gear system is connected with drive mechanism elements (motor, coupling, etc.), and it is known that stiffness of the shafts affects vibration characteristics of the total system. Accordingly, when we evaluate vibration and noise of a gear system in an industrial machine, it is important not only to consider the total transmission system (that is, the gear drive) but also consider the whole structure including the drive mechanism and the transmission system. We therefore comprehensively analyzed the vibration of a total practical drive mechanism using a gear transmission system consisting of spur or helical gears. And we developed a new method to establish a three-dimensional vibration model of complex gear-drive systems such as multi-stage gears and branched gears including drive mechanism elements.

A major source of gear noise is structure-borne noise generated by the excitation force passing through shafts and bearings. And this force is converted from torque fluctuation and modified by the tooth mesh. Therefore, it is necessary to build a vibration model considering tooth meshing in detail[3]-[6]. The normal load on the tooth surface of helical gears is slanted by the helix angle and the excitation force caused by the eccentricity of gears, and a profile error acts on the gears three dimensionally. Therefore, it is effective in gear modelling to consider six degrees of freedom: three translational and three rotational motions. We assumed that the effects of tooth spring and bearing spring are concentrated at the center of
gravity of the gear shafts, and the effect of the drive mechanism elements (which are coupled to the input and output shafts) is torsional vibration only. Finally, we propose a new method that synthesizes each gear element block, in order to establish a vibration model of a total drive system.

2. Nomenclature

- $b$: face width of the $j$-th gear pair
- $D_i$: direction vector of $\Delta l_{jm}$ of the $j$-th gear pair $D_j= R(y, \gamma) d_i$
- $d_i$: direction vector of deflection of tooth spring of the $j$-th gear pair (where $\gamma=0$)
- $d_i = N_i (a_{ix}, a_{iy}, a_{iz}) T = (n_{ai} S_{ai} S_{i} \beta_{ai}) n_{ai} S_{ai} C_{i} \beta_{ai}$
- $J_i$: polar moment of inertia of the $i$-th axis
- $J_{ax}$: inertia moment of the $i$-th axis around the center of gravity
- $k_{ix}$: stiffness of tooth spring of the $j$-th gear pair
- $k_{xx}$, $k_{xx}$: bearing stiffness of $x$-direction of the $i$-th axis
- $L_{bi}$, $L_{ei}$: distance between the center of gravity and the center of the bearings
- $l_{ax}$: distance between the center of the face width and the center of gravity of the $j$-th gear pair, $k$-th gear
- $m_i$: mass of the $i$-th gear shaft
- $n_i$: number of the $j$-th gear pair
- $n_{ax}$, $n_{ay}$, $n_{az}$: positive or negative sign of components of the direction vector of the $i$-th tooth spring
- $N_i= \text{diag}(n_{ax}, n_{ay}, n_{az}) \in R^{3 \times 3}$
- $p_i$: position vector of the center of gravity of the $i$-th axis $p_i= (x_i, y_i, z_i)^T$
- $R(y, \gamma)$: rotation transformation matrix
- $r_{ai}$: base circle radius of the $j$-th gear pair, $k$-th gear
- $S_{ax}$, $S_{ay}$: positive or negative sign, which is determined by the combination type of the $j$-th gear pair, $k$-th gear
- $S_{ai}$, $S_{ai}$, $C_{ai}$, $C_{ai}$: sin $\alpha_i$, sin $\beta_{ai}$, cos $\alpha_i$, cos $\beta_{ai}$
- $\alpha_{ai}$: transverse pressure angle of the $j$-th gear pair
- $\beta_{ai}$: base helix angle of the $j$-th gear pair
- $\Delta l_{jm}$: deflection of tooth spring of the $j$-th gear pair
- $\gamma$: slant angle between the axial plane (plane that includes two center lines of meshing gears) and the base coordinate system of the $j$-th gear pair
- $\theta_i$: rotational vector of the $i$-th axis $\theta_i= (\phi_i, \theta_i, \phi_i)^T$
- $\oplus$: direct sum of the sub space (a combination of matrices is shown)
- $U \oplus V = \begin{pmatrix} U & 0 \\ O & V \end{pmatrix} \in R^{(i+1)(i+1)}$
- Subscripts:
  - $i$: shows that the symbol is connected with the input shaft (cf. Fig. 5)
  - $j$: axis number ($i=1, 2, \ldots$)
  - $k$: gear pair number ($j=1, 2, \ldots$)
  - $l$: index which shows the drive type ($k=1$: in case of the driving gear; $k=2$: in case of the driven gear)
  - $m$: tooth spring number ($m=1, 2$)
  - $O$: shows that the symbol is connected with the output shaft (cf. Fig. 5)

3. Vibration Model of a Multi-Stage Gear-Drive System

A method for building a vibration model of the total multi-stage gear system including drive mechanism elements was developed. Since one of our objectives is to show the matrix generation method, we only discuss a mass matrix and a stiffness matrix and
omit a damping matrix in order to avoid equations being so complicated.

### 3.1 Model of each gear shaft

In a helical gear meshing, excitation force generates vibration of gears three dimensionally. Therefore, the model of each gear shaft in multi-stage gear system has six degrees of freedom: three translational and three rotational motions. Let us assume that bending deflection of each gear shaft is added to each bearing deflection, and the tooth spring of the gear meshing is modeled in terms of two parallel springs directed normal to the tooth surface and lying in the plane of action. When the $i$-th axis exists between the $(j-1)$-th and the $j$-th gear pair as illustrated in Fig. 1, the equations of motion at the center of gravity for the $i$-th axis can be expressed as follows:

$$
\sum_{p=1}^{2} \sum_{j=1}^{m_i} \left( -1 \right)^{p+1} b_{i,p} \frac{K_{p}}{2} \Delta L_{pmn} \partial_{i,p} L_{pmn} = 0,
$$

(2)

$$
\sum_{p=1}^{2} \sum_{j=1}^{m_i} \left( -1 \right)^{p+1} b_{i,p} \frac{K_{p}}{2} \Delta L_{pmn} \partial_{i,p} L_{pmn} = 0,
$$

(3)

$$
\sum_{p=1}^{2} \sum_{j=1}^{m_i} \left( -1 \right)^{p+1} b_{i,p} \frac{K_{p}}{2} \Delta L_{pmn} \partial_{i,p} L_{pmn} = 0,
$$

(4)

where $A_i=\text{diag}[m_i, m_i, m_i, J_{i}, J_{i}, J_{i}] \in \mathbb{R}^{6 \times 6}$, $B_i=\sum_{p=1}^{2} C_{p} \left( \Delta L_{i} \right)$, $C_{p} = (-1)^{p+1} b_{p} \frac{K_{p}}{2} D_{pm} \in \mathbb{R}^{6 \times 2}$.

Equation (8) can be defined as a substructure in the vibration model of multi-stage gears. By taking an example of the vibration model of third-stage reduction gears with a branched gear as shown in Fig. 2, and using Eq. (8), we get vibration models of each gear shaft as

$$
A_i \ddot{x}_i + B_i x_i + C_{i} \left( \Delta L_{i} \right) = 0,
$$

(8)

$$
A_3 \ddot{x}_3 + B_3 x_3 + C_3 \left( \Delta L_{3} \right) + C_2 \left( \Delta L_{2} \right) + C_1 \left( \Delta L_{1} \right) = 0,
$$

(9)

$$
A_2 \ddot{x}_2 + B_2 x_2 + C_2 \left( \Delta L_{2} \right) = 0,
$$

(10)

$$
A_1 \ddot{x}_1 + B_1 x_1 + C_1 \left( \Delta L_{1} \right) = 0,
$$

(11)

where $X_i=(p_{i}, \theta_{i})^{T} \in \mathbb{R}^{6}$.

Regarding a vibration model of each gear shaft as a substructure in this way we can constitute simultaneous equations of total system easily. And also by introducing the tooth spring deflection as a parameter, we can consider the excitation force of gear meshing easily.

### 3.2 Derivation of tooth spring deflection

Here we derive tooth spring deflection of the $j$-th gear pair $\Delta L_{pm}$. Figure 3 (left) shows the tooth spring model of the $j$-th gear pair, and Fig. 3 (right) shows the displacement of the joint of the tooth spring at $d_{21}$ ($j$-th gear pair, driven gear, and 1st tooth spring). When the joint of the tooth spring makes a minute
displacement from \( \mathbf{a}_{j,n} \) to \( \mathbf{b}_{j,n} \), the displacement of the tooth spring can be written as \( \| \mathbf{b}_{j,n} - \mathbf{a}_{j,n} \| \). Then, the following equation is obtained from geometry:

\[
\| \mathbf{b}_{j,n} - \mathbf{a}_{j,n} \| = \| \mathbf{b}_{j,n} - \mathbf{a}_{j,n} \|^T (\mathbf{b}_{j,n} - \mathbf{a}_{j,n}) . \tag{9}
\]

Using the direction vector of the tooth spring \( \mathbf{D}_n \), the displacement vector \( \mathbf{b}_{j,n} - \mathbf{a}_{j,n} \) in Fig. 3 (right) is expressed as

\[
\mathbf{b}_{j,n} - \mathbf{a}_{j,n} = \mathbf{b}_{j,n} - \mathbf{a}_{j,n} \mathbf{D}_n . \tag{10}
\]

Furthermore, when the displacement vector of the joint of the tooth spring is defined as \( \Delta \mathbf{P}_{j,n} \), it can be expressed as

\[
\Delta \mathbf{P}_{j,n} = \mathbf{b}_{j,n} - \mathbf{a}_{j,n} = (\Delta \mathbf{x}_{j,n} \Delta \mathbf{y}_{j,n} \Delta \mathbf{z}_{j,n})^T . \tag{11}
\]

Here, introducing \( \| \mathbf{b}_{j,n} - \mathbf{a}_{j,n} \| = \mathbf{l}_{j,n} \) and using Eqs. (9), (10) and (11) gives \( \mathbf{l}_{j,n} \) as

\[
\mathbf{l}_{j,n} = \mathbf{D}_n \Delta \mathbf{P}_{j,n} . \tag{12}
\]

Therefore, the tooth spring deflection \( \Delta \mathbf{l}_{j,n} \) can be derived as

\[
\Delta \mathbf{l}_{j,n} = \mathbf{l}_{j,n} - \mathbf{l}_{j,n} . \tag{13}
\]

On the other hand, \( \Delta \mathbf{P}_{j,n} \) can be written using \( \mathbf{p}_i \) and \( \mathbf{q} \) as

\[
\Delta \mathbf{P}_{j,n} = \mathbf{E}_{j,n} \mathbf{X}_i . \tag{14}
\]

where

\[
\mathbf{E}_{j,n} = \begin{pmatrix}
\mathbf{e}_{j,n} & \mathbf{s}_{z_{j,n}} \mathbf{r}_{z_{j,n}} \mathbf{s}_{z_{j,n}} & 0 \\
0 & 0 & 0
\end{pmatrix} \in \mathbb{R}^{2 \times 3} .
\]

Hence, \( \Delta \mathbf{l}_{j} \) and \( \Delta \mathbf{l}_{j} \) are written as

\[
(\Delta \mathbf{l}_{j}) = \mathbf{F}_j \mathbf{X}_i , \tag{15}
\]

\[
(\Delta \mathbf{l}_{j}) = \mathbf{E}_{j,n} \mathbf{X}_i .
\]

where

\[
\mathbf{F}_j = \begin{pmatrix}
\mathbf{D}_j^T & 0 \\
0 & \mathbf{D}_j
\end{pmatrix} \in \mathbb{R}^{2 \times 12}.
\]

There are eight combinations of gear pairs as shown in Fig. 4, and \( n_{x_{j}}, n_{y_{j}}, n_{z_{j}}, s_{x_{j}}, s_{y_{j}}, \) and \( s_{z_{j}} \) determined by the combination of gear pairs are listed in Table 1.

### Table 1: Signs of eight types of helical gear pairs

<table>
<thead>
<tr>
<th>( n_{x_{j}} )</th>
<th>( n_{y_{j}} )</th>
<th>( n_{z_{j}} )</th>
<th>( s_{x_{j}} )</th>
<th>( s_{y_{j}} )</th>
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</table>

Fig. 4 Eight types of helical gear pairs

\[
G_j = \begin{pmatrix}
G_{x_{j}} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
G_{y_{j}} & \mathbf{0} & \cdots & \mathbf{0}
\end{pmatrix} \in \mathbb{R}^{12 \times 12}, \quad m \geq 2 , \tag{18}
\]

Overlapping \( G_j \) in consideration of each stage coupling, we can obtain equations of motion of a total multi-stage gear system.

As the first example, we consider the \( n \)-th stage reduction gears that only have one gear pair between two gear shafts and no branch. The total gears model for the 1, 2, ..., \((n+1)\)-th axes is expressed by using Eqs. (1), (8), and (16) as

\[
\mathbf{A} \mathbf{X} + (\mathbf{B} + \mathbf{G}) \mathbf{X} = \mathbf{0} , \tag{17}
\]

where

\[
\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_{n+1} \in \mathbb{R}^{(n+1) \times (n+1)} ,
\]

\[
\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \cdots + \mathbf{B}_{n+1} \in \mathbb{R}^{(n+1) \times (n+1)} ,
\]

\[
\mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2 + \cdots + \mathbf{G}_{n+1} \in \mathbb{R}^{(n+1) \times (n+1)} ,
\]

\[
\mathbf{X} = (X_1 \mathbf{X}_2 \cdots \mathbf{X}_{n+1})^T \in \mathbb{R}^{(2n+1)} .
\]

The following describes the next example of a branched gear system. At the fork which distributes power to two gears, three gear shafts are coupled. However, after the fork, each branched gears is not coupled. Therefore, in the stiffness matrix of the total gears model, depending on the arrangement of the parameters in the vector \( \mathbf{X} \), one branched stiffness matrix must be coupled with the main gears so that it strikes the stiffness matrix of the other branch. Hence, we define the matrix that overlaps the others as

\[
G_{j,m} = \begin{pmatrix}
G_{11} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
G_{21} & \mathbf{0} & \cdots & \mathbf{0}
\end{pmatrix} \in \mathbb{R}^{12 \times 12}, \quad m \geq 2 , \tag{18}
\]
where
\[ G_i = \begin{pmatrix} G_{i1} & G_{i2} \\ G_{i2} & G_{i1} \end{pmatrix} \in \mathbb{R}^{8 \times 12}, \quad G_{j1}, \ldots, G_{j6} \in \mathbb{R}^{8 \times 6}. \]

Using Eqs. (1) and (18) enables easy modeling of the complex multi-stage gear systems. Here, we assume the vibration model of multi-stage branched gears has \( n \)-th stage gears with \((n+1)\) axes and only one gear pair exists between two gear shafts. We also assume that the model has branches from the \( l \)-th axis that have \( n_0 \) gear pairs to the end of the branch. Then we number gear pairs and gear shafts (axes) as shown in Fig. 2; i.e., number first to the end of one branch, and then number the other.

Defining all of the gear pair number as \( n_{all} = n + n_0 \), we get a total \( n_{all} \)-th stage gears model for 1, 2, \( \cdots \), \( (n_{all} + 1) \)-th axes as
\[
A \dot{X} + (B + G)X = -\alpha,
\] (19)
where
\[
A = A_1 \oplus A_2 \oplus \cdots \oplus A_{n_{all} + 1} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
\[
B = B_1 \oplus B_2 \oplus \cdots \oplus B_{n_{all} + 1} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
\[
G = G_1 \oplus G_2 \oplus \cdots \oplus G_{n_{all} + 1} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
\[
G_A = G_{a1} \oplus G_{a2} \oplus \cdots \oplus G_{a(n_{all} + 1)} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
\[
G_A = G_{a1} \oplus G_{a2} \oplus \cdots \oplus G_{a(n_{all} + 1)} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
\[
G_X = G_{x1} \oplus G_{x2} \oplus \cdots \oplus G_{xn_{all} + 1} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
\[
X = (X_1^T \ X_2^T \ \cdots \ X_{n_{all} + 1}^T)^T \in \mathbb{R}^{(n_{all} + 1) \times 1}.
\]

3.4 Modelling of a total gear-drive system

By using Eq. (17) or Eq. (19), we can make the matrix for the total multi-stage gear system. However, an actual gear-drive system includes drive mechanism elements with input and output shafts, and they affect torsional vibration for the vibration of the gear drive system. We therefore make the model of a multi-stage gear system including drive mechanism elements as some disks exist on both sides of the input and output shafts of the gears in series as shown in Fig. 5.

Here, each symbol is defined as follows:
\( n_a \): number of disks which exists on the left or right side of the gear shaft.
\( J_{all} \): polar inertia moment of the \( i \)-th disk from the gear shaft (\( i = 1, 2, \ldots, n_a \)).
\( k_{all} \): torsional stiffness between the \((i-1)\)-th and the \( i \)-th disks (\( i = 1, 2, \ldots, n_a \)).

Subscript \( a \) is \( IL, IR, OL, \) or \( OR \), according to the position of the shaft. When the shaft is on the left input axis; \( a = IL \), on the right input axis; \( a = IR \).

Generally, in case of torsional vibration of disks combined in series, the stiffness matrix becomes a triple diagonal matrix. However, when the disks are in a branched system, like a branched system of gears, at the fork, coupling with three gear shafts exists and after the fork, no coupling take place with each branch. Hence, we define a matrix that overlaps the others like Eq. (18) as

\[
H_i = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]

Next, we set each vector as
\[
Z = (\theta_{b1} \ \theta_{b2} \ X_1^T \ X_2^T \ \theta_{b1}^T)^T \in \mathbb{R}^{N_{all}},
\]
where
\[
\theta_b = (\theta_{b1} \ \theta_{b2} \ \cdots \ \theta_{b(n_{all} + 1)}) \in \mathbb{R}^{n_{all} \times n_a};
\]
\[
N_{all} = n_{IL} + n_{IR} + 6(n + 1) + n_{aL} + n_{aR}.
\]

Then, we generate the stiffness matrix of the total gear drive system. Let the stiffness matrix of all the disks connected to the input shaft be \( K_i \), it can be written as
\[
K_i = K_{IL} \oplus n_{aL + 1} K_{IL} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
where
\[
K_{IL} = (K_{IL1} \oplus K_{IL2} \oplus \cdots \oplus K_{IL(n_{all} + 1)}) \oplus n_{aL} K_{IL1} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
\[
K_{IR} = (K_{IR1} \oplus K_{IR2} \oplus \cdots \oplus K_{IR(n_{all} + 1)}) \oplus n_{aR} K_{IR1} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
where the matrices which generate the coupled terms of each disk are
\[
K_{IL1} = K_{IL1} H_{IL1} + 5 H_{IL1} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
\[
K_{IR1} = K_{IR1} H_{IR1} \oplus n_{aR} H_{IR1} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
\[
K_{IL2} = K_{IL2} H_{IL2} \oplus n_{bL} H_{IL2} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
\[
K_{IR2} = K_{IR2} H_{IR2} \oplus n_{bR} H_{IR2} \in \mathbb{R}^{(n_{all} + 1) \times (n_{all} + 1)},
\]
Because \( \theta_b \) is the fifth term of the displacement vector \( X \) of the gears, five rows and five columns of coupled terms with \( \theta_b \) are added to \( K_{IL} \) and \( K_{IR} \). In the same way we generate the stiffness matrix \( K_{all} \) of the total gear–drive system as
\[
K_{all} = K_i \oplus (B + G) \oplus G \in \mathbb{R}^{N_{all} \times N_{all}}.
\]
In the same way, the stiffness matrix can be generated when the system of gears is a branched
system.

The mass matrix \( M_{w} \) is given as
\[
M_{w} = J_{f} + A + J_{o} \in \mathbb{R}^{n_{m} \times n_{m}}
\]
where
\[
J_{f} = \text{diag}[J_{f1}, \ldots, J_{fn_{f}}, J_{o1}, \ldots, J_{on_{o}}] \in \mathbb{R}^{n_{f} \times n_{f}}, \quad n_{f} = n_{f1} + n_{f2},
\]
\[
J_{o} = \text{diag}[J_{o1}, \ldots, J_{o,n_{o}}, J_{o1}, \ldots, J_{on_{o}}] \in \mathbb{R}^{n_{o} \times n_{o}}, \quad n_{o} = n_{o1} + n_{o2}.
\]
Finally, the vibration model of the total gear drive system can be expressed as
\[
M_{w} \ddot{Z} + K_{w} Z = 0.
\]

4. Example

Here we apply the matrix-generation method to an actual machine. First, we analyze eigenvalues and examine the correlation between the measured vibration of an actual machine in order to verify the vibration model. The details of the experiment on the vibration will be reported in the future.

4.1 Modelling

The machine that we modelled is illustrated in Fig. 6. The vibration model of the machine is shown in Fig. 7. In the model, the number of the gear pairs \( n \) is 2, that of gear shafts is \( n+1 = 3 \), and that of disks is \( n_{d1} = 1, n_{d2} = 3, \) and \( n_{o1} = 1 \). The vibration model of the total gear drive is written as
\[
Z = (\theta_{d1}, \theta_{d2}, \theta_{o1} X_{d}^{T} X_{o}^{T} \theta_{o1})^{T}
\]
\[
\in \mathbb{R}^{3},
\]
\[
M_{w} = J_{f} + A + J_{o} \in \mathbb{R}^{n_{m} \times n_{m}}, \quad J_{f} = \text{diag}[J_{f1}, J_{f2}, J_{o1}, J_{o2}] \in \mathbb{R}^{4 \times 4},
\]
\[
K_{w} = (K_{d1} \otimes K_{d2} \otimes (B + G) \otimes K_{o1}) \in \mathbb{R}^{3 \times 3}, \quad (B + G) \in \mathbb{R}^{3 \times 3},
\]
\[
K_{d1} = K_{d11} K_{d12} \otimes K_{d13} \in \mathbb{R}^{3 \times 3},
\]
where the matrices which generate the coupled terms of each disk are
\[
K_{d1} = K_{d11} K_{d12} \otimes K_{d13} \in \mathbb{R}^{3 \times 3}, \quad (i = 2, 3),
\]
\[
K_{o1} = K_{o11} K_{o12} \otimes K_{o13} \in \mathbb{R}^{3 \times 3}.
\]

4.2 Numerical analysis

We used above equations to numerically calculate theoretical natural frequencies and vibration mode shapes. In order to calculate accurately, it is important not only to use valid equations but also to input appropriate values of masses, moments of inertias, etc. Here, we describe how to determine these values. The coupling at the input end of the input shaft was made of rubber, so it was assumed that the translational motions were isolated at the coupling. Therefore, the half mass of the coupling was added to the mass of the input gear shaft. In the same way, the moment of inertia of the half of the coupling was added to that of input gear shaft. However, polar moment of inertia of the half of the coupling was treated as an independent inertia. The moments of the output gear shaft are determined in the same way. For the intermediate gear shaft whole mass and moment of inertia were taken into account. Table 2 lists the gear shaft data used in the analyses. Tooth stiffness was calculated by using the equations proposed by Ishikawa(10). The value used was that of the virtual spur teeth meshing at the pitch point.

Three-dimensional vibration mode shapes are very complicated. Therefore, in order to recognize them, a three-dimensional graphic display program was coded (see Appendix). Figure 8 is a mode-shape graphic showing the yaw—motion of the second axis.

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<thead>
<tr>
<th>Table 2 Gear shaft data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>( m_{i} ) (kg)</td>
</tr>
<tr>
<td>91.7</td>
</tr>
<tr>
<td>( J_{i} ) (kgm²)</td>
</tr>
<tr>
<td>( J_{i} ) (kgm²)</td>
</tr>
<tr>
<td>( L_{i} ) (mm)</td>
</tr>
<tr>
<td>( L_{i} ) (mm)</td>
</tr>
<tr>
<td>( k_{i1,i1} ) (Nm/rd)</td>
</tr>
<tr>
<td>( k_{i2,i2} ) (Nm/rd)</td>
</tr>
<tr>
<td>( k_{g1} ) (Nm/rd)</td>
</tr>
<tr>
<td>( k_{g2} ) (Nm/rd)</td>
</tr>
</tbody>
</table>
4.3 Experiments

Torque fluctuation caused by torsional vibration was measured by strain gauges on shaft A between the gear and the coupling (Fig. 7). The load was applied in 15 steps from the no-load condition to the full load by using a dynamometer. Torque fluctuation in each load was measured and order ratio analyses were done using an FFT analyzer.

4.4 Results and discussion

Figure 9 compares the measured and calculated natural frequencies and the vibration mode shapes. Bars with black circles show the resonance frequency bands of tooth contact frequency of the first-stage (gear pair) and bars with black squares show those of the second-stage. Experimental mode shapes are torsional dominant modes. Vertical broken lines are theoretical natural frequencies that have torsional dominant mode shapes.

Experimental results show that torque fluctuation amplitude is a maximum at 389 to 403 Hz of tooth contact frequency of the first-stage and 364 to 375 Hz of tooth contact frequency of the second stage. While theoretical results show that the model has natural frequencies of 390.1 Hz and 374.3 Hz, and the mode shapes of these two orders have large relative torsional displacement at the input coupling. The other natural frequencies also agree well, as can be seen in Fig. 9. Thus, the validity of the developed program and the model are confirmed.

5. Summary

A multi-stage gear-drive system was numerically analyzed and this analysis was experimentally validated. The main findings are summarized below:

- A vibration model, which includes multi-stage gears and has six degrees of freedom per gear shaft, and drive and/or driven mechanism elements were developed.
- Natural frequencies and mode shapes obtained by using the above model agree well with the resonances of the measured torque fluctuation.

Appendix

The three-dimensional graphic display program was coded so that we could easily recognize the mode shapes. The process to display three-dimensional graphics is as follows.

A tooth of one gear is expressed as a trapezoid, which consists of four coordinate as shown in Fig. 10. These gear models are used to display vibration mode shapes. Since the model has six degrees of freedom, we can apply the concept of direct kinematics in robotics. So, we define coordinate systems and vectors as follows:

- \( B \) : base coordinate system of the gear shaft
- \( C \) : universe coordinate system of the gear shaft
- \( D \) : coordinate system of the gear shaft including translational mode
- \( E, F, G \) : coordinate system of the gear shaft including translational mode and rotational mode
- \( H \) : coordinate system which forms the side surface of the tooth of the gear shaft including vibration mode

\( \vec{r}_p \) : position vector to the center of gravity of the gear shaft from the origin of the base coordinate system \( \vec{r}_p \in \mathbb{R}^3 \)

\( \omega, \theta, \phi \) : component of the rotational vector in the direction of \( \phi, \theta \) or \( \phi \)
$\mathbf{c}_r$: translational mode vector $\mathbf{c}_r=(u_1, u_2, u_3)^T \in \mathbb{R}^3$

$\mathbf{c}_r$: vector which shows the side surface of the gear from the center of gravity of the gear shaft $\mathbf{c}_r=(0, b_x + \frac{b_y}{2}, 0)^T \in \mathbb{R}^3$

Then, each coordinate system is expressed by using a homogeneous transformation matrix as

$$
\mathbf{e}_A = \begin{pmatrix} 1 & 0 & \mathbf{c}_r \\ 0 & 1 & \mathbf{0} \\ 0 & 0 & 1 \end{pmatrix}, \\
\mathbf{e}_B = \begin{pmatrix} 1 & 0 & \mathbf{c}_r \\ 0 & 1 & \mathbf{0} \\ 0 & 0 & 1 \end{pmatrix}, \\
\mathbf{e}_C = \begin{pmatrix} 1 & 0 & \mathbf{c}_r \\ 0 & 1 & \mathbf{0} \\ 0 & 0 & 1 \end{pmatrix}, \\
\mathbf{e}_D = \begin{pmatrix} 1 & 0 & \mathbf{c}_r \\ 0 & 1 & \mathbf{0} \\ 0 & 0 & 1 \end{pmatrix}.
$$

Therefore, the coordinate system which forms the side of the gear viewed from coordinate system $B$ as shown in Fig. 11 is

$$\mathbf{C}_A = \mathbf{C}_B \mathbf{A}_B \mathbf{A}_C \mathbf{C}_C \mathbf{A}_D. \quad (26)$$

Let the vector which forms the side surface of the gear in the coordinate system $H$ be $\mathbf{a}_r$, and the vector which forms the same surface viewed from coordinate system $B$ be $\mathbf{b}_r$, we obtain the following equation:

$$\mathbf{u}_r = \mathbf{A}_B \mathbf{A}_C \mathbf{A}_D \mathbf{A}_C \mathbf{A}_C \mathbf{A}_B.$$

To display the gear side surface, we rotate vector $\mathbf{a}_r$ around the axis of the gear shaft (while changing its length) like a rectangular wave. The track of the tip of vector $\mathbf{b}_r$ forms a gear model. Thus, we can see the vibration mode-shapes of gears three dimensionally.

References


