A Step-by-Step Integration Scheme Utilizing the Cardinal B-splines*

Takumi INOUE** and Atsuo SUEOKA**

The authors present a new step-by-step integration scheme by utilizing the cardinal B-splines. The new method organizes conventional implicit methods such as Newmark-β method and Wilson-θ method and so on, and provides a simple computation procedure so that the step-by-step integration can be carried out efficiently. In addition, when we analyze a nonlinear system with discontinuity the computational accuracy can be improved by the approximate detection of the discontinuous points by making good use of the two-scale relation. In this paper, we formulate an algorithm of a time historical response analysis of a straight-line beam structure as an elementary example of multi-degree of freedom system besides a simple single degree of freedom system. The cardinal B-splines used here are only the ones of orders 3 and 4 but the other cardinal B-splines are also available for the step-by-step integration in the same way.

**Key Words**: Computer Aided Analysis, Transient Response, Nonlinear Vibration, Cardinal B-splines, Step-by-Step Time Integration, Discontinuous Functions

1. Introduction

Numerical simulations such as seismic responses of large scaled and complicated structures are frequently carried out in cooperation with the development of computers\(^\text{1-10}\). Various step-by-step integration schemes for the numerical simulations have been proposed\(^\text{11-22}\). Particularly some methods classified into implicit methods such as the Newmark-β or the Wilson-θ method are frequently employed. The implicit method is good for a practical use because it provides a stable solution in the case of a linear system and the accuracy of the solution satisfy usual requirements in engineering. However, it still involves a great deal of computational time to deal with a large scaled structure and often happens a numerical divergence when a strong nonlinearity exists in a system\(^\text{5\&6}\).

In this paper, a new step-by-step integration scheme of an ordinary differential equation is proposed by applying the cardinal B-splines\(^\text{7\&8}\). The cardinal B-splines are used in interpolations, boundary value problems and wavelet basis. A distinctive feature of this method is an algorithm that the response is represented as a linear series of a cardinal B-spline and the coefficients of the series are subsequently obtained at each time step. This algorithm brings a successful simplification in the numerical algorithm and improves the computational speed compared with the conventional implicit methods. The present method provides various types of step-by-step integration schemes according to the order of cardinal B-splines. Some of these schemes are equal to such step-by-step integration schemes as the Newmark-β and Wilson-θ method. In other word, the present method is regarded as a reconsideration of the conventional implicit methods and unifies these operations.

Two-scale relation\(^\text{9\&10}\) is an inherent character of the cardinal B-splines. When a system contains such a strong nonlinearity as having a discontinuity, the
present method is able to find an approximate point that the step-by-step response passes through the discontinuity by utilizing the two-scale relation. This device prevents a numerical solution from diverging which often occurs in the use of conventional method and exceedingly contributes to the improvement of the computational accuracy.

2. Cardinal B-splines

Functions called \( n \)-th order cardinal spline are organized by evenly spaced connections of \( (n-1) \)-th order polynomial functions and \( (n-2) \)-th differentiable at each connecting point. The basis of the space formed by the \( n \)-th order cardinal splines are called \( n \)-th order cardinal B-splines. The \( n \)-th order cardinal B-spline \( N_n(x) \) only has non-zero values at \( 0 < x < n \) and consists of \( (n-1) \)-th piecewise continuous functions which are different from one another at each section \( i < x < i+1 \) (\( i = 0, \ldots, n-1 \)). This function is represented in unified form as follows although it is a piecewise continuous function.

\[
N_n(x) = \frac{1}{(n-1)!} \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} (x-i)^{i-1}
\]  

where the function \((x-i)^{i-1}\) is called truncated powers and the original form \(x^i\) is defined as:

\[
x^i = \begin{cases} 
\max(0, x) & n = 1, 2, \ldots \\
(x-1)^{i-1} & n = 2, 3, \ldots
\end{cases}
\]  

For example, the 4th order cardinal B-spline \( N_4(x) \) consists of following piecewise cubic functions and depicted in Fig. 1(a).

\[
\begin{align*}
0, & \quad x < 0, 4 \leq x \\
\frac{x^3}{6}, & \quad 0 \leq x < 1 \\
\frac{-3x^3+12x^2-12x+4}{6}, & \quad 1 \leq x < 2 \\
\frac{3x^3-24x^2+60x-44}{6}, & \quad 2 \leq x < 3 \\
\frac{-x^3+12x^2-48x+64}{6}, & \quad 3 \leq x < 4
\end{align*}
\]  

The first and second derivatives of \( N_n(x) \), \( N'_n(x) \) and \( N''_n(x) \) are derived from differentiating Eq.(3). The functions are piecewise quadratic and piecewise linear function as shown Fig. 1(b) and (c) respectively. As to the 3rd order cardinal B-spline \( N_3(x) \), its derivatives \( N'_3(x) \) and \( N''_3(x) \) are piecewise quadratic, linear and constant function as shown Fig. 1(d), (e) and (f) respectively.

On the other hand, the cardinal B-splines have an inherent character, two-scale relation given as:

\[
N_n(x) = \sum_{i=0}^{n} p_i N_n(2x-i), \quad p_i = \frac{1}{2^{n-i}} \binom{n}{i}
\]  

3. Application to the Step-by-Step Integration

Although the cardinal B-splines could be defined infinitely according to the order \( n \) of Eq.(1), the step-by-step integration schemes discussed here are restricted to the ones utilizing the 3rd and the 4th order cardinal B-splines because these are widely applicable to practical use. An algorithm formulated in this section is mainly based on the 4th order cardinal B-spline and deals with a single degree of freedom system (called SDOF system).

3.1 Step-by-step integration scheme

An equation of motion of a linear SDOF system is

\[
md(t) + cd(t) + kd(t) = q(t)
\]  

where \( m, c, k \), \( d(t) \) and \( q(t) \) are mass, viscous damping coefficient, spring constant, displacement and external force respectively. The system parameters \( m, c \) and \( k \) are assumed to be constant.

In this paper, the displacement \( d(t) \) is represented as a series of the cardinal B-spline. If the 4th order cardinal B spline is adopted, the displacement \( d(t) \) is defined as follows.

\[
d(t) = \sum a_i N_i(t)
\]  

A diagrammatic representation of Eq.(6) is given in Fig. 2.

The displacement \( d(t) \), which is illustrated as a bold broken line, represents a superimposition of the 4th order cardinal B splines. Displacements at discrete time \( d_i = d(t_i) \) are successively obtained through the step-by-step time integration scheme, where \( t_i = i \Delta t \) (\( i = 1, 2, 3, \ldots \)) is a discrete time of \( i \)-th step. Only the three 4th order cardinal B splines illustrated as

\[
\begin{align*}
da_1 N_i(t/\Delta t - i+2) & \quad T_1 \\
da_2 N_i(t/\Delta t - i+1) & \quad T_2 \\
da_3 N_i(t/\Delta t - i) & \quad T_3
\end{align*}
\]  

Fig. 2 Schematic diagram of displacement
solid lines in Fig. 2 define the displacement $d_i$ by taking the domain of $N_i(x)$ into consideration. It follows that the displacement $d_i$ can be represented as a following simple form by substituting actual values of $N_i(x)$.

$$d_i=a_{i-1}N_i(1)+a_{i-2}N_i(2)+a_{i-3}N_i(3)-\frac{(a_{i-1}+4a_{i-2}+a_{i-3})}{6}$$

(7)

Velocity $\ddot{d}_i=\ddt{d}(t)$ and acceleration $\dddot{d}_i=\ddt{\ddt{d}(t)}$ are also represented in the same manner by the actual values of $N_i(x)$ and $N_i'(x)$, respectively.

$$\ddot{d}_i=(a_{i-2}-a_{i-3})/2\ddt{t}$$

(8)

$$\dddot{d}_i=(a_{i-3}-2a_{i-2}+a_{i-3})/\ddt{t}^2$$

(9)

Equations (7)–(9) show that physical quantities $d_i$, $\ddot{d}_i$ and $\dddot{d}_i$ are represented as a combinations of $a_{i-1}$, $a_{i-2}$ and $a_{i-3}$. On the other hand, the inverse relations to Eqs. (7)–(9) are also derived as follows.

$$a_{i-1}=d_i+\ddt{d}_i+\ddt{t}^2\dddot{d}_i/3, \quad a_{i-2}=d_i-\ddt{t}^2\dddot{d}_i/6$$

(10)

It is obvious that the relationship ($d_i$, $\ddot{d}_i$, $\dddot{d}_i$) and ($a_{i-1}$, $a_{i-2}$, $a_{i-3}$) is a one-to-one linear mapping from Eqs. (7)–(10). For example, the coefficients $a_{i-1}$, $a_{i-2}$ and $a_{i-3}$ are known at the beginning of the integration ($i=0$). These coefficients are given by Eq. (10) with replacing $i$ by $0$, where the displacement $d_0$ and velocity $\ddot{d}_0$ are given as initial values, and the acceleration $\dddot{d}_0$ is derived from substituting $d_0$ and $\ddot{d}_0$ into Eq. (5).

Supposing the responses $d_i$, $\ddot{d}_i$ and $\dddot{d}_i$ are known, the coefficients $a_{i-1}$, $a_{i-2}$ and $a_{i-3}$ will be also known from Eq. (10). Then, the responses of the next time step $d_{i+1}$, $\ddot{d}_{i+1}$ and $\dddot{d}_{i+1}$ are sought. Calculating these values corresponds to the calculation of the coefficients $a_i$, $a_{i-1}$ and $a_{i-2}$. Since $a_{i-1}$ and $a_{i-2}$ are already known, the coefficient calculated at the next time step is only $a_i$. This is calculated from following equation, which is derived from substituting Eqs. (7)–(9) into Eq. (5) with replacing $i$ to $i+1$.

$$\left(\frac{m}{\ddt{t}^2}+\frac{c}{\ddt{t}}+\frac{k}{6}\right)a_i=q(t_{i+1})+\left(\frac{2m}{\ddt{t}^2}+\frac{2k}{3}\right)a_{i-1}-\left(\frac{m}{\ddt{t}^2}+\frac{c}{\ddt{t}}+\frac{k}{6}\right)a_{i-2}$$

(11)

Once the coefficient $a_i$ is obtained, it immediately follows that the responses $d_{i+1}$, $\ddot{d}_{i+1}$ and $\dddot{d}_{i+1}$ are obtained from Eqs. (7)–(9) with replacing $i$ to $i+1$. In other words, obtaining the responses $d_{i+1}$, $\ddot{d}_{i+1}$ and $\dddot{d}_{i+1}$ is replaced by obtaining the coefficient $a_i$. The step-by-step procedure begins with the initial coefficients $a_{i-1}$, $a_{i-2}$ and $a_{i-3}$ then recurrent process to obtain $a_i(i=0, 1, 2, \cdots)$ follows [Eq. (11)].

Although the conventional implicit methods obtain $d_{i+1}$, $\ddot{d}_{i+1}$ and $\dddot{d}_{i+1}$ directly from the responses at previous time step, the present method uses the coefficients of the spline series intermediatively and the step-by-step procedure is carried out only on the coefficients $a_i$. This unique style of the algorithm simplifies the process of the calculation and makes the computational speed increase. In addition, full physical quantities $d_{i+1}$, $\ddot{d}_{i+1}$ and $\dddot{d}_{i+1}$ are not necessarily calculated from Eqs. (7)–(9). Only some of them needed in actual use should be calculated. This contributes to computational efficiency furthermore.

In the step-by-step integration scheme using the $n$-th order cardinal B-splines, a recurrent formula to calculate the coefficients $a_0, a_1, \cdots$ is generally represented as follows

$$h_{i+1}=q(t_{i+1})+h_{i-1}a_{i-1}+\cdots+h_{i-n}a_{i-n+1}+\dddot{h}$$

(12)

The actual values for $h_i \sim h_n, \dddot{h}$ are immediately obtained from Eq. (11) and listed in $N_i$ line in Table 1.

### 3.2 3rd order cardinal B-spline

In the case of using the 3rd order cardinal B-spline, the formulation of the algorithm is basically the same as the one using the 4th order. The displacement $d(t)$ is defined by replacing $N_i$ by $N_i$ in Eq. (6) and the discrete displacement $d_i$ and velocity $\ddot{d}_i$ are represented as follows.

$$d_i=\frac{(a_{i-1}+a_{i-2})}{2}, \quad \ddot{d}_i=\frac{(a_{i-1}-a_{i-2})}{\ddt{t}}$$

(13)

The acceleration is a step function depicted in Fig. 1 (f) and hold a constant value $\left(\frac{a_{i-1}-2a_{i-2}+a_{i-3}}{\ddt{t}^2}\right)$ at $t_i<t<i+1$. The constant acceleration between $t_i$ and $t_{i+1}$ is assumed to be a mean value between $d_i$ and $\ddot{d}_{i+1}$ as:

$$\left(\ddot{d}_i+\ddot{\ddot{d}}_{i+1}\right)/2=\frac{(a_{i-1}-2a_{i-2}+a_{i-3})}{\ddt{t}^2}$$

(14)

Solving $\ddot{d}_{i+1}$ from Eq. (14), replacing $i$ to $i+1$ and substituting them into Eq. (5) gives the same form as Eq. (12) to provide $a_i$. The values of $h_i$ etc. are listed in $N_i$ line in Table 1.

In the case of using the 3rd order cardinal B-spline, the acceleration $d_i$ is involved in the step-by-step integration as shown in Table 1. It can be given from Eq. (14) by replacing $i$ to $i-1$ after calculating

<table>
<thead>
<tr>
<th>$N_3$</th>
<th>$h_0$</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$h_3$</th>
<th>$\dddot{h}$</th>
</tr>
</thead>
</table>
| $N_4$ | 2m + c + k/2 | 4m + c + k/2 | 2m + 2k | 0 | m $\ddt{d}_i$
| $N_{4b}$ | m + c + k/2 | 2m + 2k | m + c + k/2 | m + c + k/2 | 0 |

<table>
<thead>
<tr>
<th>$N_3$</th>
<th>$h_0$</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$h_3$</th>
<th>$\dddot{h}$</th>
</tr>
</thead>
</table>
| $N_4$ | 2m + c + k/2 | 4m + c + k/2 | 2m + 2k | 0 | m $\ddt{d}_i$
| $N_{4b}$ | m + c + k/2 | 2m + 2k | m + c + k/2 | m + c + k/2 | 0 |

**Table 1** Coefficients for different solution schemes

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The initial acceleration is given by Eq. (5) and the initial coefficients \( a_{-1} \) and \( a_{-2} \) are given by:
\[
\begin{align*}
a_{-1} &= d_0 + \Delta t d_0/2, \\
a_{-2} &= d_0 - \Delta t d_0/2.
\end{align*}
\] (15)

### 3.3 Relation to the conventional integration schemes
Replacing \( i \) by \( i+1 \) on \( a_{i-2} \) and \( a_{i-3} \) of Eq. (10) gives
\[
\begin{align*}
a_{i-1} &= d_{i-1} - \Delta t^2 d_{i-1}/6, \\
a_{i-2} &= d_{i-1} - \Delta t^2 d_{i-2} + \Delta t^2 d_{i-1}/3
\end{align*}
\] (16)

Equating \( a_{i-2} \) and \( a_{i-3} \) in Eq. (10) and (16) yields following relations.
\[
\begin{align*}
d_{i-1} &= d_i + \Delta t d_i + \Delta t^2 (d_{i-1} + 2d_i)/6, \\
d_{i-1} &= d_i + \Delta t (d_{i-1} + d_i)/2.
\end{align*}
\] (17)

This formula corresponds to the relationship of the Newmark-\( \beta \) method of \( \beta = 1/6 \). The acceleration of the 4th order polynomial B-spline is a discontinuous linear function as shown in Fig. 1(c), so that the physical property of the present method is equivalent to that of the Newmark-\( \beta \) method of \( \beta = 1/6 \) in which the acceleration is assumed to vary linearly between \( t_i \) and \( t_{i+1} \).

The present method using the 3rd order cardinal B-spline, which assures the constant acceleration between \( t_i \) and \( t_{i+1} \) to be a mean value between \( d_i \) and \( d_{i+1} \), similarly brings the relationship of the Newmark-\( \beta \) method of \( \beta = 1/4 \).

The present method yields various types of step-by-step integration schemes according to the order of cardinal B-splines. Since a portion of them, the ones using 3rd and 4th order cardinal B-splines, are equivalent to the Newmark-\( \beta \) method of \( \beta = 1/4 \) or \( 1/6 \) respectively, it can be said that the present method includes some of the conventional implicit methods.

### 3.4 Wilson-\( \theta \)-like algorithm
The step-by-step integration scheme using the 4th order cardinal B-spline is equivalent to the Newmark-\( \beta \) method of \( \beta = 1/6 \) which is conditionally stable. This brings a possibility of numerical divergence as the highest natural frequency becomes higher due to an increase of degree of freedom unless smaller time step is used. Therefore, an improvement to keep a numerical stability is desired.

The well-known Wilson-\( \theta \) method is the one to reinforce the numerical stability of the Newmark-\( \beta \) method of \( \beta = 1/6 \). This method yields the responses from a balance of force at \( t_{i+1} = t_i + \Delta t (\theta > 1) \) not \( t_{i+1} = t_i + \Delta t \) and becomes absolutely stable under \( \theta > 1.37 \). This concept of the Wilson-\( \theta \) method is incorporated into the present method using the 4th order cardinal B-spline.

The formulation is done in the same manner as section 3.1. Representing the responses \( d_{i-4}, d_{i-4} \) and \( d_{i+5} \) at time \( t_{i+5} \) as combination of \( a_0, a_{i-1}, \ldots \) and substituting them into Eq. (5) gives a same form as Eq. (12) to provide \( a \). The values of \( \kappa_0 \) etc. corresponding to \( \theta = 1.4 \), which is usually chosen in the Wilson-\( \theta \) method, are listed in \( N_{a_0} \) line in Table 1.

It has been also confirmed that the relationship of the Wilson-\( \theta \) method is derived from the method formulated in this section, that is, the present method unifies the Wilson-\( \theta \) method in addition to the Newmark-\( \beta \) method.

### 3.5 Algorithm for a system having discontinuous points
Examples of nonlinear elements including some discontinuous points are illustrated in Fig. 3 as restoring forces against displacement \( d \) or velocity \( \dot{d} \). A piecewise linear system is shown in Fig. 3(a) where \( k_1 \sim k_5 \) are spring constants at each section and \( a, b \) are switch points. Coulomb's friction is shown in Fig. 3(b) where \( R \) is amplitude of frictional force. The discontinuity happens at \( d = a, b \) in (a) and \( d = 0 \) in (b). This type of nonlinearity can be treated as linear systems in each segment sectioned by the discontinuous points. But much attention should be paid on a treatment of the time step in which the response \( d \) or \( \dot{d} \) passes through the discontinuous points.

Generally, smaller time step is required at least in the vicinity of the discontinuous points in order to obtain an adequate solution when the system has such nonlinearities, so that an increase of the computational time is unavoidable. In this section, a simple procedure to approximate an approximate time at which the response passes through the discontinuous points is suggested by making good use of the two-scale relation.

Supposing that a displacement \( d \) passes through a discontinuous point \( g \) between the time \( t_i \) and \( t_{i+1} \) as shown in Fig. 4. When the 4th order cardinal B-spline
is adopted, the displacement $d$ at an arbitrary moment is given by Eq.(6). Substituting the two-scale relation Eq.(4) into (6) yields the following relationship.

$$d(t)=\sum a_i N_i \left( \frac{2t}{\Delta t} - l \right), \quad a_i=\sum a_{i-20}$$ (18)

Let the midpoint between $t_i$ and $t_{i+1} = t_i + \Delta t/2$ be represented as $t_{i+1/2}$. The displacement at $t_{i+1/2}$, which is represented as $d_{i+1/2}$, is derived from substituting $t = t_{i+1/2}$ into Eq.(18) and making use of the actual values of $N_i(x)$ and $p_{i-20}$ [see Eq.(4)].

$$d_{i+1/2} = (a_{i+1} N_{i+1} + a_{i-1} N_{i-1})/5$$ (19)

where

$$a_{i+1} = (a_{i+2} + 6a_{i+1} + a_{i-2})/8$$

$$a_{i-1} = (a_{i+2} + a_{i-2})/2$$

$$a_{i-1} = (a_{i+2} + 6a_{i-1} + a_{i-3})/8$$ (20)

Replacing $d_{i+1} \sim d_{i+2}$ by $a_{i+1} \sim a_{i+3}$ in Eq.(20) gives the same formula as $d_i$ in Eq.(7). Then, the two-scale relation is still able to apply to $d_{i+1/2}$, and the displacement at $t_i + \Delta t/4$ or $t_i + 3 \Delta t/4$ is subsequently obtained. Repeating this procedure brings the approximate time at which $d(t) = g$ like the bisection method. Calculation of the velocity and acceleration of the discontinuous point follows the detection of $d(t) = g$. It can be immediately done from Eqs.(8) and (9) and the step-by-step integration restarts from the discontinuous point by using the calculated displacement, velocity and acceleration as initial values. The procedure is available for the other order cardinal B-spline except changing $N_i$ and $p_{i-20}$ in Eq.(18).

Note that the detection of the discontinuous point obtained by the two-scale relation is not so precise because it is based on an interpolation. However, it doesn’t require a recalculation with changing $\Delta t$ so that the computational efficiency is exceedingly good. It will be shown that the accuracy of this method is sufficiently acceptable in Section 5.

4. Step-by-Step Integration of a Multi-Degree-of-Freedom System

In Section 3, a step-by-step integration scheme by utilizing the 3rd and 4th order cardinal B-splines on a SDOF system has been presented. The present method is easily extended to a multi-degree-of-freedom system (called MDOF system). The equation of motion for a MDOF system is generally represented as:

$$M\ddot{d}(t) + C\dot{d}(t) + Kd(t) = q(t)$$ (21)

The displacement vector at the $i$-th step of integration is also represented as a series of the $n$-th order cardinal B-spline.

$$d_i = \sum a_i N_i \left( \frac{2i}{\Delta t} - l \right)$$ (22)

where $a_i$ is a coefficient vector of which dimension is equal to the degree of freedom of the system. The theory presented in Section 3 is also available in the case of MDOF system by replacing $a_i$ with $a_i$. The operation corresponding to Eq.(12) becomes following simultaneous equations.

$$H\ddot{a}_i = q(t_i) + H_{a_{i-1}} a_{i-1} + \cdots + H_{a_{i-n+1}} a_{i-n+1} + \tilde{H}$$ (23)

Matrices $H_a \sim H_{a_{i-1}}$ correspond to $\tilde{h}_a \sim \tilde{h}_{a_{i-1}}$ in Table 1 by replacing $m$, $c$ and $k$ with $M$, $C$ and $K$ respectively. The Matrix $\tilde{H}$ also corresponds to $\tilde{h}$ by replacing $m\ddot{d}_i$ with $\tilde{M}\ddot{d}_i$. To solve Eq.(23) at each time step yields the coefficient vectors $a_i$ successively. When the system involves discontinuity, the algorithm utilizing the two-scale relation mentioned in Section 3 is still available.

5. Numerical Computational Results

In order to demonstrate the feasibility of the present method, some numerical simulations have been performed. The simulations have been done on a 233 MHz Pentium PC with 32 MB RAM running Windows 95. The programming source is written in Fortran 90 with double precision variables.

At first, free vibration of a SDOF system is done for an investigation of the computational accuracy of the present methods. Then, a straight-line beam structure is dealt with as a basic example of a MDOF system and computational speed and numerical stability is tested. The present method using the 3rd and 4th order cardinal B-splines are called Spline 3 and Spline 4, respectively. The Wilson-β like algorithm presented in Section 3.4 is called Spline 4β. The Spline 4β adopts $\beta = 1.4$.

5.1 Single-degree-of-freedom systems

As mentioned in Section 3.3 and 3.4, the physical property of the Spline 3, Spline 4 and Spline 4β are equal to the Newmark-β method of $\beta = 1/4$, $\beta = 1/6$ and Wilson-β method, respectively. Actually, numerical simulations give the same results each other on the physically same methods. Therefore, the computational accuracy and the numerical stability of the Spline 3, Spline 4 and Spline 4β follow the current theory under a restriction that only a linear system is provided to a simulation. However, an alternate evaluation of the computational accuracy involved in nonlinear systems is needed because an effective method to deal with the nonlinearity having discontinuous points is devised in Section 5.5. We propose a following occasional method for the evaluation of the computational accuracy.

A free vibration of a linear SDOF system is governed by following equation.

$$d(t) + 2\omega_n^2 \ddot{d}(t) + \omega_n^2 d(t) = 0$$ (24)

where $\omega_n$ is a natural frequency and $\zeta$ is a damping
ratio. A response vector of Eq. (24) is defined as \( \mathbf{x}(t) = [d(t), d'(t)] \) and the one originated from an initial vector \( \mathbf{x}(0) \) is given as \( \mathbf{x}(t) = \mathbf{W}(t)\mathbf{x}(0) \), where \( \mathbf{W}(t) \) is a transition matrix. The transition matrix is given as \( \mathbf{W}(t) = [\mathbf{u}(t) \cdot \mathbf{u}_d(t)] \), where the two responses \( \mathbf{u}(t) \) and \( \mathbf{u}_d(t) \) are originated from two initial vectors \( \mathbf{x}(0) = (1, 0) \) and \( \mathbf{x}(0) = (0, 1) \) respectively. When \( \mathbf{u}(t) \) and \( \mathbf{u}_d(t) \) are derived analytically, the eigenvalues of \( \mathbf{W}(t) \) are \( \omega = \exp(-\omega_\infty t + j\omega_d t) \) in which \( \omega = \omega_\infty \sqrt{1-\xi^2} \). On the other hand, another pair of eigenvalues \( \omega = \exp(-\omega_\infty t \pm j\omega_d t) \) are given when \( \mathbf{u}(t) \) and \( \mathbf{u}_d(t) \) are obtained numerically by using the step-by-step integration schemes presented in this paper. The computational accuracy of the present methods can be evaluated by comparing the exponents of the numerically obtained eigenvalues to the analytically derived ones.

\[
e = \sqrt{\left(\omega_\infty - \omega_\infty^2\right)^2 + \left(\omega_d - \omega_\infty^2\right)^2 / \omega_d^2} \quad (25)
\]

where \( e \) is a relative error of the exponents. There is no problem on methods when only a linear system is evaluated. But this evaluation uses two initial vectors, so that a direct use of this method to a nonlinear system is not available because the initial values are essentially associated with the characteristics of the response in the case of nonlinear system. Therefore, we propose an occasional method to provide a transition matrix \( \mathbf{W}(t) \) by using a response originated from one initial vector. If the response \( \mathbf{x}(t) \) is obtained from the initial vector \( \mathbf{x}(0) \), two points of the response \( \mathbf{x}(t) \) and \( \mathbf{x}(t) \) at fixed time \( t \) and \( t \) are chosen arbitrarily. Then other two responses \( \mathbf{x}(t+\Delta t) \) and \( \mathbf{x}(t+\Delta t) \) at following time \( t+\Delta t \) and \( t+\Delta t \) are picked up, where \( \Delta t \) is a specified time interval. The relationships \( t \), \( t \), \( t+\Delta t \), \( t+\Delta t \) are diagrammatically illustrated in Fig. 5.

Supposing that \( \mathbf{x}(t) \) and \( \mathbf{x}(t) \) are defined as initial values, a quasi-transition matrix to provide a response at \( t \) and \( t+\Delta t \) can be represented as follows.

\[
\mathbf{W}(t; t, t) = [\mathbf{x}(t), \mathbf{x}(t)][\mathbf{x}(t), \mathbf{x}(t)]^{-1} \quad (26)
\]

The initial time \( t \) and \( t \) should be chosen as \( [d(t) = 0, d'(t) = 0] \) and \( [d(t) = 0, d'(t) = 0] \) so that the matrix \([x(t), x(t)]\) is not singular (See Fig. 5). It is to be desired that \( t \) and \( t \) are as close as possible in order that the characteristics of the response between \( t \) and \( t+\Delta t \) are not much different from one another. Since the matrix \( \mathbf{W}(t; t, t) \) can be the alternative to a transition matrix \( \mathbf{W}(t) \), the eigenvalues of \( \mathbf{W}(t; t, t) \) are represented as exp \((\omega_\infty t \pm j\omega_d t)\) and the exponents \( \omega_\infty \) and \( \omega_d \) are provided to Eq. (25). It is difficult to obtain the analytical values of \( \omega_\infty \) and \( \omega_d \) in a nonlinear system, so that \( \omega_\infty \) and \( \omega_d \) obtained by a highly accurate response are used in Eq. (25) in place of \( \omega_\infty \) and \( \omega_d \).

Note that the evaluation of computational accuracy by Eq. (25) is a restricted and expedient procedure only to deal with a specified initial value and segment of the response because the choice of the initial values seriously affects \( \omega_\infty \) and \( \omega_d \) when a nonlinear system is treated.

Relative errors of the present step-by-step integration scheme to time step size \( \Delta t \) are calculated on the nonlinear SDOF systems illustrated in Fig. 3 Supposing that the piecewise linear system is symmetric \( b = -a \), a non-dimensional equation of motion is given as:

\[
\begin{align*}
\dot{X} + 2\omega_\infty X + a\omega^2 f(X) &= 0 \\
K_0 X - K_0 X &= 1 X > 1 \\
K_0 X - K_0 X &\leq 1 X < -1 \\
F(X) &= \begin{cases} \end{cases} \\
X &= d(t)/a \quad K_0 = k_0/k_0, \quad K_0 = k_0/k_0 \\
K_0 X = K_0 X &= 5 \quad (27)
\end{align*}
\]

where the parameters are assumed \( \omega_\infty = 1.0, \xi = 0.01 \) and \( K_0 = k_0 = 5 \). An equation of motion for a system having the Coulomb's friction is

\[
\dot{d}(t) + sgn(d(t))R + \omega^2 d(t) = 0 \quad (28)
\]

where \( \omega^2 = 1.0 \), Coulomb's friction force \( R = 0.04 \). The time step size is given by dividing one period \( T = 2\pi/\omega \) into \( M \). The choice for \( \mathbf{W}(t; t, t) \) is \( x(0) = (0, 0) \), \( t = \pi/6, b = 0 \) and \( t = 10 T \) for the piecewise linear system and \( x(0) = (1, 0), h = 1, h = \pi/2 \) and \( t = 6 T \) [the response sticks at \( t > 6 T \), see Fig. 7 (b)] for the Coulomb's friction respectively. Since piecewise analytical solutions can be obtained to the two systems, the connection of the analytical solutions can be regarded as an exact solution. The exponents \( \omega_\infty \) and \( \omega_d \) derived from the exact solution are used in Eq. (25) as \( \omega_\infty \) and \( \omega_d \).

The results of the piecewise linear system and the Coulomb's friction are plotted in Figs. 6 (a) and (b) respectively. Marks denoted by \( \circ \) show the results with the detection of the discontinuous points by the two-scale relation and the others by \( \cdot \) show the results without the detection. The detection is estimated by \( |X - 1| < \epsilon \) for the piecewise linear system and \( |d| < \epsilon \) for the Coulomb's friction, where \( \epsilon = 10^{-3} \). It has been confirmed that the results do not change by using smaller \( \epsilon \). The detection of the discontinuous
points improves the accuracy for the most part of $\Delta t$. Although the accuracy of the Coulomb's friction without the detection is nearly saturated, the one with the detection is uniformly improved as $\Delta t$ decreases.

Good examples of the improvement by the detection of the discontinuous points are shown in Fig. 7. Displacements of the piecewise linear system by the Spline 4 of $M=32$ and the ones of the Coulomb's friction by the Spline 4 of $M=8$ are plotted in Figs. 7 (a) and (b) respectively. The horizontal axes show non-dimensional time $\tau=\omega n t$. The bold dashed line plots the exact solution obtained by the connection of the piecewise analytical solutions, the solid line plots the step-by-step response with the detection and the fine dashed line plots the one without the detection. As shown in Fig. 7, both the amplitudes and the phases are exceedingly improved.

5.2 Multi-degree-of-freedom systems

Step-by-step integration of a MDOF system is performed. Bending vibration of a straight-line beam structure as shown in Fig. 8 is treated. The structure is modeled as a lumped system and each rigid element is called node 0 ~ node $r$ from the left hand side to the right hand side of the system. A beam element between node $s$ and $s-1$ is called the $s$-th beam element and variables with subscript $s$ represent physical quantities of node $s$ or $s$-th beam element. Lateral displacement $y_s$ and rotation $\theta_s$ of the rigid element form the response vector. Mass and moment of inertia of the rigid element is denoted by $m_s$ and $J_s$, external force and moment is by $q_{s,v}$ and $q_{s,\gamma}$. Spring constants and viscous damping coefficients of the base support element (BSE) are denoted by $k_s$ and $c_s$ on the lateral motion, $K_s$ and $C_s$ on the rotational motion. The elements of Eq. (21) on this system are represented as:

$$
M = \text{diag}(m_0, m_1, \ldots, m_r, f_r) \\
C = \text{diag}(c_0, c_1, \ldots, c_r, c_r) \\
K = \begin{bmatrix}
Y_0 & X_1 & 0 \\
X_0 & Y_1 & X_2 \\
& \ddots & \ddots \\
& & X_{r-1} & Y_{r-1} & X_r \\
0 & \cdots & 0 & Y_r & X_r
\end{bmatrix}
$$

$$
d(t) = \begin{bmatrix} d_0, d_1, \ldots, d_r, \theta_r \end{bmatrix}^T, \\
q(t) = \begin{bmatrix} q_{0,v}, q_{0,\gamma}, \ldots, q_{r,v}, q_{r,\gamma} \end{bmatrix}^T
$$

$$
L_s = \begin{bmatrix} 1 & 0 \\
0 & l_s & 1 \end{bmatrix}, \\
\gamma = \begin{bmatrix} \alpha & \gamma \\
\gamma & \beta \end{bmatrix}, \\
K_s = \begin{bmatrix} k & 0 \\
0 & K_s \end{bmatrix}, \\
(a, \beta, \gamma)_s = (l^3/3EI + l|xGA, lEI, l^2/2EI)^s
$$

where $l$, $(EI, G, A, k$ and $x$ are length, flexural elasticity, shear modulus, the sectional area and the numerical factor of the cross section of the beam elements respectively. This model is constructed by a series of

Fig. 7 Transient response of nonlinear SDOF systems, ---: by using the two-scale relation, \hlc{---}: constant time step

Fig. 8 Analytical model of a straight-line beam structure
connection of the elements “1”~“4” as shown in Fig. 8 and total length is 1 m. Each element is a solid steel shaft of diameter 10 mm and equally divided at a rate of 3 : 3 : 1 : 3 (the number of total division is r). The properties $E=2.06 \times 10^{11}$ N/m², $G=7.92 \times 10^{9}$ N/m² and $x=0.886$ are chosen to each beam element. Only the lateral element is considered in the BSE. While a linear element of $k=10^{9}$ N/m, $c=1.0$ Ns/m is in BSE 1, 2 and 4, a piecewise linear system of $a=-b=1$ mm, $k_0=0$ N/m, $k_0-k=-10^{6}$ N/m (symmetric clearance) is in BSE 3. A linear damping between the system and the environment $c_a=0.1$ Ns/m is assumed to be at each node. The primary natural frequency of this model is $f_1=10.8$ Hz. An external harmonic excitation $q_0(t)=Q \sin \omega t$ of amplitude $Q=10$ N and frequency $\omega/2\pi=10$ Hz ($\approx f_1$) acts on the point indicated in Fig. 8. Duration of the excitation is only 3 periods of itself, no excitation acts after that. The time step size $\Delta t$ is given by dividing one period of excitation $T_f=2\pi/\omega$ into $M$.

The total numbers of partitions of the system is fixed to be $r=10$ at first. The response of the node supported by BSE 3 is plotted in Fig. 9. The horizontal axes show non-dimensional time $\tau=\omega t$ and the duration of $q_0(t)$ corresponds to $\tau=0-6\pi$. The time step size is given by $M=32$. Only the responses by the Spline 3 and Spline 4θ are plotted in Fig. 9 because the response by the Spline 4 diverges to the time step size. The thin solid line and dashed line plot the step-by-step responses with the detection of the discontinuity and without the one respectively. The bold dashed line plots the step-by-step responses obtained by a sufficiently small time step size $M=10^4$ and this solution is regarded as trustable. The responses begin to diverge as soon as these pass through the first discontinuous point unless the point is detected. Although the Spline3 and Spline4θ provide a stable solution to a linear system, a numerical divergence happens in the case of a strong nonlinear system unless a sufficiently small time step is chosen. On the other hand, the solutions with the detection of the discontinuous point are sufficiently acceptable as compared with trustable ones. This fact shows that the simple detection of the discontinuous points by the two-scale relation is an effectual tool to improve the accuracy or stabilize the solution because it does not need much computational labor such as a recalculation with a smaller time step.

Investigation of computational speed follows. The transition of the computational time to the numbers of partitions $r$ is taken up. The number of partitions $r$ indicates the degree of freedom of the system by $2(r+1)$. The Newmark-β method of $\beta=1/4$, $\beta=1/6$ and the Wilson-θ method of $\theta=1.4$, which provide the same solutions as the Spline 3, the Spline 4 and the Spline 4θ respectively, are also used here. A small time step size $(M=10^4)$ enough to provide a numerically stable condition to the Newmark-β method of $\beta=1/6$ and the Spline 4 is chosen. When a piecewise linear system is treated, the number of times that the response passes through the discontinuous points is not constant on each integration method according to its accuracy. Therefore, a linear system without the nonlinear BSE 3 is treated in order to provide a same computational condition to each method.

The CPU time of the step-by-step integration through the ten-period long forced excitation is plotted in Fig. 10. The present methods are compared with the physically same conventional ones. While the Spline 4θ gains $25\sim30\%$ in computational speed to the Wilson-θ method, the Spline 3 and the Spline4 gain $30\sim40\%$ and $30\sim35\%$ to the Newmark-β method of $\beta=1/4$ and $1/6$ respectively. The improvement of the computational speed is derived from the simplicity of the algorithm. The present method is expected to show an outstanding performance when a complicated and large-scaled structure is treated.
6. Conclusions

A new step-by-step integration scheme utilizing the cardinal B-splines is proposed. The results are summarized as follows.

(1) The present method includes some conventional step-by-step integration schemes in a physical meaning. In other words, the present method is regarded as a reconsideration of the conventional implicit methods and unifies these operations.

(2) When a nonlinear system having discontinuity is treated, the present method is able to easily find an approximate point that the response passes through the discontinuity by making good use of two-scale relations and improves the accuracy of the solutions efficiently.

(3) The algorithm of the present method is simpler than that of the conventional methods and considerably improves the computational speed.

References


