Reduced-Order Nonlinear Modal Equations of Plates Based on the Finite Element Method*

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This paper presents a procedure to derive reduced-order nonlinear modal equations of plates using mode vectors obtained by the finite element method. Nonlinear finite element formulation is derived by the principle of virtual work taking into account geometrical nonlinearity. Modal analysis is applied to the nonlinear finite element equation by using mode vectors obtained by finite element analysis. Reduced-order modal equations are derived by using modal coordinates of several dominant modes and neglecting in-plane inertia. Nonlinear modal equations of rectangular plates are derived, and numerical results for the fundamental out-of-plane mode and an internal resonance show good agreement with those presented in other papers.

Key Words: Nonlinear Vibration, Finite Element Method, Modal Analysis, Reduced-Order Model

1. Introduction

Nonlinear vibration of continuous systems with geometrical nonlinearity has been studied by many researchers⁽¹⁾. The perturbation method⁽²⁾ has been widely used to analyze weakly nonlinear systems. Padvan⁽³⁾ obtained good results for continuous systems by using the perturbation method. If the nonlinearity is not weak, the Ritz method, the Galerkin method or the harmonic balance method (HBM) is generally used^{(4),(5)}. The physical meanings of solutions obtained by using those theoretical methods are always clear. However, solutions are very complicated and sometimes can not be obtained under certain kinds of boundary conditions. On the other hand, there have been many studies, such as that by Rao et al.⁽⁶⁾, based on the finite element method (FEM) taking into account geometrical nonlinearity. By using such a numerical method, a solution can be obtained under any kind of boundary conditions.

However, the physical meaning of the solution may not be clear. Methods combining theoretical and numerical approaches have recently been proposed by several researchers. Singh et al.⁽⁷⁾ obtained a single d.o.f. equation of motion by introducing a spatial deformations to a dynamic nonlinear FEM and solved the equation by numerical integration. They solved the dynamic nonlinear FEM iteratively to obtain the spatial deformations. Ribeiro and Petyt^{(8),(9)} used the hierarchical finite element method (HFEM) to obtain an accurate spatial model with a small number of d.o.f. since the shape functions for the HFEM are derived from Rodorigues' form of Legendre polynomials that satisfy the clamped boundary condition, the polynomials should be modified to apply the HFEM to other boundary conditions. Shi and Mei⁽¹⁰⁾ obtained a reduced-order model by introducing modal expansion to the nonlinear FEM. They used only linear out-of-plane modes to approximate in-plane modes.

This paper presents a procedure to derive reduced-order nonlinear modal equations of plates using mode vectors obtained by the FEM. Nonlinear finite element formulation is derived by the principle of virtual work taking into account geometrical

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$$\{\varepsilon\}_i = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_i = [\mathbf{Z}]\{\varepsilon^*\}_i, \quad (6)$$

where

$$[\mathbf{Z}] = \begin{bmatrix} 1 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & z \end{bmatrix}$$

$$\{\varepsilon^*\}_i = \{\{\varepsilon_x\}_i + \{\varepsilon_y\}_i + \{\gamma_{xy}\}_i\}, \quad (7)$$

and z is the distance from the middle plane, and $\{\varepsilon_x\}_i$, $\{\varepsilon_y\}_i$, $\{\gamma_{xy}\}_i$ and $\{\varepsilon^*\}_i$ are defined as

$$\{\varepsilon_x\}_i = \begin{Bmatrix} u_{i,x} \\ -w_{i,xx} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2}(w_{i,x})^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$= [[\hat{\varepsilon}_{xL}]_i + [\hat{\varepsilon}_{xNL}(\mathbf{V})]_i]\{\mathbf{V}\}_i, \quad (8)$$

$$\{\varepsilon_y\}_i = \begin{Bmatrix} 0 \\ 0 \\ v_{i,y} \\ -w_{i,yy} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2}(w_{i,y})^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$= [[\hat{\varepsilon}_{yL}]_i + [\hat{\varepsilon}_{yNL}(\mathbf{V})]_i]\{\mathbf{V}\}_i, \quad (9)$$

$$\{\gamma_{xy}\}_i = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_{i,y} + v_{i,x} \\ -2w_{i,xy} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ w_{i,x}w_{i,y} \\ 0 \end{Bmatrix}$$

$$= [[\hat{\gamma}_L]_i + [\hat{\gamma}_{NL}(\mathbf{V})]_i]\{\mathbf{V}\}_i, \quad (10)$$

$$\{\varepsilon^*\}_i = [[\hat{\varepsilon}_L^*]_i + [\hat{\varepsilon}_{NL}^*(\mathbf{V})]_i]\{\mathbf{V}\}_i, \quad (11)$$

where

$$[\hat{\varepsilon}_{xL}]_i = \begin{Bmatrix} \{1 \ 0 \ 0\}[N]_{i,x}[C]_i \\ -\{0 \ 0 \ 1\}[N]_{i,xx}[C]_i \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix}$$

$$[\hat{\varepsilon}_{xNL}(\mathbf{V})]_i = \begin{Bmatrix} \frac{1}{2}\{\mathbf{V}\}_i^T[C]_i^T[N]_{i,x}^T\{0 \ 0 \ 1\}^T\{0 \ 0 \ 1\}[N]_{i,x}[C]_i \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix}, \quad (12)$$

$$[\hat{\varepsilon}_{yL}]_i = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0 \ 1 \ 0\}[N]_{i,y}[C]_i \\ -\{0 \ 0 \ 1\}[N]_{i,yy}[C]_i \\ \{0\} \\ \{0\} \end{Bmatrix}$$

$$[\hat{\varepsilon}_{yNL}(\mathbf{V})]_i = \begin{Bmatrix} \{0\} \\ \{0\} \\ \frac{1}{2}\{\mathbf{V}\}_i^T[C]_i^T[N]_{i,y}^T\{0 \ 0 \ 1\}^T\{0 \ 0 \ 1\}[N]_{i,y}[C]_i \\ \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix}, \quad (13)$$

$$[\hat{\gamma}_L]_i = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{1 \ 0 \ 0\}[N]_{i,y}[C]_i + \{0 \ 1 \ 0\}[N]_{i,x}[C]_i \\ -2\{0 \ 0 \ 1\}[N]_{i,xy}[C]_i \end{Bmatrix}$$

$$[\hat{\gamma}_{NL}(\mathbf{V})]_i = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \frac{1}{2}\{\mathbf{V}\}_i^T[C]_i^T[N]_{i,x}^T\{0 \ 0 \ 1\}^T\{0 \ 0 \ 1\}[N]_{i,y}[C]_i \\ + \frac{1}{2}\{\mathbf{V}\}_i^T[C]_i^T[N]_{i,y}^T\{0 \ 0 \ 1\}^T\{0 \ 0 \ 1\}[N]_{i,x}[C]_i \\ \{0\} \end{Bmatrix} \quad (14)$$

$$[\hat{\varepsilon}_L^*]_i = [\hat{\varepsilon}_{xL}]_i + [\hat{\varepsilon}_{yL}]_i + [\hat{\varepsilon}_{NL}(\mathbf{V})]_i$$

$$[\hat{\varepsilon}_{NL}^*(\mathbf{V})]_i = [\hat{\varepsilon}_{xNL}(\mathbf{V})]_i + [\hat{\varepsilon}_{yNL}(\mathbf{V})]_i + [\hat{\gamma}_{NL}(\mathbf{V})]_i, \quad (15)$$

The stress-strain relationship may be expressed in the form

$$\{\sigma\}_i = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_i$$

$$= [\mathbf{E}^*] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_i = [\mathbf{E}^*][\mathbf{Z}]\{\varepsilon^*\}_i, \quad (16)$$

where

$$[\mathbf{E}^*] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \quad (17)$$

The work done by the inertia and internal and external forces due to a virtual displacement are respectively given by

$$\delta W_{in} = -\rho h \sum_i \iint_R \{\delta \mathbf{d}\}_i^T \{\ddot{\mathbf{d}}\}_i dx dy$$

$$= -\{\delta \mathbf{V}\}^T [\mathbf{M}] \{\ddot{\mathbf{V}}\}, \quad (18)$$

$$\begin{aligned}\delta W_V &= -\sum_i \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_R \{\delta \varepsilon\}_i^T \{\varepsilon\}_i dx dy dz \\ &= -\sum_i \int_R \{\delta \varepsilon^*\}_i^T [\mathbf{E}] \{\varepsilon^*\}_i dx dy \\ &= -\{\delta \mathbf{V}\}^T ([\mathbf{K}_1] + [\mathbf{K}_2(\mathbf{V})] + [\mathbf{K}_3(\mathbf{V})] \\ &\quad + [\mathbf{K}_4(\mathbf{V}, \mathbf{V})]) \{\mathbf{V}\},\end{aligned}\quad (19)$$

$$\delta W_{ex} = \{\delta \mathbf{V}\}^T \{\mathbf{F}\}, \quad (20)$$

where $[\mathbf{M}]$, $[\mathbf{K}_1]$ and $[\mathbf{K}_i]$ ($i=2, 3, 4$) are the mass matrix, the linear stiffness matrix and the nonlinear stiffness matrix, respectively, and defined as

$$\begin{aligned}[\mathbf{E}] &= \int_{-\frac{h}{2}}^{\frac{h}{2}} [\mathbf{Z}]^T [\mathbf{E}^*] [\mathbf{Z}] dz \\ [\mathbf{M}] &= \rho h \sum_i \int_R [\mathbf{C}]_i^T [\mathbf{N}]_i^T [\mathbf{N}]_i [\mathbf{C}]_i dx dy \\ [\mathbf{K}_1] &= \sum_i \int_R [\varepsilon_L^*]_i^T [\mathbf{E}] [\varepsilon_L^*]_i dx dy \\ [\mathbf{K}_2(\mathbf{V})] &= \sum_i \int_R [\varepsilon_L^*]_i^T [\mathbf{E}] [\varepsilon_{NL}^*]_i(\mathbf{V}) dx dy \\ [\mathbf{K}_3(\mathbf{V})] &= \sum_i \int_R 2[\varepsilon_{NL}^*]_i(\mathbf{V})^T [\mathbf{E}] [\varepsilon_L^*]_i dx dy \\ [\mathbf{K}_4(\mathbf{V}, \mathbf{V})] &= \sum_i \int_R 2[\varepsilon_{NL}^*]_i(\mathbf{V})^T [\mathbf{E}] \\ &\quad \times [\varepsilon_{NL}^*]_i(\mathbf{V}) dx dy.\end{aligned}\quad (21)$$

Matrices $[\mathbf{K}_2]$ and $[\mathbf{K}_3]$ depend on $\{\mathbf{V}\}$ linearly, $[\mathbf{K}_4]$ depends on $\{\mathbf{V}\}$ quadratically (For convenience of notation, $[\mathbf{K}_4(\mathbf{V}, \mathbf{V})]$ will be written as $[\mathbf{K}_4(\mathbf{V})]$); $[\mathbf{K}_3]$ and $[\mathbf{K}_2]$ are related by $[\mathbf{K}_3] = 2[\mathbf{K}_2]^T$.

Substituting Eqs.(18)-(20) into the principle of virtual work,

$$\delta W_{in} + \delta W_V + \delta W_{ex} = 0, \quad (22)$$

we obtain

$$\begin{aligned}[\mathbf{M}]\{\dot{\mathbf{V}}\} + ([\mathbf{K}_1] + [\mathbf{K}_2(\mathbf{V})] + [\mathbf{K}_3(\mathbf{V})] \\ + [\mathbf{K}_4(\mathbf{V})])\{\mathbf{V}\} = \{\mathbf{F}\}.\end{aligned}\quad (23)$$

Considering harmonic excitation,

$$\{\mathbf{F}\} = \{\mathbf{f}\} \cos \omega t, \quad (24)$$

and only the response of fundamental harmonics,

$$\{\mathbf{V}\} = \{\mathbf{v}\} \cos \omega t, \quad (25)$$

the acceleration is obtained as

$$\{\ddot{\mathbf{V}}\} = -\omega^2 \{\mathbf{V}\}. \quad (26)$$

Substituting Eq.(26) into Eq.(23) results in

$$\begin{aligned}(-\omega^2 [\mathbf{M}] + [\mathbf{K}_1] + [\mathbf{K}_2(\mathbf{V})] + [\mathbf{K}_3(\mathbf{V})] \\ + [\mathbf{K}_4(\mathbf{V})])\{\mathbf{V}\} = \{\mathbf{F}\}.\end{aligned}\quad (27)$$

3. Introduction of Modal Coordinates

Using the in-plane modal matrix $[\phi]_{pi}$, the out-of-plane modal matrix $[\varphi]_{po}$, the in-plane modal coordinates vector $[\xi]$ and the out-of-plane modal coordinates vector $[\eta]$, the node value vector $\{\mathbf{V}\}$ is assumed to have the form

$$\begin{aligned}\{\mathbf{V}\} &= [[\phi]_{pi} [\varphi]_{po}] \begin{Bmatrix} \{\xi\} \\ \{\eta\} \end{Bmatrix} \\ &= [\phi] \{\xi\},\end{aligned}\quad (28)$$

where pi , po are the number of modes. Substituting Eq.(28) into Eq.(23) results in

$$[\mathbf{M}][\phi] \{\ddot{\xi}\} + ([\mathbf{K}_1] + [\mathbf{K}_2(\xi)] + [\mathbf{K}_3(\xi)]$$

$$+ [\mathbf{K}_4(\xi)])[\phi] \{\xi\} = \{\mathbf{F}\}. \quad (29)$$

Multiplying Eq.(29) by $[\phi]^T$, we obtain

$$\begin{aligned}[\mathbf{m}]\{\ddot{\xi}\} + ([\mathbf{k}] + [\mathbf{k}_{m2}(\xi)] + [\mathbf{k}_{m3}(\xi)] + [\mathbf{k}_{m4}(\xi)])\{\xi\} \\ = [\phi]^T \{\mathbf{F}\},\end{aligned}\quad (30)$$

or from Eq.(27),

$$\begin{aligned}(-\omega^2 [\mathbf{m}] + [\mathbf{k}] + [\mathbf{k}_{m2}(\xi)] + [\mathbf{k}_{m3}(\xi)] \\ + [\mathbf{k}_{m4}(\xi)])\{\xi\} = [\phi]^T \{\mathbf{F}\},\end{aligned}\quad (31)$$

where

$$\begin{aligned}[\mathbf{m}] &= [\phi]^T [\mathbf{M}] [\phi], \\ [\mathbf{k}] &= [\phi]^T [\mathbf{K}_1] [\phi], \\ [\mathbf{k}_{m2}(\xi)] &= [\phi]^T [\mathbf{K}_2(\xi)] [\phi], \\ [\mathbf{k}_{m3}(\xi)] &= [\phi]^T [\mathbf{K}_3(\xi)] [\phi], \\ [\mathbf{k}_{m4}(\xi)] &= [\phi]^T [\mathbf{K}_4(\xi)] [\phi].\end{aligned}\quad (32)$$

We call the method based on Eq.(30) Modal FEM in the following analysis.

4. Applications

In this section, we apply the modal FEM to a simply supported plate (plate 1), a clamped plate (plate 2) and a clamped plate with 1:3 internal resonance (plate 3). Table 1 shows the specifications of the plates. The present method is examined by comparing the results obtained by other researchers with the results obtained by the modal FEM using Eq.(30). Table 2 shows the magnitudes of external forces. For plates 1 and 2, concentrated external forces acts on the center of the plate in the z direction, and the force is assumed to be distributed in the case of plate 3. We investigate nonlinear resonance about the fundamental out-of-plane mode in detail.

Table 1 Specifications of the plates

Plate 1	
Length-x	$a = 0.3$ [m]
Length-y	$b = 0.3$ [m]
Thickness	$h = 0.001$ [m]
Young's Modulus	$E = 70$ [GPa]
Density	$\rho = 2778$ [kg/m ³]
Boundary Condition	SSSS
Number of Elements	$16 \times 16 \times 2$
Plate 2	
Length-x	$a = 0.3$ [m]
Length-y	$b = 0.3$ [m]
Thickness	$h = 0.001$ [m]
Young's Modulus	$E = 70$ [GPa]
Density	$\rho = 2778$ [kg/m ³]
Boundary Condition	CCCC
Number of Elements	$20 \times 20 \times 2$
Plate 3	
Length-x	$a = 0.3$ [m]
Length-y	$b = 0.32$ [m]
Thickness	$h = 0.001$ [m]
Young's Modulus	$E = 70$ [GPa]
Density	$\rho = 2778$ [kg/m ³]
Boundary Condition	CCCC
Number of Elements	$20 \times 20 \times 2$

Table 2 Specifications of the external forces

Plate 1	0.2 [N]	at $(x,y) = (a/2, b/2)$
Plate 2	0.2 [N]	at $(x,y) = (a/2, b/2)$
Plate 3	5.0 [N/m ²]	

4.1 Plates 1 and 2 without internal resonance

4.1.1 Modal equation Based on the authors' previous work⁽¹²⁾, several dominant modes to derive the reduced-order modal equation can be obtained by a systematic procedure. The following is an outline of the procedure and derivation of the modal equation. The frequency range of ω_l to ω_u is considered.

(i) From Eq.(27), an eigenvalue equation of linear free vibration is given by

$$(-\omega^2[\mathbf{M}] + [\mathbf{K}_l])\{\mathbf{V}\} = 0. \quad (33)$$

From Eq.(33), the linear fundamental out-of-plane mode vector $\{\varphi_l\}$ and natural frequency ω_l are obtained.

(ii) Equation (31) is rewritten by using only the linear fundamental out-of-plane mode vector, $[\phi] = [\varphi]_1 = \{\varphi_l\}$. Assuming that $\eta (= \{\xi\})$ is independent of time, Eq.(31) can be treated as a nonlinear algebraic equation, and $\eta|_{\omega_u}$ is obtained by applying the Newton-Raphson method to the equation for the frequency range $\omega_l \leq \omega_1 \leq \omega_u$.

(iii) Equation (27) is assumed to be a nonlinear algebraic equation, and $\{\mathbf{V}\}|_{\omega_u}$ is obtained by applying the Newton-Raphson method to the equation at $\omega = \omega_u$ with the initial condition $\{\mathbf{V}\}_{ini} = \{\varphi_l\}\eta|_{\omega_u}$.

(iv) Nonlinear in-plane pseudo-mode vector $\{\phi^*\} (= [\phi]_1)$ is obtained by making out-of-plane components of $\{\mathbf{V}\}|_{\omega_u}$ zero.

(v) $[\phi]$ is obtained as $[[\phi]_1, [\varphi]_1]$.

By using $[\phi]$ and $\{\xi\} = \{\xi_1\eta_1\}^T$, Eq.(30) is rewritten in the form

$$\begin{bmatrix} m_i & 0 \\ 0 & m_o \end{bmatrix} \begin{bmatrix} \ddot{\xi}_1 \\ \ddot{\eta}_1 \end{bmatrix} + \begin{bmatrix} k_{1i} & 0 \\ 0 & k_{1o} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix} + \begin{bmatrix} 0 & k_{2io}\eta_1 \\ 2k_{2io}\eta_1 & k_{4o}\eta_1^2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}. \quad (34)$$

Neglecting the in-plane inertia and eliminating ξ_1 from Eq.(34), we obtain

$$m_o \ddot{\eta}_1 + k_{1o}\eta_1 + \beta\eta_1^3 = f, \quad (35)$$

where

$$\beta = k_{4o} - 2 \frac{k_{2io}^2}{k_{1i}}. \quad (36)$$

4.1.2 Comparison with results in other papers

Convergence characteristics of the fundamental natural frequencies obtained by Eq.(33) and coefficients in Eq.(35) for plate 1 and plate 2 calculated with $\omega_u = 2\pi \times 75 \approx 1.42\omega_1$ for plate 1 and $\omega_u = 2\pi \times 112 \approx 1.16\omega_1$ for plate 2 are given in Tables 3 and 4 and in Tables 5 and 6, respectively. The difference between two adjacent values is shown as $|\Delta|[\%]$. With an increase in the number of elements, each value converges gradually.

Table 3 Convergence characteristics of linear natural frequency for plate 1

$N_x \times N_y$	ω_l [rad/s]	$ \Delta \omega_l $ [%]
8×8	329.03	
12×12	330.70	0.506
16×16	331.33	0.191
20×20	331.64	0.092

Table 4 Convergence characteristics of coefficients of Eq.(35) for plate 1

$N_x \times N_y$	m_o [kg]	$ \Delta m_o $ [%]
8×8	6.2913×10^{-2}	
12×12	6.2717×10^{-2}	0.313
16×16	6.2632×10^{-2}	0.134
20×20	6.2591×10^{-2}	0.067
$N_x \times N_y$	k_{1o} [N/m]	$ \Delta k_{1o} $ [%]
8×8	6.8110×10^3	
12×12	6.8590×10^3	0.699
16×16	6.8760×10^3	0.248
20×20	6.8841×10^3	0.118
$N_x \times N_y$	β [N/m ³]	$ \Delta \beta $ [%]
8×8	10.0481×10^9	
12×12	9.8177×10^9	2.347
16×16	9.7289×10^9	0.913
20×20	9.6861×10^9	0.442

Table 5 Convergence characteristics of linear natural frequency for plate 2

$N_x \times N_y$	ω_l [rad/s]	$ \Delta \omega_l $ [%]
12×12	602.30	
16×16	604.09	0.298
20×20	604.91	0.136
24×24	605.34	0.040

Table 6 Convergence characteristics of coefficients of Eq.(35) for plate 2

$N_x \times N_y$	m_o [kg]	$ \Delta m_o $ [%]
12×12	4.1372×10^{-2}	
16×16	4.1308×10^{-2}	0.156
20×20	4.1276×10^{-2}	0.077
24×24	4.1260×10^{-2}	0.040
$N_x \times N_y$	k_{1o} [N/m]	$ \Delta k_{1o} $ [%]
12×12	1.5008×10^4	
16×16	1.5074×10^4	0.439
20×20	1.5104×10^4	0.195
24×24	1.5119×10^4	0.100
$N_x \times N_y$	β [N/m ³]	$ \Delta \beta $ [%]
12×12	8.0396×10^9	
16×16	7.8835×10^9	1.979
20×20	7.8083×10^9	0.963
24×24	7.7665×10^9	0.538

From Tables 4 and 6, the coefficients in Eq.(35) converge sufficiently if we choose 16×16 and 20×20 models, respectively. Figure 2 and Table 7 show the backbone curve obtained by applying elliptic integra-

Table 7 Frequency ratios for plate 1

η_1/h	ω/ω_1						
Present	Ref.(6)	Ref.(10)	Ref.(13)	Ref.(14)	Ref.(16)	Ref.(17)	Ref.(18)
0.2	1.0210	1.0185	1.0218	1.0197	1.0195	1.0196	1.0189
0.4	1.0811	1.0717	1.0832	1.0758	1.0765	1.0763	1.0771
0.6	1.1737	1.1534	1.1774	1.1625	1.1658	1.1645	1.1669
0.8	1.2914	1.2566	1.2973	1.2737	1.2796	1.2779	1.2813
1.0	1.4277	1.3753	1.4363	1.4025	1.4163	1.4109	1.4141

Fig. 2 Backbone curve for plate 1

Fig. 3 Backbone curve for plate 2

tion to Eq.(35) using a 16×16 model, Figure 3 and Table 8 show the backbone curve obtained when a 20×20 model is used. The present results show good agreement with those presented in other papers.

4.2 Plate 3 with internal resonance

4.2.1 Modal equation The 1st and 5th natural frequencies of plate 3 are related by $3\omega_1 \approx \omega_5$, so we calculate $\{\phi_i^*\}$, $\{\phi_5^*\}$ as well as $\{\phi_i^*\}$, $\{\phi_5^*\}$ and obtain $[\phi]$ as $[\{\phi_i^*\}\{\phi_5^*\}\{\phi_i^*\}\{\phi_5^*\}]$. By using $[\phi]$ and $\{\xi\} = \{\xi_1 \xi_5 \eta_1 \eta_5\}^T = \{\{\xi\}^T \{\eta\}^T\}^T$, Eq.(30) is rewritten in the form

$$\begin{aligned} & \begin{bmatrix} [m_i] & [0] \\ [0] & [m_o] \end{bmatrix} \begin{Bmatrix} \{\dot{\xi}\} \\ \{\dot{\eta}\} \end{Bmatrix} + \begin{bmatrix} [k_{1i}] & [0] \\ [0] & [k_{1o}] \end{bmatrix} \\ & + \eta_1 \begin{bmatrix} [0] & [k_{22}] \\ 2[k_{22}]^T & [0] \end{bmatrix} + \eta_2 \begin{bmatrix} [0] & [k_{23}] \\ 2[k_{23}]^T & [0] \end{bmatrix} \\ & + \eta_1^2 \begin{bmatrix} [0] & [0] \\ [0] & [k_{47}] \end{bmatrix} + \eta_1 \eta_2 \begin{bmatrix} [0] & [0] \\ [0] & [k_{48}] \end{bmatrix} \end{aligned}$$

Table 8 Frequency ratios for plate 2

η_1/h	ω/ω_1				
	Present	Ref. (6)	Ref. (15)	Ref. (16)	Ref. (17)
0.2	1.0077	1.0070	1.0070	1.0073	1.0072
0.4	1.0305	1.0276	1.0276	1.0291	1.0286
0.6	1.0672	1.0608	1.0607	1.0648	1.0632
0.8	1.1163	1.1047	1.1044	1.1138	1.1098
1.0	1.1761	1.1578	1.1573	1.1762	1.1667

Table 9 Convergence characteristics of 1st linear natural frequency for plate 3

$N_x \times N_y$	ω_1 [rad/s]	$ \Delta \omega_1 $ [%]
12×12	566.37	
16×16	568.07	0.300
20×20	568.85	0.137
24×24	569.25	0.071

Table 10 Convergence characteristics of 5th linear natural frequency for plate 3

$N_x \times N_y$	ω_5 [rad/s]	$ \Delta \omega_5 $ [%]
12×12	1975.0	
16×16	1976.2	0.061
20×20	1977.4	0.062
24×24	1978.2	0.043

$$+ \eta_2^2 \begin{bmatrix} [0] & [0] \\ [0] & [k_{49}] \end{bmatrix} \begin{Bmatrix} \{\xi\} \\ \{\eta\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{f\} \end{Bmatrix}. \quad (37)$$

Assuming that the in-plane modal matrix $[\phi]_2 = [\{\phi_i^*\}\{\phi_5^*\}]$ is orthogonal to $[M]$ and $[K_1]$, $[m_i]$ and $[k_{1i}]$ become diagonal matrices. Neglecting the in-plane inertia and eliminating $\{\xi\}$ from Eq.(37), we obtain

$$[m_o]\{\ddot{\eta}\} + ([k_{1o}] + [\beta])\{\eta\} = \{f\}, \quad (38)$$

where

$$\begin{aligned} [\beta] = & \eta_1^2 [k_{47}] + \eta_1 \eta_2 [k_{48}] + \eta_2^2 [k_{49}] - 2(\eta_1 [k_{22}] \\ & + \eta_2 [k_{23}]^T [k_{1i}]^{-1} (\eta_1 [k_{22}] + \eta_2 [k_{23}])). \end{aligned} \quad (39)$$

4.2.2 Comparison with results in other papers

Tables 9 and 10 show convergence characteristics of the 1st and 5th natural frequencies with an increase in the number of elements. As shown by the results for plates 1 and 2, an accurate modal equation can be obtained if the variation of natural frequency obtained by Eq.(33) with an increase in number of elements is less than 0.2% . Therefore, we can obtain an accurate 2 d.o.f. modal equation for the 1st and 5th modes by using a 20×20 model. By calculating Eq.(38) with $\omega_u = 2\pi \times 104 \approx 1.15\omega_1$ for the 1st mode and with $\omega_u = 2\pi \times 312 \approx 3 \times 1.15\omega_1$ for the 5th mode by using a 20×20 model and by adding a mass proportional damping term as same as the term used by Ribeiro and Petyt⁽⁸⁾ to Eq.(38), we obtain

$$[m_o]\{\ddot{\eta}\} + \frac{0.01\omega_1^2}{\omega} [m_o]\{\dot{\eta}\} + ([k_{1o}] + [\beta])\{\eta\} = \{f\}. \quad (40)$$

Fig. 4 Amplitude of 1st harmonic of plate 3 at $(x, y) = (a/2, b/2)$

Fig. 6 Amplitude of 3rd harmonic of plate 3 at $(x, y) = (a/2, b/2)$

Fig. 5 Amplitude of 1st harmonic of plate 3 at $(x, y) = (a/4, b/4)$

Fig. 7 Amplitude of 3rd harmonic of plate 3 at $(x, y) = (a/4, b/4)$

Considering harmonic excitation,

$$\{\mathbf{f}\} = \{\hat{\mathbf{f}}\} \cos \omega t, \quad (41)$$

the steady state response is assumed to have the form

$$\{\eta\} = \begin{Bmatrix} \eta_{1c1} \\ \eta_{5c1} \end{Bmatrix} \cos \omega t + \begin{Bmatrix} \eta_{1s1} \\ \eta_{5s1} \end{Bmatrix} \sin \omega t + \begin{Bmatrix} \eta_{1c3} \\ \eta_{5c3} \end{Bmatrix} \cos 3\omega t + \begin{Bmatrix} \eta_{1s3} \\ \eta_{5s3} \end{Bmatrix} \sin 3\omega t. \quad (42)$$

Substituting Eqs.(41) and (42) into Eq.(40) and applying HBM, we obtain the amplitudes of the 1st and 3rd harmonics. Figures 4 - 7 show the results of HBM, where

$$W_1 = \sqrt{(w_1^i \eta_{1c1} + w_5^i \eta_{5c1})^2 + (w_1^i \eta_{1s1} + w_5^i \eta_{5s1})^2}, \quad (43)$$

$$W_3 = \sqrt{(w_1^i \eta_{1c3} + w_5^i \eta_{5c3})^2 + (w_1^i \eta_{1s3} + w_5^i \eta_{5s3})^2}, \quad (44)$$

and w_j^i means the value of w at node j in $\{\varphi_i\}$. FRFs obtained by the modal FEM are almost the same as the results obtained by Ribeiro and Petyt⁽⁸⁾. Therefore, coefficients in Eq.(38) for the clamped plate with internal resonance obtained from a 20×20 model can be approximated very well by using the present method.

5. Conclusions

Modal analysis was applied to a nonlinear finite element equation by using mode vectors obtained by finite element analysis. Reduced-order modal equations were derived by using modal coordinates of several dominant modes and neglecting in-plane inertia. Dominant linear out-of-plane modes are obtained by solving an eigenvalue equation of linear free vibration and dominant nonlinear in-plane pseudo-modes were obtained by an approximate approach using the Newton-Raphson method. Nonlinear modal equations of rectangular plates were derived, and the numerical results for the fundamental out-of-plane mode and an internal resonance showed good agreement with results presented in other papers. Thus, the present method can be applied to nonlinear vibration problems of continuous systems with complex boundary conditions.

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