Kinematic Analysis and Derivation of Equations of Motion for Mechanisms with Loops of Different Motion Spaces* (Theoretical Analysis)

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Kinematic analysis and derivation of equations of motion for a mechanism with loops of different motion spaces are developed. First, it is shown that screw coordinates of a joint which expresses relative motion between two links must be expressed by the basis vectors of motion spaces of the loops to which it belongs. Then, it becomes possible to derive the same number of linear equations for joint velocities and accelerations of a loop as the dimension of its motion space, and joint velocities and accelerations can be determined by solving these linear equations. Next, it is described that the force analysis can be performed by expressing the joint wrench as a linear combination of basis vectors of a reciprocal space of a motion space and other linearly independent screw coordinates. Using the developed theorem, the computational scheme for kinematic and force analysis is derived. The method to derive the equations of motion for a mechanism is also clarified.

Key Words : Dynamics, Robot, Modeling, Kinematics, Force Analysis, Lie Algebra, Screw Theory

1. Introduction

A spatial mechanism has three translational and three rotational motion elements. Expressing the motion elements by a vector, the vectors which a spatial mechanism can have span a six dimensional linear vector space. This vector space is a Lie algebra of the special Euclidian group $SE(3)$ (affine motion group). In this paper, the linear space spanned by motion elements is said to be a motion space. A Lie algebra is a special motion space in which the calculation of the Lie bracket is defined.

In the conventional kinematics, a mechanism, all the loops in which have a same motion space, is mainly considered. Recently, however, mechanisms having different motion groups in their loops are utilized in some robot manipulators and parallel actuated mechanisms. In them, though the whole

motion group is the six dimensional Lie algebra, some of loops have the motion space of a sub Lie algebra. For instance, some industrial robot mechanisms have a parallel link loop with the Lie algebra of a planar motion space to drive their forearm. There exists a spatial parallel actuated mechanism having connecting chains with the loops of planar motion groups. Another example is found in a multi DOF mechanism for translational motions adopting loops of cylindrical motion groups to drive it.

The kinematic and dynamic analysis of the mechanism having several different motion groups in its loops is very complicated compared with a mechanism with single motion group, because the dimension of the vectors expressing kinematic and dynamic conditions in one loop is different from others.

In the past, for these kinds of mechanisms, analysis procedures were found out individually according to the mechanism. On the other hand, to develop the computer program which can calculate kinematic and dynamic problems of various types of mechanisms including above mentioned types, the unified method for a kinematic and dynamic analysis is required. But a study on this problem was hardly accomplished. In

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this paper, a novel algorithm based on the Lie algebra (screw theory) for the kinematic and dynamic analysis of the mechanisms having loops with different motion groups is introduced. The procedure for the analysis is expressed in the matrix form for the easy treatment in a computer programming.

In the force analysis and derivation of equations of motion for a mechanism, wrenches acting on links and joints of a mechanism are represented by six dimensional vectors (screw coordinates). If the motion group of a mechanism is a six dimensional Lie algebra, all the six parameters of each wrench are related with its motion. When a mechanism is a planar mechanism with a three dimensional Lie algebra, the three in six parameters are related with the motion. Remaining parameters are not determined by the motion. They are influenced by the forces and reaction forces between parts of a mechanism and are hardly determined. Thus the analysis of force related with the motion of a planar mechanism requires only three parameters for each wrench.

When a mechanism has loops with different motion groups, sets of parameters of wrenches which are related with its motion may be different in every loop. Some links and joints of a mechanism belong to several loops and, in this case, the problems how the parameters of wrenches on links and joints should be decided occurs. When the wrenches are represented by six parameters, some of them are not related with the motion and they are determined by solving a redundant problem. A redundant problem in the motion analysis makes the analysis complicated and it is better for us to avoid solving a redundant problem if possible. Conventional kinematic and dynamic analysis of some types of mechanisms, such as planar mechanisms, shows that solving a redundant problem is not necessary for their motion analysis.

Therefore, in this paper, it was studied to analyze mechanisms with different motion group loops without solving a redundant problem. It is derived that this type of analysis can be done by determining the parameters of wrenches based on the concept of the reciprocal space of a motion space and its complement space in the screw theory. Procedures for force analysis and derivation of equations of motion are also derived.

2. Motion Space

Let two links, i and j be connected by a joint with single degree of freedom. Also, let the velocity and acceleration of the links i, j be represented by screw coordinates V_i, A_i, V_j, A_j respectively. The relative velocities and accelerations between the links are represented by using a displacement θ about a joint

axis M as $\dot{\theta}M, \ddot{\theta}M$. M is a screw coordinates representing a unit motion by the joint and is said to be a joint axis in this paper. Then the relations of velocities and accelerations between the links can be written in the following equations.

$$V_j = \dot{\theta}M + V_i \quad (1)$$

$$A_j = \ddot{\theta}M + V_i \times \dot{\theta}M + A_i \quad (2)$$

The multiplication in Eq.(2) is said to be an outer product of screw coordinates⁽¹⁾.

A set of all the screw coordinates of velocities is a six dimensional vector space and is said to be a motion space. The six dimensional motion space has its sub-space which is spanned by five or less linearly independent screw coordinates.

A motion space which is closed to the calculations of the outer product is a Lie algebra. The six dimensional motion space is a Lie algebra and some of the sub-motion spaces are Lie algebras. The physical meaning of a Lie algebra in kinematics is as follows. Screw coordinates is simply called a vector in the following expressions.

Let M be a joint axis (a six dimensional vector) fixed on a moving rigid body with velocity vector V . Let W be a constant screw on a moving rigid body. The derivative of W about a small period dt is $V \times Wdt$ and W changes to $W + V \times Wdt$. When the vectors W and V are elements of a Lie algebra S , then, $V \times W$ is also its element and $W + V \times Wdt$ is also in S . It means that even if a rigid body has any velocity in a Lie algebra S , W always belong to S . The motion space of a mechanism is one spanned by all its joint axes. The velocity of its link is a linear combination of its joint axes and, therefore, is in its motion space. When the motion space is a Lie algebra, any joint axes are stayed in it by any motion of the mechanism. Thus, a Lie algebra is closed to any motion of its element. When the motion space of a mechanism is a Lie algebra, the dimensions of the motion space keep constant (except at a singular point) while in its motion.

Following motion spaces are Lie algebras⁽¹⁾.

1. any one dimensional screw motion space (1D)
2. cylindrical motion space (2D)
3. planar translational motion space (2D)
4. spherical motion space (3D)
5. planar motion group (3D)
6. three dimensional translational motion space (3D)
7. one rotational and three translational motion space (4D)
8. spatial motion space (6D)

The last one is the six dimensional Lie algebra and all remainders are its sub-algebras.

All the loops of a structure which is composed of

links connected by joints to each other has the same Lie algebra and the numbers of links and joints satisfies the condition of degrees of freedom. That is the sufficient condition for the structure to be a mechanism. In the conventional kinematic analysis, a mechanism should satisfy this sufficient condition. Mechanisms considered in this paper do not satisfy this condition. Its loops have different Lie algebras.

Because a joint with multi degrees of freedom can be expressed by a serial connection of joints with single degree of freedom, only joints with single degree of freedom is considered in this paper. An analysis of a mechanism having loops with same Lie algebra is a special case of the computational scheme described in this paper.

There exist another types of special mechanisms such as Sarrus mechanism and Bennet Mechanism. They do not have loops of Lie algebras. The Sarrus mechanism is obtained by connecting two loops of different Lie algebras and it is possible to apply the method described in this paper. This procedure will be introduced in the next paper. A motion space of the Bennet mechanism changes as its motion but the dimension is kept constant. It may be possible to apply the method of this paper to the Bennet mechanism by determining the basis of the motion space at any instance of its motion. But its procedure will be very complicated and, therefore, the analysis of the Bennet mechanism is excluded in this paper.

3. Degrees of Freedom of a Mechanism

Firstly let consider the degrees of freedom of a mechanism from a view point of velocity and acceleration analysis. The parameters to be determined in this analysis are joint velocities $\dot{\theta}$ and accelerations $\ddot{\theta}$. The number of parameters is p which is equal to the number of joints. Joint axes in a loop must satisfy the condition that they are linearly dependent. For example, a single loop mechanism shown in Fig. 1 must satisfy the following relation.

$$\dot{\theta}_1 \mathbf{M}_1 + \dot{\theta}_2 \mathbf{M}_2 + \cdots + \dot{\theta}_p \mathbf{M}_p = \mathbf{0} \quad (3)$$

where $\mathbf{M}_i$ is an axis of the joint i .

On the other hand, when a dimension of the motion space is d , the linear combination is d linear homogeneous equations on d number of parameters of joints in a loop. The same relations exist in all the loops, and $\sum_{i=1}^a d_i$ number of linear homogeneous equations are obtained. Where d_i is the dimension of a motion space of the loop i and a is a number of loops. The number of parameters is equal to the number of joints p . The degree of freedom ζ of a mechanism is a number of parameters required to solve the linear equations and is given by the following equation.

Fig. 1 Single loop mechanism

$$\zeta = p - \sum_{i=1}^a d_i \quad (4)$$

The condition for joint accelerations in a loop is:

$$\ddot{\theta}_1 \mathbf{M}_1 + \ddot{\theta}_2 \mathbf{M}_2 + \cdots + \ddot{\theta}_p \mathbf{M}_p + \sum_{i=2}^p \left\{ \left(\sum_{j=1}^{i-1} \dot{\theta}_j \mathbf{M}_j \right) \times \dot{\theta}_i \mathbf{M}_i \right\} = \mathbf{0} \quad (5)$$

And the same result on the degree of freedom is obtained.

The loops of a mechanism should be selected such that they are the basis vectors of the motion space of a loop⁽¹⁾, the summation of their dimensions should be minimum, and their motion spaces should be Lie algebras. In the following procedures, the suitable loop basis is supposed to be determined beforehand and the locations of links and joints are already determined by some location analysis⁽²⁾.

By expressing all the joint axes by coordinate vectors of a basis of the motion spaces, the equations for determining joint velocities and accelerations can be derived using the Eqs. (3) and (5).

4. Analysis of Velocities and Accelerations

A structure of a mechanism can be represented by means of a kinematic graph. A mechanism with two closed loops are shown in Fig. 2(a). Circles in the figure show joints and lines and the areas indicated by slanted lines show links. This structure can be represented by a graph in Fig. 2(b) where vertices show links and edges show joints. This graph can be numerically expressed by an incidence matrix and a loop matrix. In this case, a basis of the loop space of a mechanism is supposed to be composed of loops 1 and 2.

For the kinematic analysis of a mechanism, the directions of joint motions must be determined beforehand. In this paper, the positive direction of a joint displacement θ is taken from a smaller number link to a larger number link.

For expressing the direction of joint displacements, a directional graph and its matrices are utilized. In the incidence matrix, its (ij) th element is set

Fig. 2 Closed loop mechanism

1 when a link number i connected to the joint j is smaller than that of another link connected to it j , and -1 when it is larger. In the loop matrix, set its element 1 when link numbers connected by a joint are increasing order in accord with the loop direction, and -1 when they are decreasing order. Then, matrices of the graph in Fig. 2 become:

incidence matrix:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

loop matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

The matrix removing the row of a stationary link from the incidence matrix is utilized in the following expressions and is represented by Φ , and the loop matrix is represented by Λ .

The relations for velocities of a mechanism can be expressed in the following form.

$$\Lambda M \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_p \end{bmatrix} = 0 \quad (6)$$

$$\text{where: } M = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_p \end{bmatrix} \quad (7)$$

M_i , $i=1, 2, \dots, p$ is the axis of a joint i , and θ_i is its displacement.

Dividing joints into passive joints and active joints. Add super script ' p ' on the passive joint axes, and ' a ' on the active joint axes. All the joint are renumbered sequentially such that both passive joint numbers and active joint numbers begin from 1. And one of the numbers of two links connected by the joint

is also attached. For example, $M_{j,m}^p$ is the axis of a passive joint j connected by link m . By using these notations, Eq.(6) can be rewritten in the following equation.

$$\Lambda_p \begin{bmatrix} M_{1,a}^p & 0 & \cdots & 0 \\ 0 & M_{2,a}^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{(p-\zeta),c}^p \end{bmatrix} \dot{\theta}_p + \Lambda_a \begin{bmatrix} M_{1,d}^a & 0 & \cdots & 0 \\ 0 & M_{2,e}^a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{\zeta,f}^a \end{bmatrix} \dot{\theta}_a = 0 \quad (8)$$

where Λ_p and Λ_a are a passive joint part and an active joint part respectively of the loop matrix. Equation (8) can be simplified in the following form.

$$\Lambda_p M_p \dot{\theta}_p + \Lambda_a M_a \dot{\theta}_a = 0 \quad (9)$$

For the computation of Eq.(9), the joint axes in a row of the matrices $\Lambda_p M_p$ and $\Lambda_a M_a$ should be expressed as a coordinate vectors with the same dimension d as that of the motion space of a corresponding loop. Then an original row vector composed of joint axes becomes a matrix with d number of rows. Then, $\Lambda_p M_p$ becomes $\sum d_i \times \sum d_i$ square matrix. By solving Eq.(9), joint velocities can be given as a linear function of velocities of active joints by the following equation.

$$\dot{\theta}_p = -(\Lambda_p M_p)^{-1} \Lambda_a M_a \dot{\theta}_a \quad (10)$$

Let

$$\theta = \begin{pmatrix} \theta_p \\ \theta_a \end{pmatrix} \quad (11)$$

then, Eq.(10) can be written in the following expression.

$$\dot{\theta} = \begin{bmatrix} -(\Lambda_p M_p)^{-1} \Lambda_a M_a \\ E \end{bmatrix} \dot{\theta}_a \quad (12)$$

where E is a unit matrix. From Eq.(12), all the joint velocities can be determined.

By taking the time derivative of Eq.(12), the following relation is obtained.

$$\ddot{\theta} = \begin{bmatrix} -(\Lambda_p M_p)^{-1} \Lambda_a M_a \\ E \end{bmatrix} \ddot{\theta}_a + \begin{bmatrix} -(\Lambda_p M_p)^{-1} \{ \Lambda_a \dot{M}_a - \Lambda_p \dot{M}_p (\Lambda_p M_p)^{-1} \Lambda_a M_a \} \\ 0 \end{bmatrix} \dot{\theta}_a \quad (13)$$

where,

$$\dot{M}_p = \begin{bmatrix} V_a \times M_{1,a}^p & 0 & \cdots & 0 \\ 0 & V_b \times M_{2,b}^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_c \times M_{(p-\zeta),c}^p \end{bmatrix} \quad (14)$$

$$\dot{M}_a = \begin{bmatrix} V_d \times M_{1,d}^a & 0 & \cdots & 0 \\ 0 & V_e \times M_{2,e}^a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_f \times M_{\xi,f}^a \end{bmatrix} \quad (15)$$

An element of a row of the matrices $\dot{M}_p$, and $\dot{M}_a$ is a linear combination of joint axes in a same loop and should be represented as a coordinate vector. By solving Eq.(13), accelerations of joints can be expressed as a function of velocities and accelerations of the active joints.

On the other hand, deriving the relation of Eq. (1) for all the links except a stationary link, the following equation is obtained.

$$\Phi^T \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{n-1} \end{bmatrix} = M\ddot{\theta} \quad (16)$$

where, V_i is a velocity vector of the link i . Elements of the diagonal matrix M are six dimensional vectors expressing joint axes. From a condition of joint velocities in a loop, one joint axis in it is linearly dependent to other axes. Therefore, one joint axis from each loop can be excluded. Eliminate d rows from Eq.(16) and the number of rows of the matrix Φ^T becomes $p-\alpha$. The number of columns is $n-1$ and is equal to the number of rows. Let the matrices Ψ^T and N be obtained by eliminating α number of joint axes from the matrices Φ^T and M . Then Ψ^T is a square matrix, and the velocities of links can be determined by the following equation.

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{n-1} \end{bmatrix} = \Psi^{-T} N \ddot{\theta} \quad (17)$$

The accelerations of links can be obtained by differentiating Eq.(17) :

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{bmatrix} = \Psi^{-T} N \ddot{\theta} + \Psi^{-T} \dot{N} \dot{\theta} \quad (18)$$

where, $\dot{N}$ can be derived by the same way for Eqs(14) and (15).

5. Force Analysis

First, a single loop mechanism with $d(d < 6)$ dimensional motion space will be considered. A degree of freedom ζ of the mechanism is $p-d$. Let $p-1$ wrenches acting on links be $F_i, i=1, 2, \dots, p-1$. A wrench acting on a stationary link is excluded. Wrenches $H_j, j=1, 2, \dots, p$ are acting on both links connected by a joint j . The wrench H is said to be a joint wrench. When a joint is passive, its joint wrench

must satisfy the following condition.

$$M_j \circ H_j = 0 \quad (19)$$

On the other hand, the equilibrium condition of wrenches on a link with two pair elements j and $j+1$ is given by the following equation.

$$H_j + F_i - H_{j+1} = 0 \quad (20)$$

Equations (19) and (20) are equations with unknown parameters of joint wrenches H_s . The number of vector equations given by Eq.(20) is $p-1$ and the number of linear equations given by Eq.(19) is $p-\zeta=d$. Using these equations, the values of joint wrenches must be determined first. When wrenches are represented by β dimensional coordinate vectors, the total number of linear equations given by (19) and (20) are $\beta(p-1)+d$, and the number of unknown parameters is βp . Therefore, in order to determine the values for joint wrenches, β should be equal to d . That is, wrenches should be represented by d dimensional coordinate vectors.

When a dimension of a motion space is d , there exist a $6-d$ dimensional linear space of wrenches. The work between a velocity in the motion space and a wrench in the linear space is always zero. This linear space is said to be the reciprocal space of the motion space. And a linear space of wrenches is said to be a wrench space.

When a joint wrench is expressed by a linear combination of six linearly independent wrenches, $6-d$ wrenches must be the basis vectors of a reciprocal space. The basis vectors of the reciprocal space vanishes when taking reciprocal product with any velocity vectors in the motion space. And the coefficients of remaining d basis vectors can be determined using Eqs.(19) and (20). Here, the linear space spanned by these d basis vectors is said to be a compliment space. Wrenches which are related with the motion of a mechanism can thus be represented by the basis of the compliment space.

This concept can be derived only by using the screw coordinates which represent rotational and translational velocities, and also represent a force and a couple simultaneously. The concept of the basis for representing a wrench can only be derived by considering the linear spaces of twists and wrenches.

For the analysis of planar mechanisms, wrenches have been expressed by three dimensional coordinate vectors of the basis with two forces parallel to the plane, and one moment about the perpendicular axis to the plane. Forces acting on existing planar mechanisms are not always parallel to the plane and many directional forces such as a gravity force and a inertial force are acting. However, for the kinematic and dynamic analysis, to express the wrenches which has an influence on the motion this kind of three dimen-

Fig. 3 Closed loop

sional vectors are enough. Remaining three parameters are indeterminate by a kinematic analysis. Such an concept can be used for the analysis of mechanisms having several loops with different Lie algebra.

When a mechanism is in motion, gravity forces and inertial forces are acting on links. The inertial wrench acting on the link i is designated by $\mathbf{D}_i$ and is given by the following equation⁽¹⁾.

$$\mathbf{D}_i = -\mathbf{B}\mathbf{A}_i - \mathbf{V}_i \times \mathbf{B}\mathbf{V}_i \quad (21)$$

where $\mathbf{B}$ is a 6×6 tensor said to be an inertia binor. $\mathbf{V}_i$ and $\mathbf{A}_i$ can be determined by Eqs.(17) and (18). The wrench acting on the link i is a sum of these wrenches and is designated by $\mathbf{F}_i$.

Select one loop from a mechanism. The loop l is illustrated in Fig. 3. A wrench $\mathbf{F}_i$ is acting on a link i . Joint wrenches in a loop are designated by $\mathbf{H}_s$. These wrenches are in the equilibrium condition. Let the dimension of this loop l be d_l . There are p_l joints in the loop numbered $1, 2, \dots, p_l$ ($p_l=6$ in Figure) sequentially. p_l links are also in the loop and they are also numbered $1, 2, \dots, p_l$ in the same way. Wrenches $\mathbf{F}_s$ are acting on the links and, wrenches $\mathbf{H}_{out-j}$ ($j=1, 2$ in Figure) on some of them are also acting from joints in another loop. As an example, the equilibrium equation for link ④ becomes:

$$\mathbf{H}_4 - \mathbf{H}_3 = \mathbf{F}_4 + \mathbf{H}_{out_2} \quad (22)$$

These equations can be obtained about p_l links in the loop.

Next, when equilibrium conditions for a loop are fixed, it can be considered as a link for deriving equilibrium equations for next loop. Then the equilibrium equation on it is given in the following form.

$$\sum_i \mathbf{F}_i + \sum_j \mathbf{H}_{out-j} = \mathbf{0} \quad (23)$$

The equation can be obtained by adding all the equations of (22) in a loop. Therefore, Eq.(22) and Eq. (23) are linearly dependent, and, to avoid this problem, a equilibrium equation for one link (stationary link if it is in a loop) must be excluded and the number

of equilibrium equations for a loop becomes $p_l - 1$.

Thus, the equations of the number with the small of one from the number of joints are obtained for a loop. Derive equilibrium equations for a loop, replace this loop with a link and repeat this operation to the last loop. In the last loop there should be a stationary link or a link obtained by replacing a loop with a stationary link. Therefore, the equations of the number with the small of one from the number of joints are obtained for a last loop too.

A mechanism with α loops is considered. Let the dimension and the number of joints in loops be p_l, d_l , $l=1, 2, \dots, \alpha$. Equations for a loop l are $p_l - 1$ number of vector equations and the dimension of the vectors is d_l . Then the total number c_f of equations for a mechanism become:

$$c_f = \sum_{l=1}^{\alpha} d_l (p_l - 1) \quad (24)$$

And the number c_v of equations, obtained from the condition that the work between a passive joint axis and its joint wrench is zero, become:

$$c_v = \sum_{l=1}^{\alpha} p_l - \zeta = \sum_{l=1}^{\alpha} d_l \quad (25)$$

On the other hand, the number of unknown parameters u is equal to the number of elements in joint wrenches. Joint wrenches can be represented by means of coordinate vectors based on the compliment space and this coincides the number of dimensions of the motion space. Therefore:

$$u = \sum_{l=1}^{\alpha} d_l p_l \quad (26)$$

and

$$u = c_f + c_v \quad (27)$$

are obtained. Then the number of unknown parameters and the number of conditions become equal, and the values for joint wrenches can be determined.

A joint wrench can be represented by means of six basis vectors including basis vectors of compliment space derived from the motion space of a loop to which it belongs. Only coefficients of basis vectors of the motion space are related with the motion. Furthermore, a joint wrench defined in one loop is not related with the motion of other loops. When a joint wrench belongs to the intersection of loops, coefficients of the basis vectors of the intersection are related with the motion. Thus a joint wrench represented by the basis vectors of the intersection of the loops to which it belong is enough for the analysis. A joint wrench does not have an influence on motions of loops to which it does not belong. This is because a joint wrench is closed in a loop

Based on the above concept, the calculation scheme can be derived as follows.

By writing down the equilibrium equations for all

the links except the stationary one, next expression can be obtained.

$$\Phi \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \\ \vdots \\ \mathbf{H}_p \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \vdots \\ \mathbf{F}_{n-1} \end{bmatrix} \quad (28)$$

Select one joint from each loop, and add them to the row of Eq.(28). These selected joints are said to be cut joints. When a joint $s1$ is selected from a loop, the row in which the $s1$ th element is 1 and the other elements are zero is added at the bottom of the matrix, and $\mathbf{H}_{s1}$ is added in the right hand side of Eq.(28). Add the same number of joint wrenches $\mathbf{H}_{s1} \cdots \mathbf{H}_{sa}$, and then the matrix changes to the square matrix γ . By multiplying the inverse matrix γ^{-1} , the following equation can be obtained.

$$\begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \\ \vdots \\ \mathbf{H}_p \end{bmatrix} = \gamma^{-1} \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \vdots \\ \mathbf{F}_{n-1} \\ \mathbf{H}_{s1} \\ \vdots \\ \mathbf{H}_{sa} \end{bmatrix} \quad (29)$$

Delete the rows of the active joint wrenches from Eq.(29) and renumber a remaining joint wrenches as $\mathbf{H}_a \cdots \mathbf{H}_b$. Also divide the right hand side matrix into a matrix composed of wrenches $\mathbf{F}_1 \cdots \mathbf{F}_{n-1}$ acting on links and a matrix composed of the cut joints $\mathbf{H}_{s1}, \cdots \mathbf{H}_{sa}$. Then, the following equation is obtained.

$$\begin{bmatrix} \mathbf{H}_a \\ \vdots \\ \mathbf{H}_b \end{bmatrix} = A \begin{bmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_{n-1} \end{bmatrix} + B \begin{bmatrix} \mathbf{H}_{s1} \\ \vdots \\ \mathbf{H}_{sa} \end{bmatrix} \quad (30)$$

where A is a $(p-\zeta) \times (n-1)$ matrix and B is a $(p-\zeta) \times a$ matrix. Remainings are the vectors composed of passive joint wrenches, the wrenches act on links, and cut joint wrenches. Let G be a diagonal matrix the i th element of which is the $(DM_i)^T$, where M_i is the joint axis corresponding the i th wrench of $[\mathbf{H}_a^T \cdots \mathbf{H}_b^T]^T$ to Eq.(30). Multiply the matrix G to Eq.(30). All the terms in the left hand side become zero, and the following equation is obtained.

$$GB \begin{bmatrix} \mathbf{H}_{s1} \\ \vdots \\ \mathbf{H}_{sa} \end{bmatrix} = -GA \begin{bmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_{n-1} \end{bmatrix} \quad (31)$$

The matrix GB becomes $\sum d_i \times (p-\zeta)$ and square by expressing its row by the same number of scalars as the dimension of the motion space of the loop of its row. That is, by representing the cut joint wrenches in the left hand side of Eq.(31) by means of a linear combination of $6-d$ number of basis vectors of the reciprocal space and d number of basis vectors of the complement spaces, the left hand side becomes $p-\zeta$ number of linear expressions with $p-\zeta$ number of

unknowns and the right hand side becomes a $p-\zeta$ dimensional vector with constant elements. By solving Eq.(31), the following expression is obtained.

$$\begin{bmatrix} \mathbf{H}_{s1} \\ \vdots \\ \mathbf{H}_{sa} \end{bmatrix} = -(GB)^{-1}GA \begin{bmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_{n-1} \end{bmatrix} \quad (32)$$

Thus the values for the cut joint wrenches are determined. Values for all the joint wrenches can be determined by substituting this result for Eq.(29). The torque τ required to the active joint can be obtained by a reciprocal product between its axis M_c and wrench H_c as:

$$\begin{aligned} \tau_c &= M_c \circ H_c \\ &= (DM_c)^T H_c \end{aligned} \quad (33)$$

6. Expression of Inertia Wrenches for Deriving a Equation of Motion

When inertia wrenches are expressed as functions of active joint accelerations and velocities, the equations of motion can be obtained by Eq.(33). When a sequence of joints from a stationary link to the link i is determined and their axes are $N_1, N_2, \cdots, N_q$ with joint velocities $\dot{\beta}_1, \dot{\beta}_2, \cdots, \dot{\beta}_q$ respectively, the velocity V_i and the acceleration A_i of the link i become:⁽¹⁾

$$V_i = [N_1 \ N_2 \cdots N_q] \dot{\beta} \quad (34)$$

$$\begin{aligned} A_i &= [N_1 \ N_2 \cdots N_q] \ddot{\beta} \\ &+ \dot{\beta}^T \begin{bmatrix} 0 & N_1 \times N_2 & N_1 \times N_3 & \cdots & N_1 \times N_q \\ 0 & 0 & N_2 \times N_3 & \cdots & N_2 \times N_q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N_{q-1} \times N_q \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \dot{\beta} \end{aligned} \quad (35)$$

where

$$\dot{\beta} = [\dot{\beta}_1 \ \dot{\beta}_2 \ \cdots \ \dot{\beta}_q]^T \quad (36)$$

Select the suitable row of Eq.(12), $\dot{\beta}$ can be written in the following expression.

$$\dot{\beta} = C_i \dot{\theta}_a \quad (37)$$

Therefore, V_i and A_i are expressed as follows.

$$V_i = P_i \dot{\theta}_a \quad (38)$$

$$A_i = P_i \ddot{\theta}_a + \dot{\theta}_a^T Q_i \dot{\theta}_a \quad (39)$$

where P_i is the Jacobian of link i and Q_i is a matrix whose elements are vectors. As a result, the multiplication of the inertia binor B_i and the acceleration A_i of the link i becomes:

$$B_i A_i = B_i P_i \ddot{\theta}_a + \dot{\theta}_a^T S_i \dot{\theta}_a \quad (40)$$

where, S_i is obtained by multiplying B_i to all the elements of Q_i .

On the other hand, the velocity V_i is a linear combination of the column vectors of the matrix P_i , and it can be written in the following expression.

$$V_i = \dot{\theta}_{a1} P_{i1} + \dot{\theta}_{a2} P_{i2} + \cdots + \dot{\theta}_{as} P_{is} \quad (41)$$

Therefore,

$$V_i \times B_i V_i$$

$$= \dot{\theta}_a^T \begin{bmatrix} P_1 \times B_i P_1 & P_1 \times B_i P_2 & \cdots & P_1 \times B_i P_\zeta \\ P_2 \times B_i P_1 & P_2 \times B_i P_2 & \cdots & P_2 \times B_i P_\zeta \\ \vdots & \vdots & \ddots & \vdots \\ P_\zeta \times B_i P_1 & P_\zeta \times B_i P_2 & \cdots & P_\zeta \times B_i P_\zeta \end{bmatrix} \dot{\theta}_a \quad (42)$$

Thus the inertia wrench can be expressed by the summation of a linear form of $\dot{\theta}_a$ and a quadratic form of $\dot{\theta}_a$. And the equations of motion can be obtained by taking reciprocal products between the wrenches and axes of active joints.

7. Conclusions

The computational scheme for the force analysis and the dynamic analysis of mechanisms having loops with different motion spaces is developed. First, it is shown that the degree of freedom of a mechanism can be calculated by subtracting the summation of dimensions of the motion spaces of loops from the summation of joint freedoms. It is also shown that the same number of equations as the dimension of a loop can be derived by expressing the joint axes as the coordinate vector of the basis of a motion space of a loop. Joint velocities and accelerations can be expressed as the function of those of active joints by solving these equations. Further, it is shown that the force analysis and derivation of equations of motion can be done by expressing a joint wrench as a linear combination of the basis vectors of the reciprocal space to the motion space and the basis vectors of its complement space. The procedure for the analysis is also presented.

All the procedure of the analysis shown in this paper are written in linear matrix equations in closed forms. This procedure is directly derived based on the theory of the virtual work and extra unknown parameters are not needed to be introduced. Utilizing these

characteristics, it is possible to develop a computer program for the analysis of the mechanisms. The procedure can be easily applied to the ordinal mechanisms having loops with the same motion space.

Motion spaces of the Sarrus mechanism and the Bennett mechanism are not Lie algebras. The Sarrus mechanism can be considered as the combination of two Lie algebras⁽⁶⁾ and can be applied the method described in this paper by connecting an output link and a stationary link by a hypothetical prismatic joint. The Bennett mechanism is a quite different type and its analysis requires another study.

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