Variable Structure Controller Design for Output Tracking of a Class of Discrete-time Nonlinear Systems*

Teh-Lu LIAO** and Tsun-I CHIEN**

In this paper, a robust controller based on the variable structure control technique for the output tracking of a class of discrete-time uncertain nonlinear systems is developed. An upper bound on the uncertainties is derived to ensure the existence of a quasi-sliding mode in these uncertain systems. Analysis results indicate that the output of the closed-loop system tracks a desired output trajectory and, simultaneously, all signals inside the loop are bounded. Furthermore, the tracking error depends on the proposed controller's parameters. Two examples are given to demonstrate the effectiveness of the proposed control scheme.

Key Words: Discrete-time Variable Structure Control, Discrete-time Nonlinear Systems, Output Tracking

1. Introduction

The pioneering works of Brockett⁽²⁾ and Krener⁽¹⁸⁾ led to the development of several differential geometric methods capable of synthesizing a nonlinear feedback control system. Many of those methods focused on input-output (I/O) and/or states linearization^{(13),(14),(16)}, approximate linearization by state feedback and coordinate change for continuous-time nonlinear systems, as well as discrete-time systems^{(10),(19),(22),(23)}. Correspondingly, several investigators^{(11),(12)} derived solutions based on an I/O linearization or system inversion technique to the output tracking problem of the continuous-time nonlinear control systems without parameterization error and/or unpredicted disturbances. Meanwhile, real control systems were applied in many investigations^{(20),(27)}. However, the feasibility of applying discrete-time nonlinear systems to the same problem has seldom been mentioned. Castillo and Gennaro⁽³⁾ attempted, for the first time, to employ a nonlinear regulation scheme⁽¹⁵⁾ to achieve asymptotic output tracking.

Developing accurate mathematical descriptions for many physical systems is rather difficult, thereby accounting for why inevitable uncertainties arise in these constructed models. Variable structure control with a sliding mode is a robust control approach, which encompasses uncertain nonlinear systems by exploiting the variable structure concept. This approach initially defines the sliding surface in the error state space and, then, forms a switching state feedback control using the bounds on the uncertainties. The high-speed switching state feedback forces the error states to move towards the switching surface and slide along that surface until these states converge and the tracking purpose is ultimately attained. While in the sliding mode, the system remains insensitive to uncertainties⁽⁶⁾. However, variable structure control designed for continuous-time systems is typically implemented using a digital computer⁽⁴⁾. When applying a discrete control algorithm, the sampling rate may alter the controlled system's performance and even cause chattering and instability.

Discrete-time variable structure control (VSC) has received increasing attention, along with various conditions proposed for the existence and reachability of discrete-time VSC^{(8),(17),(21),(25),(28)}. Results obtained from discrete-time VSC differ from those for continu-

* Received 20th November, 2000

** Department of Engineering Science, National Cheng Kung University, Tainan, Taiwan 701, R.O.C. E-mail: tlliao@mail.ncku.edu.tw

ous-time systems. The sliding mode characteristics of discrete-time VSC, commonly termed as a quasi-sliding mode, have also been thoroughly studied^{(1),(9),(17),(26)}. Sira-Ramirez⁽²⁶⁾ closely examined the relationship between the features of quasi-sliding and the notion of relative degree in the nonlinear control systems. Later, Gao, Yang and Homaifa⁽⁹⁾ presented some attributes and definitions of quasi-sliding band and reaching conditions. Based on the reaching law proposed in that study, a discrete-time VSC was developed for robust control⁽¹⁾.

In light of above developments, this work constructs a robust tracking controller for a class of discrete-time nonlinear minimum-phase systems⁽²²⁾ that, at the same time, stabilizes the system. State feedback linearization of the output equation is assumed herein for the nominal system and a discrete-time VSC is also proposed to alleviate the effects of uncertainties. In addition, the proposed control law fulfills the reaching law of the discrete-time uncertain nonlinear systems so that the system trajectory crosses the switching surface and begins the motion of quasi-sliding mode in finite time. Moreover, an upper bound on the uncertainties is derived to ensure the existence of the quasi-sliding mode, thereby resolving the output tracking problem.

The rest of this paper is organized as follows. Section 2 formulates the control problem with some technical assumptions. Section 3 presents a discrete VSC law that achieves the uniform boundedness of the overall system. In Section 4, two numerical examples demonstrate the proposed control scheme's effectiveness. Conclusions are finally made in Section 5.

2. Problem Statement

Consider a class of single-input single-output discrete-time uncertain nonlinear systems of the following form:

$$\begin{aligned} x(k+1) &= f(x(k), u(k), w(k)), \quad k=0, 1, 2, \dots \\ y(k) &= h(x(k)), \end{aligned} \quad (1)$$

where $x \in X \subset \mathcal{R}^n$ denotes the state vector, $u \in \mathcal{R}$ represents the control input by which the state response can be affected, $w \in W \subset \mathcal{R}$ is a deterministic system uncertainty, and $y \in \mathcal{R}$ is the system output. The mapping $f: X \times \mathcal{R} \times W \rightarrow X$ and the function $h: X \rightarrow \mathcal{R}$ are assumed to be sufficiently smooth. Moreover, assume that $(x, u, w) = (0, 0, 0)$ is an equilibrium point for the system (1), i.e. $f(0, 0, 0) = 0$, $h(0) = 0$.

The nominal model is then defined as follows (i.e. with $w(k) = 0$ in (1)):

$$\begin{aligned} x(k+1) &= f(x(k), u(k), 0), \quad k=0, 1, 2, \dots \\ &\equiv f_0(x(k), u(k)) \\ y(k) &= h(x(k)). \end{aligned} \quad (2)$$

The usual composition of functions denoted by $\circ$ can be expressed as $h \circ f_0(x, u) \equiv h(x)$. High-order composition can be defined recursively as $h \circ f_0^j(x, u) \equiv (h \circ f_0^{j-1}(x, u)) \circ f(x, u)$. Herein, the nominal system (2) is assumed to possess the relative degree r , i.e. $\frac{\partial(h \circ f_0^j(x, u))}{\partial u} \equiv 0$ for $j=0, 1, \dots, r-1$ and $\frac{\partial(h \circ f_0^r(x, u))}{\partial u} \neq 0$.

Associated with the relative degree $r \leq n$ and denoted by $\bar{f}_0(x) \equiv f_0(x, 0)$, a local state transformation:

$$\begin{aligned} Z &= \Phi(x) = (z_1(x)^T, z_2(x)^T)^T \\ &\equiv (h(x), h \circ \bar{f}_0(x), \dots, h \circ \bar{f}_0^{r-1}(x), \\ &\quad z_{2,1}(x), \dots, z_{2,(n-r)}(x))^T \end{aligned} \quad (3)$$

with $\frac{\partial}{\partial u}(z_{2,i} \circ f_0(x, u)) = 0$, $i=1, \dots, (n-r)$, is a coordinate transformation, which transforms the nominal system (2) into the following form:

$$\begin{aligned} z_{1,1}(k+1) &= z_{1,2}(k) \\ &\vdots \\ z_{1,r-1}(k+1) &= z_{1,r}(k) \\ z_{1,r}(k+1) &= (h \circ \bar{f}_0^{r-1}(z_1(k), z_2(k))) \circ f_0(z_1(k), \\ &\quad z_2(k), u(k)) \\ z_2(k+1) &= \Psi_1(z_1(k), z_2(k)) \\ y(k) &= z_{1,1}(k). \end{aligned} \quad (4)$$

Remark 1: For a controllable system of order n , the relative degree r determines the inherent delay between the control input and the output. It will take r delays of the output for the control input to appear, namely, the input $u(k)$ affects the output $y(t+r)$ ⁽¹⁷⁾.

Many uncertainties in the constructed models arise in a less general form, thereby satisfying some structural conditions. Herein, we consider only a class of uncertain dynamics in which the uncertainty satisfies the following assumption:

A1. The condition: $\frac{\partial f(x, 0, w)}{\partial w} \in \text{Im}\left(\frac{\partial f(x, u, 0)}{\partial u}\right)_{u=w}$ holds for all $(x, u, w) \in X \times \mathcal{R} \times W$. Moreover, assume that $(h \circ f^{r-1}(x, u, w)) \circ f(x, u, w) = a(x) + b(x)u + c(x)w$ and that $b(x)$ is nonsingular for all $(x, u, w) \in X \times \mathcal{R} \times W$.

Remark 2: Assumption A1 allows us to easily verify that the relative degrees of system (1) r and d with respect to the control and the disturbance, respectively, are equal to each other, i.e. $r=d$. Under this condition, which is generally referred to as a *matching condition*, a control input u can be designed to compensate for the disturbance so that the control objectives of the resulting system can be achieved.

By the assumption A1, the uncertain nonlinear system (1) can be expressed in the new coordinate as

$$\begin{aligned}
z_{1,1}(k+1) &= z_{1,2}(k) \\
&\vdots \\
z_{1,r-1}(k+1) &= z_{1,r}(k) \\
z_{1,r}(k+1) &= (h \circ f_0^{-1}(z_1(k), z_2(k))) \circ f(z_1(k), \\
&\quad z_2(k), u(k), w(k)) \\
&= a(x) + b(x)u + c(x)w(k) \\
&= a(z_1(k), z_2(k)) + b(z_1(k), z_2(k))u(k) \\
&\quad + c(z_1(k), z_2(k))w(k) \\
z_2(k+1) &= \Psi(z_1(k), z_2(k)) \\
y(k) &= z_{1,1}(k)
\end{aligned} \quad (5)$$

where $a(z_1, z_2)$, $b(z_1, z_2)$ and $c(z_1, z_2)$ denote $a(x)$, $b(x)$, and $c(x)$, respectively, in the (z_1, z_2) coordinate. Moreover, the dynamics of the following difference equation

$$z_2(k+1) = \Psi(0, z_2(k)) \quad (6)$$

characterizes the *zero dynamics* of system (1). Namely, the zero dynamics is the dynamics of the nonlinear systems (1) subject to the constraint that the output be identically zero. Furthermore, the zero dynamics determines the internal state behavior of the system (1).

To obtain the objective of output tracking with internal stability, a common minimum phase assumption involving the system is required. This is stated in the following assumption.

A2. The nonlinear system (1) is an exponentially minimum phase, i.e. the equilibrium point 0 of the zero dynamics (6) is exponentially stable. Moreover, $\Psi(z_1, z_2)$ satisfies a *sector condition* and is Lipschitz with respect to z_1 for all $z_2 \in X_2 \subset X$. Those are $\|\Psi(z_1, z_2)\| \leq L(\|z_1\| + \|z_2\|)$ and $\|\Psi(z_1, z_2) - \Psi(0, z_2)\| \leq L\|z_1\|$ for all $z_1 \in X_1 \subset \mathcal{R}^r$ and $z_2 \in X_2 \subset \mathcal{R}^{n-r}$, respectively, where L denotes the Lipschitz constant of Ψ .

This investigation largely focuses on the following objective: given a desired trajectory $y_d(k) \in \mathcal{R}$, we want to design a robust control law so that the tracking error $e(k) = y(k) - y_d(k)$ eventually falls into a bounded set around zero whose size can be designed, i.e. $\limsup_{k \rightarrow \infty} |e(k)| = \limsup_{k \rightarrow \infty} |y(k) - y_d(k)| < \kappa$ where κ denotes a positive constant chosen at our disposal. Furthermore, all signals inside the closed-loop system are bounded.

3. Variable Structure Control Law for Output Tracking

As pointed out in Gao et al.⁽⁹⁾, the reaching laws for discrete-time variable structure control systems have the following characteristics: (i) the system trajectory moves monotonically towards the desired switching surface and crosses it from any initial state; (ii) The trajectory, once having crossed the surface at a certain step, crosses the surface again in every successive step, subsequently producing a zigzag

Fig. 1 Trajectory of a second order discrete variable structure system

motion around the switching surface; and (iii) The size of each zigzagged step is nonincreasing and the trajectory remains within a specific band. A *quasi-sliding mode* refers to the motion of a discrete-time variable structure system satisfying characteristics (ii) and (iii). Moreover, a discrete-time VSC is said to satisfy the *reaching condition* if the resulting system possesses all three characteristics: (i), (ii), and (iii).

The reaching law introduced by Gao et al. is

$$s(k+1) = \alpha s(k) - \beta \operatorname{sgn}(s(k)) \quad (7)$$

where $0 < \alpha < 1$ and $\beta > 0$. It can be easily verified that (7) satisfies all characteristics (i) - (iii) and thereby obtaining the *quasi-sliding mode* of a discrete-time variable structure system. The specific band in the state space, which contains the quasi-sliding mode, can be expressed as

$$\Omega_x = \left\{ x(k) \in X \mid -\frac{\beta}{1+\alpha} < s(x(k)) < \frac{\beta}{1+\alpha} \right\}. \quad (8)$$

Once the state trajectory enters the Ω_x , it remains within Ω_x . Figure 1 depicts one possible state trajectory of a second order discrete-time variable structure system. Generally, the switching surface is defined by a linear combination of state x and passes the origin of the state space. Furthermore, the evolution of x depends on the time instant k , thereby implying that s can also be expressed in terms of k . Therefore, the switching surface divides the state space into three subspaces: $s(k) = s(x(k)) > 0$, $s(k) = s(x(k)) \equiv 0$, and $s(k) = s(x(k)) < 0$.

In this section, we present a discrete-time variable structure control using the linearizing control law along with a reaching law for output tracking. The former attempts to obtain a linear output equation, whereas the latter tries to obtain a quasi-sliding mode despite the uncertainties; meanwhile, it attempts to

provide a bounded tracking force. Two steps are involved in constructing the switching control: in the first step; a switching surface which achieves the desired tracking objective is constructed; the second step entails deriving an appropriate control law so as to attain the reaching law.

Next, allow $y_d(k)$ to be the desired output trajectory and the i -th time-ahead of $y_d(k)$, $y_d(k+i)$, $i=0, \dots, r$, be bounded signals and available. Define a 'surface' $s(k) \in \mathcal{R}$ in the output error space as follows:

$$s(k) = e(k+r-1) + \sum_{i=1}^{r-1} \gamma_i e(k+r-i-1) \quad (9)$$

where $e(i) = y(i) - y_d(i)$, $i=k, \dots, k+r-1$, denote the output errors and γ_i , $i=1, \dots, r-1$, are appropriate constants chosen so that the following polynomial of a complex variable z

$$H(z) = z^{(r-1)} + \gamma_1 z^{(r-2)} + \dots + \gamma_{r-2} z + \gamma_{r-1} \quad (10)$$

is an exponentially stable (Hurwitz) polynomial, i.e. the roots of the equation $H(z)=0$ lie inside the unit circle in the complex z -plane.

Equation (9) can be rewritten as follows:

$$e(k+r-1) + \gamma_1 e(k+r-2) + \dots + \gamma_{r-2} e(k+1) + \gamma_{r-1} e(k) = s(k) \quad (11)$$

in which $s(k)$ can be regarded as the input of a linear system with transfer function $H(z)$, subsequently producing the output $e(k)$ through the system $H(z)$. Hence, by applying the bounded input and bounded output (BIBO) theorem⁽⁷⁾, if the input $s(k)$ is bounded and converges to a constant, i.e. $s(k) \in l_\infty$ and $\lim_{k \rightarrow \infty} s(k) = s_d$ where s_d denotes a constant in $\mathcal{R}$. Then, the output $e(k)$ is also bounded and converged, i.e. $e(k) \in l_\infty$ and $\lim_{k \rightarrow \infty} e(k) = H(1)s_d$. Similarly, if the reaching law is satisfied, then the trajectory $s(k)$, $k \geq 0$ in the output error space reaches the neighborhood of the sliding surface $s=0$ and remains within a bounded set $\Omega_s = \{s(k) \in \mathcal{R} \mid |s(k)| < \Delta\}$ (defined in (44)). This occurrence implies that $e(k)$ falls into a bounded set around zero as k tends to infinity. This obviously includes the control objective $|e(k)| = |y(k) - y_d(k)| < \kappa = H(1)\Delta$ as $k \rightarrow \infty$.

According to the definition of $s(k)$ and (5), the following can be easily obtained:

$$\begin{aligned} s(k+1) &= y(k+r) - y_d(k+r) \\ &+ \sum_{i=1}^{r-1} \gamma_i (y(k+r-i) - y_d(k+r-i)) \\ &= a(x(k)) + b(x(k))u(k) + c(x(k))w(k) \\ &- y_d(k+r) + \sum_{i=1}^{r-1} \gamma_i (y(k+r-i) \\ &- y_d(k+r-i)). \end{aligned} \quad (12)$$

To maintain the trajectory $s(k)$ in the output error space in a quasi-sliding mode, a discrete-time variable structure control law based on input-output linearization technique is designed as follows:

$$u(k) = b(x(k))^{-1} \{-a(x(k)) + as(k) + \xi(k+r) - \beta \operatorname{sgn}(s(k))\} \quad (13)$$

where α and β are feedback gains which yet to be determined, $\operatorname{sgn}(s)$ is sign function of s , and $\xi(k+r)$ can be shown that

$$\begin{aligned} \xi(k+r) &= y_d(k+r) - \sum_{i=1}^{r-1} \gamma_i e(k+r-i) \\ &= y_d(k+r) - \sum_{i=1}^{r-1} \gamma_i (h^0 \bar{f}_0^{r-1}(x(k)) \\ &- y_d(k+r-i)) \end{aligned} \quad (14)$$

which is an implementable signal. With the construction of $u(k)$ in (13), the dynamics of (12) becomes

$$s(k+1) = c(x(k))w(k) + as(k) - \beta \operatorname{sgn}(s(k)), \quad (15)$$

which can be stated as the reaching law for the uncertain systems (1) and (12). Notably, Eq. (15) is critical for robustness analysis, which will be developed below.

An appropriate value of feedback gains α and β must be chosen to fulfill the reaching condition and, then, to ensure the existence of the quasi-sliding mode for a discrete-time variable structure system (15). Before proceeding with this task, an additional assumption is deemed necessary.

A3. A positive constant ρ exists such that, for all $(x(k), w(k)) \in X \times W$ and $k \geq 0$,

$$\|c(x(k))w(k)\| < \rho. \quad (16)$$

Next, the main theorem is stated as follows.

Main Theorem: Consider the system (1) satisfying assumptions **A1** - **A3**. Given the bounded desired output y_d with the aforementioned properties, the proposed control law in (13) with the feedback gains $0 < \alpha < 1$, $0 < \beta$, and $\rho < \frac{(1-\alpha)\beta}{1+\alpha}$ ensure the uniform boundedness of all signals inside the closed-loop system and of the tracking error. Furthermore, the tracking error eventually falls into a bounded set around zero whose size is at our disposal, i.e. $\limsup_{k \rightarrow \infty} |y(k) - y_d(k)| < \kappa = \frac{2H(1)\beta}{1+\alpha}$ where $H(1)$, α and β are the parameters chosen by a designer.

Proof: Details can be found in Appendix A.

Remark 3: By the main theorem, the robustness properties of the proposed control scheme can be further explained as follows: (1) If the control gains α and β are designed in advance, then the allowable upper bound on the uncertainties, ρ , must satisfy $\rho < \frac{(1-\alpha)\beta}{1+\alpha}$ to ensure the stability of the closed-loop system. In general, for obtaining the better output-tracking performance, the value of α is chosen sufficiently close to 1 and the value of β is chosen as small as possible. However, in this case, the robustness is too conservative. (2) If the upper bound on the uncertainties, ρ , is known in advance, then the

gains α and β must be chosen such that the above inequality is satisfied. If the bound ρ increases, the gain β must increase to accommodate the effect of uncertainties, which will destroy the tracking performance.

4. Numerical Examples

In this section we present two numerical examples to demonstrate the effectiveness of the proposed control scheme.

Example 1: Consider a discrete-time nonlinear system in the absence of disturbance as follows:

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} &= \begin{bmatrix} x_2(k) + (x_3(k) - x_1(k)^2 + u(k))^2 \\ x^3(k) - x_1(k)^2 + u(k) \\ x_1(k) + 2x_2(k)(x_3(k) - x_1(k)^2 + u(k))^2 + (x_3(k) - x_1(k)^2 + u(k))^4 + 0.8(x_3(k) - x_1(k)^2) \end{bmatrix} \\ &= f_0(x(k), u(k)) \\ y &= h(x(k)) = x_1(k) - x_2(k)^2. \end{aligned} \quad (17)$$

Notably,

$$\begin{aligned} h \circ f_0(x, u) &= x_2 \\ (h \circ f_0(x, u)) \circ f_0(x, u) &= x_3 - x_1^2 + u = a(x) + b(x)u \end{aligned} \quad (18)$$

so that $r=2$ and $b(x)=1$ which is nonsingular for all $x \in X \subset \mathcal{R}^3$. By defining the coordinate transformation as

$$\begin{aligned} z_1 &= \begin{bmatrix} z_{1,1} \\ z_{1,2} \end{bmatrix} = \begin{bmatrix} h(x) = x_1 - x_2^2 \\ h \circ f_0(x, 0) = x_2 \end{bmatrix} \\ z_2 &= x_3 - x_1^2 \end{aligned} \quad (19)$$

and by applying the above transformation and state feedback control law lead to

$$u(k) = -a(x(k)) + v(k) = -(x_3(k) - x_1(k)^2) + v(k) \quad (20)$$

so that the new state equations become

$$\begin{aligned} z_{1,1}(k+1) &= z_{1,2}(k) \\ z_{1,2}(k+1) &= v(k) \\ z_2(k+1) &= z_{1,1}(k) + 0.8z_2(k) \equiv \Psi(z_1(k), z_2(k)) \end{aligned} \quad (21)$$

and the zero dynamics is

$$z_2(k+1) = \Psi(0, z_2(k)) = 0.8z_2(k) \quad (22)$$

which is asymptotically stable. Next, defining the sliding surface as follows:

$$s(k) = y(k+1) - y_d(k+1) + 0.2(y(k) - y_d(k)) = x_2(k) - y_d(k+1) + 0.2(y(k) - y_d(k)) \quad (23)$$

leads to

$$\xi(k) = y_d(k+2) - 0.2(x_2(k) - y_d(k+1)) \quad (24)$$

which is an available signal. The feedback control law $v(k)$ in (20) can be chosen as

$$v(k) = \alpha s(k) - \beta \operatorname{sgn}(s(k)) + \xi(k). \quad (25)$$

To obtain the numerical results, we set $\alpha=0.5$, $\beta=0.01$ and the desired output $y_d(k)=\sin(0.1 \times k)$. Figure 2(a) depicts the evolution of the output tracking error in a closed-loop system. According to this figure, the tracking error eventually falls into a bounded set whose size depends on the magnitude of parameters of the control law. Figure 2(b) presents the control input's history. Figure 2(c) displays the states trajectories, and the $s(k)$ for $60 \leq k \leq 100$ is given in Fig. 2(d).

Example 2: Consider a nonlinear system with disturbance as follows:

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} x_2(k) \\ x_1(k) - x_2(k)^2 + (1 + x_1(k)^2)u(k) + 0.05(1.1 + \cos(x_1(k))) \end{bmatrix} = f(x(k), u(k), w(k)) \\ y(k) &= h(x(k)) = x_1(k) \end{aligned} \quad (26)$$

where $w(k)=0.05(1.1+\cos(x_1(k)))$ is a disturbance signal. It can be easily verified that $r=2$, $d=2$ and $b(x)=1+x_1^2$ which is nonsingular for all $x \in X \subset \mathcal{R}^2$. Define the coordinate transformation as

$$z_1 = \begin{bmatrix} z_{1,1} \\ z_{1,2} \end{bmatrix} = \begin{bmatrix} h(x) = x_1 \\ h \circ f(x, 0, 0) = x_2 \end{bmatrix} \quad (27)$$

Moreover, in this case, zero dynamics do not exist since the relative degree r is equal to the dimension of the controlled system.

Next, define the sliding surface as follows:

$$s(k) = y(k+1) - y_d(k+1) + 0.2(y(k) - y_d(k)) = x_2(k) - y_d(k+1) + 0.2(y(k) - y_d(k)) \quad (28)$$

which leads to

$$\xi(k) = y_d(k+2) - 0.2(x_2(k) - y_d(k+1)) \quad (29)$$

which is an available signal. We can choose the feedback control law

Fig. 2 The simulation results of example 1

Fig. 3 The simulation results of example 2

$$u(k) = (1 + x_1(k)^2)^{-1} (-x_1(k) + x_2(k)^2 + v(k)) \quad (30)$$

and

$$v(k) = \alpha s(k) - \beta \operatorname{sgn}(s(k)) + \xi(k) \quad (31)$$

with the feedback gains α and β satisfying $0.05 < \frac{(1-\alpha)\beta}{1+\alpha}$. To obtain the numerical results, we set $\alpha = 0.5$, $\beta = 1/6$ and the desired output $y_d(k) = \sin(0.1 \times k)$. Figure 3(a) depicts the evolution of the output tracking error of the closed-loop system. According to this figure, the tracking error ultimately falls into a bounded set whose size depends on the magnitude of parameters of the control law. Figure 3(b) presents the control input's history. Figure 3(c) displays the states trajectories, and the $s(k)$ for $60 \leq k \leq 100$ is given in Fig. 3(d).

5. Conclusions

This paper attempts to resolve the robust tracking problem of a class of discrete-time uncertain

nonlinear systems that satisfy input-output linearizability, phase-minimality and the so-called *matching condition*. A reaching law of discrete-time variable structure control is also constructed for the uncertain systems. In addition, a stable robust output tracking controller is proposed to fulfill the reaching law for a class of uncertain systems. An upper bound on the uncertainties is also derived to ensure the existence of a quasi-sliding mode. Analysis results indicate that the controller guaranteed the uniform boundedness of all signals inside the closed-loop system and of the tracking error. Furthermore, the tracking error depends on the parameters, which are at our disposal. Finally, two numerical examples verify the theoretical results of the proposed scheme. However, the application of the proposed control scheme to practical systems and the enhancement of robustness will be further studied.

Acknowledgements

This research was supported by the National Science Council of the Republic of China under Grant NSC87-2218-E-006-041.

Appendix: Proof of the Main Theorem

(a) First, we claim that the state trajectory crosses the switching surface in finite time. To simplify the notation for the following derivation, let us define $\hat{f}(k) = c(x(k))w(k)$. With this notation, the solution of (7) from an initial value $s(0)$ ($s(0)$ would be positive or negative), for every $k = 1, 2, \dots, k_0$, is

$$s(1) = \alpha s(0) - \beta \operatorname{sgn}(s(0)) + \hat{f}(0) \quad (32)$$

$$\begin{aligned} s(2) &= \alpha s(1) - \beta \operatorname{sgn}(s(1)) + \hat{f}(1) \\ &= \alpha^2 s(0) - \alpha \beta \operatorname{sgn}(s(0)) - \beta \operatorname{sgn}(s(1)) \\ &\quad + \alpha \hat{f}(0) + \hat{f}(1) \end{aligned} \quad (33)$$

⋮

$$\begin{aligned} s(k_0) &= \alpha^{k_0} s(0) - \beta \sum_{i=0}^{k_0-1} \alpha^i \operatorname{sgn}(s(k_0-1-i)) \\ &\quad + \sum_{j=0}^{k_0-1} \alpha^j \hat{f}(k_0-1-j). \end{aligned} \quad (34)$$

Without loss of generality, we assume that $s(k)$, $k = 0, 1, \dots, k_0-1$, lie in the positive half-plane of s , i.e.

$$\operatorname{sgn}(s(k_0-1)) = \operatorname{sgn}(s(k_0-2)) = \dots = \operatorname{sgn}(s(0)) > 0. \quad (35)$$

Equation (35) can then be rewritten by using (16) as follows:

$$\begin{aligned} s(k_0) &\leq \alpha^{k_0} s(0) - \beta \{ \alpha^{k_0-1} + \dots + \alpha + 1 \} \\ &\quad + \sum_{i=0}^{k_0-1} \alpha^i |\hat{f}(k_0-1-i)| \\ &\leq \alpha^{k_0} s(0) - \beta \frac{1 - \alpha^{k_0}}{1 - \alpha} + \rho \frac{1 - \alpha^{k_0}}{1 - \alpha}. \end{aligned} \quad (36)$$

If $\rho < \beta$, an integer k_0 exists, i.e.

$$k_0 = \left\lceil \frac{\ln\left(1 + \frac{(1-\alpha)s(0)}{\beta-\rho}\right)}{\ln\left(\frac{1}{\alpha}\right)} \right\rceil, \quad (37)$$

where $\lceil \cdot \rceil$ denotes the smallest integer $\bar{n}$ such that $\cdot \leq \bar{n}$, such that $s(k_0) < 0$. If β and $s(0)$ are finite, then k_0 is also a finite integer. Therefore, under the condition of $\rho < \beta$, the state trajectory crosses the switching surface in finite time.

(b) Second, we derive a condition under which the system trajectory crosses the switching surface again in every successive sampling steps. This fact implies that $\text{sgn}(s(k_0+1)) = -\text{sgn}(s(k_0)) = \text{sgn}(s(k_0-1)) > 0$ must be valid.

Considering the reaching law (15) leads to

$$\begin{aligned} s(k_0+1) &= \alpha s(k_0) - \beta \text{sgn}(s(k_0)) + \hat{f}(k_0) \\ s(k_0+1) &= \text{sgn}(s(k_0-1)) \{ \alpha^2 |s(k_0-1)| \\ &\quad + (1-\alpha)\beta \} + \hat{f}(k_0) + \alpha \hat{f}(k_0-1) \end{aligned} \quad (38)$$

and by using (36), we also obtain

$$\begin{aligned} \{ \alpha^2 |s(k_0-1)| + (1-\alpha)\beta \} - (1+\alpha)\rho &< s(k_0+1) \\ &< \{ \alpha^2 |s(k_0-1)| + (1-\alpha)\beta \} + (1+\alpha)\rho \end{aligned} \quad (39)$$

If, $\rho < \frac{(1-\alpha)\beta}{(1+\alpha)}$, we can obtain $s(k_0+1) > 0$. By the induction method, we can easily infer that $\text{sgn}(s(k_0+i+1)) = -\text{sgn}(s(k_0+i))$ for all $i=1, 2, \dots$.

From (a) and (b), we can conclude that the quasi-sliding mode exists if the following condition is satisfied, i.e.

$$\rho < \frac{(1-\alpha)\beta}{(1+\alpha)} < \beta. \quad (40)$$

(c) When the system is in the quasi-sliding mode, we have, for every $m=0, 1, 2, \dots$,

$$\begin{aligned} s(k_0+m) &= \text{sgn}(s(k_0)) \left\{ \alpha^m |s(k_0)| \right. \\ &\quad \left. + (-1)^m \beta \frac{1-(-\alpha)^m}{1+\alpha} + \sum_{i=0}^{m-1} \alpha^{m-1-i} \hat{f}(k_0+i) \right\} \end{aligned} \quad (41)$$

and

$$|s(k_0+m)| \leq \left\{ \alpha^m |s(k_0)| + \beta \frac{1+\alpha^m}{1+\alpha} + \rho \frac{1-\alpha^m}{1-\alpha} \right\}. \quad (42)$$

Taking into account that $0 < \alpha < 1$, we obtain

$$\begin{aligned} \lim_{m \rightarrow \infty} |s(k_0+m)| &= \lim_{m \rightarrow \infty} \left\{ \alpha^m |s(k_0)| \right. \\ &\quad \left. + \beta \frac{1+\alpha^m}{1+\alpha} + \rho \frac{1-\alpha^m}{1-\alpha} \right\} \\ &= \beta \frac{1}{1+\alpha} + \rho \frac{1}{1-\alpha} < \frac{2\beta}{1+\alpha}. \end{aligned} \quad (43)$$

Clearly, (43) implies that $s(k)$ eventually falls into a bounded set Ω_s ,

$$\Omega_s = \left\{ s(k) \in \mathcal{R} \mid |s(k)| < \Delta = \frac{2\beta}{1+\alpha} \right\} \quad (44)$$

whose size depends on the choice of α and β . According to (11) and the BIBO stability theorem, (44) readily suggests that the tracking error eventually falls into a bounded set $\left\{ e(k) \in \mathcal{R} \mid |e(k)| < \kappa = \frac{2H(1)\beta}{1+\alpha} \right\}$

whose size κ depends on the parameters of $H(1)$, α and β which are chosen at our disposal.

Defining

$$z_{1d}(k) = (y_d(k), y_d(k+1), \dots, y_d(k+r-1))^T. \quad (45)$$

and using the definition of $s(k)$ and (44) lead to the boundedness z_1 .

Next, since the zero dynamics are exponentially stable, by using the converse theorem of Lyapunov⁽⁶⁾, a Lyapunov function $V(z_2)$ and appropriate constants $\sigma_i > 0$, $i=1, \dots, 4$ exist such that

$$\begin{aligned} \sigma_1 \|z_2(k)\|^2 &\leq V(z_2(k)) \leq \sigma_2 \|z_2(k)\|^2 \\ \Delta V(z_2(k))|_{(5)} &= V(z_2(k+1)) - V(z_2(k)) \\ &= V(\Psi(0, z_2(k))) - V(z_2(k)) \leq -\sigma_3 \|z_2(k)\|^2 \\ \left\| \frac{\partial V(z_2(k))}{\partial z_2(k)} \right\| &\leq \sigma_4 \|z_2(k)\| \end{aligned} \quad (46)$$

Taking the difference of V along the trajectory (5) yields

$$\begin{aligned} \Delta V(z_2(k))|_{(5)} &= V(z_2(k+1)) - V(z_2(k)) \\ &= \{ V(\Psi(0, z_2(k))) - V(z_2(k)) \} \\ &\quad + \{ V(z_2(k+1)) - V(\Psi(0, z_2(k))) \} \\ &= \Delta V(z_2(k))|_{(6)} + \left. \frac{\partial V(z_2(k))}{\partial z_2(k)} \right|_{z_2(k)+(1-\xi)(z_2(k+1)-z_2(k))} \\ &\quad \times (\Psi(z_1(k), z_2(k)) - \Psi(0, z_2(k))) \\ &\leq -\sigma_3 \|z_2(k)\|^2 + \sigma_4 L \|\xi z_2(k) \\ &\quad + (1-\xi)\Psi(z_1(k), z_2(k))\| \|z_1(k)\| \\ &\leq -\sigma_3 \|z_2(k)\|^2 + k_1 \|z_1(k)\| \|z_2(k)\| + k_2 \|z_1(k)\|^2 \\ &\leq -\sigma_3 \|z_2(k)\|^2 + k_1 M \|z_2(k)\| + k_2 M^2 \end{aligned} \quad (47)$$

Where the final equality and the first inequality follow from the Mean Value Theorem⁽²⁴⁾ with $0 < \xi < 1$ and from (40) as well as assumption A2. Since $z_1(k)$ is bounded for all k , a constant $M \geq 0$ exists such that $\|z_1(k)\| \leq M$ for all k . Therefore, the final inequality of (47) is obtained. Since $\Delta V|_{(5)} \leq 0$ whenever $\|z_2(k)\| \geq \frac{k_1 M + \sqrt{(k_1 M)^2 + 4k_2 \sigma_3 M^2}}{2\sigma_3}$, we can conclude that there

exists $l_b > 0$ such that $\|z_2(k)\| \leq l_b$. Consequently, we can conclude the boundedness of all state variables z . The fact that $Z = \Phi(x)$ is a coordinate transformation accounts for why the boundedness of the state x of the closed-loop system then directly follows. This completes our proof.

References

- (1) Bartoszewicz, A., Remarks on Discrete-time Variable Structure Control Systems. IEEE Trans. Ind. Electronics, Vol. IE-43 (1996), pp. 235-238.
- (2) Brockett, R.W., Feedback Invariant for Nonlinear Systems, IFAC Congress, Helsinki, (1978), pp. 1115-1120.
- (3) Castillo, B. and Gennaro, S.D., Asymptotic Output Tracking for SISO Nonlinear Discrete Systems, Proc. of the 28th IEEE Conf. on Decision and Control, Brighton, (1991), pp. 979-986.
- (4) Chan, C.Y., Robust Discrete-time Sliding Mode

- Controller, *Sys. Control Lett.*, Vol. 23 (1994), pp. 371-274.
- (5) Chen, F.C. and Khalil, H.K., Adaptive Control of Nonlinear Systems Using Neural Networks, *Int. J. Control*, Vol. 55 (1992), pp. 1299-1317.
 - (6) Delario, B.A., Zak, S.H. and Mathews, G.P., Variable Structure Control of Nonlinear Multivariable Systems: A Tutorial, *Proc. of IEEE*, Vol. 76 (1988), pp. 212-232.
 - (7) Desoer, C.A. and Vidysagar, M., *Feedback Systems: Input-output Properties*, (1975), Academic Press, New York.
 - (8) Furuta, K., Sliding Mode Control of a Discrete System, *Sys. Control Lett.*, Vol. 14 (1990), pp. 145-152.
 - (9) Gao, W., Wang, Y. and Homaifa, A., Discrete-time Variable Control Systems, *IEEE Trans. Ind. Electronics*, Vol. IE-42 (1995), pp. 117-122.
 - (10) Grizzle, J.W., Local Input-output Decoupling of Discrete-time Nonlinear Systems, *Int. J. Control*, Vol. 43 (1986), pp. 1517-1530.
 - (11) Ha, I.J. and Gilbert, G., Robust Tracking in Nonlinear Systems, *IEEE Trans. Autom. Control*, Vol. AC-32 (1987), pp. 763-771.
 - (12) Hirschorn, R., Output Tracking on Multivariable Nonlinear System, *IEEE Trans. Autom. Control*, Vol. AC-26 (1981), pp. 955-958.
 - (13) Hunt, L.R., Su, R. and Meyer, G., Global Transformation of Nonlinear Systems, *IEEE Trans. Autom. Control*, Vol. AC-28 (1983), pp. 24-31.
 - (14) Isidori, A., *Nonlinear Control System*, (1989), Springer-Verlag, New York.
 - (15) Isidori, A. and Byrnes, C.I., Output Regulation of Nonlinear Systems, *IEEE Trans. Autom. Control*, Vol. AC-35 (1990), pp. 214-224.
 - (16) Jakubczyk, B. and Respondek, W., On Linearization of Control Systems, *Bull. Acad. Polonaise Sci. Ser. Math.*, Vol. 28 (1980), pp. 517-522.
 - (17) Kotta, U., *Inversion Method in the Discrete-time Nonlinear Control Systems Synthesis Problems*, (1995), Springer-Verlag, London.
 - (18) Krener, A.J., On the Equivalence of Control Systems and Linearization of Nonlinear Systems, *SIAM J. Control Optim.*, Vol. 11 (1973), pp. 670-698.
 - (19) Lee, H.G., Arapostathis, A. and Marcus, S.I., Linearization of Discrete-time Systems, *Int. J. of Control*, Vol. 45 (1987), pp. 1803-1822.
 - (20) Luca, A.D. and Ulivi, G., Design of an Exact Nonlinear Controller for Induction Motors, *IEEE Trans. Autom. Control*, Vol. AC-34 (1989), pp. 24-31.
 - (21) Milosavljevic, D., General Conditions for Existence of a Quasi-sliding Mode on the Switching Hyperplane in Discrete Variable Structure Systems, *Autom. Remote Control*, Vol. 46 (1985), pp. 307-314.
 - (22) Monaco, S. and Normand-Cyrot, D., Minimum-phase Nonlinear Discrete-time Systems and Feedback Stabilization, In *Proc. of the 28th IEEE Conf. on Decision and Control*, Los Angeles, (1987), pp. 979-986.
 - (23) Nam, K., Linearization of Discrete-time Nonlinear Systems and a Canonical Structure, *IEEE Trans. Autom. Control*, Vol. AC-34 (1989), pp. 119-122.
 - (24) Rudin, W., *Principles of Mathematical Analysis*, (1976), McGraw-Hill, New York.
 - (25) Sarpturk, S.Z., Istefanopulos, Y. and Kaynak, O., On the Stability of Discrete-time Sliding Mode Control Systems, *IEEE Trans. Autom. Control*, Vol. AC-32 (1987), pp. 930-932.
 - (26) Sira-Ramirez, H., Nonlinear Discrete Variable Structure Systems in Quasi-sliding Mode, *Int. J. Control*, Vol. 54 (1991), pp. 1171-1187.
 - (27) Tarn, D.G., Bejczy, A.K., Isidori, A. and Chen, Y.L., Nonlinear Feedback in Robot Arm Control, In *Proc. of 23rd IEEE Conf. on Decision and Control*, Las Vegas, (1984), pp. 345-350.
 - (28) Yu, X. and Potts, R.B., Computer-controlled Variable Structure Systems, *J. Austral. Math. Soc. Ser. B*, Vol. 34 (1992), pp. 1-17.