Development of Concave Conical Gear Used for Marine Transmissions*
(2nd Report, Principal Normal Radii of Concave Conical Gear and Design of a Pair of Gears)

Hidenori KOMATSUBARA**, Ken-ichi MITOME***
and Tatsuya OHMACHI***

In the first report, the method of generating Concave conical gears was presented. In this paper, first the principal normal radii of the tooth surface generated by this method are expressed theoretically in terms of the basic dimensions of the gear and the tool. The principal normal radii of test gears are measured and found to be in good agreement with the theoretical values. Secondly, the allowable normal load of a pair of Concave conical gears is obtained. Tooth bearing tests prove that the Hertzian contact ellipse is larger than that of the conventional conical gear. Thus it is proven that the method of designing a pair of Concave conical gears is of practical use.

Key Words: Gear, Grinding, Conical Involute Gear, Concave Conical Gear, Helical Concave Conical Gear, Allowable Normal Load, Design

1. Introduction

This research is aimed at the establishment of the design and production system of Concave conical gears for marine transmissions. For this purpose, the method of generating Concave conical gears was developed, as presented in the first report(1). The tooth surface generated by this method is called “Concave tooth surface”, and is shown in Fig. 1(3)-(6). Let the pitch point be the representative point of the tooth surface. Then the tooth surface near the pitch point $R$ is approximated by the hyperboloid in Fig. 2, where $R$ and $R'$ are the principal normal radii of the concave tooth surface. Hence the contact between tooth surfaces of Concave conical gear can be approximated by the contact between two hyperboloids. If $R$ and $R'$ are expressed using basic dimensions of the gear and the tool, then the allowable normal load of a pair of

---

** Niigata Engineering Co., Ltd., NICO Company, 403 Gejka, Kamo, Niigata 959-1391, Japan. E-mail: h-komatsubara@niigata-converter.co.jp
*** Faculty of Engineering, Yamagata University, 4-3-16 Jo-nan, Yonezawa, Yamagata 992-8510, Japan

---

Fig. 1 Concave tooth surface in comparison with tooth surface of conical involute gear

Fig. 2 Hyperboloid

---

3. Principal Normal Radii of Concave Tooth Surface

3.1 Principal normal radii

Let vector \( \mathbf{r} \) be the position vector of an arbitrary point on the tooth surface of the gear generated using a conical wheel, where, \( X, Y, \) and \( Z \) are given by Eq. (2).

\[
\mathbf{r} = r(u, v) = X(u, v) \mathbf{i} + Y(v, v) \mathbf{j} + Z(u, v) \mathbf{k}
\]

Here, we obtain

\[
\begin{align*}
\frac{\partial r}{\partial u} &= r_u \\
\frac{\partial r}{\partial v} &= r_v \\
\frac{\partial^2 r}{\partial u^2} &= r_{uu} \\
\frac{\partial^2 r}{\partial v^2} &= r_{vv} \\
\frac{\partial^2 r}{\partial u \partial v} &= r_{uv}
\end{align*}
\]

We discuss the pitch point \( P_b \). The unit vector perpendicular to the tooth surface is defined by Eq. (6).

\[
\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} = n_u \mathbf{i} + n_v \mathbf{j} + n_w \mathbf{k}
\]

The first fundamental quantities \( E, F, G \) and the second fundamental quantities \( L, M, N \) at the pitch point can be obtained as follows.

\[
\begin{align*}
E &= r_u^2 = r_v^2 \\
F &= \mathbf{r}_u \cdot \mathbf{r}_v = 0 \\
G &= r_v^2 = 1 \\
L &= \mathbf{n} \cdot \mathbf{r}_{uv} \\
M &= \mathbf{n} \cdot \mathbf{r}_{wu} \\
N &= \mathbf{n} \cdot \mathbf{r}_{vw}
\end{align*}
\]

The normal curvature at the pitch point \( P_b \) can be expressed by Eq. (14).

\[
\frac{1}{R} = \frac{L (du)^2 + 2M dv + N (dv)^2}{E (du)^2 + 2F dv + G (dv)^2}
\]

Setting \( du/dv = \xi \), Eq. (14) becomes

\[
\frac{1}{R} = \frac{E \xi^2 + 2F \xi + G}{E \xi^2 + 2F \xi + G}
\]

The curvature \((1/R)\) is a function of \( \xi \), so the minimum and maximum of \((1/R)\) can be determined by Eq. (15). As a result, minimum and maximum radii of the principal curvature are obtained as follows:
3.2 Directions of two principal planes of concave tooth surface

The unit vectors \( e \) and \( e' \) in the directions of the principal plane of the concave tooth surface at the pitch point are obtained as follows:

\[
\begin{align*}
e &= \frac{U}{\sqrt{1 + \xi^2}} \\
U &= \left( \zeta_1 \cos \phi + \zeta_2 \sin \phi \right)I \\
&+ \left( \zeta_3 \cos \phi + \zeta_4 \sin \phi \right)J \\
&+ \left( \zeta_5 \cos \phi + \zeta_6 \sin \phi \right)K \\
e' &= \frac{U'}{\sqrt{1 + \xi^2}} \\
U' &= \left( \zeta_1' \cos \phi + \zeta_2' \sin \phi \right)I \\
&+ \left( \zeta_3' \cos \phi + \zeta_4' \sin \phi \right)J \\
&+ \left( \zeta_5' \cos \phi + \zeta_6' \sin \phi \right)K,
\end{align*}
\]

where

\[
\begin{align*}
\xi &= \left( \frac{k_1^2 + k_2^2}{k_3} \right) \\
\zeta_1 &= k_1 \cos \phi + k_2 \sin \phi \\
\zeta_2 &= k_1 \sin \phi - k_2 \cos \phi \\
\zeta_3 &= (r_d r_b)(B_3^2 - B_2^2)/B_1 - \sin a_0 \\
\zeta_4 &= 2(r_d r_b)/B_3 \\
\zeta_5 &= (r_d^2 + r_b^2)/B_1 + \sin a_0 \\
\zeta_6 &= - (r_d^2 + r_b^2)/B_3.
\end{align*}
\]

Next we obtain the unit vectors in the directions of the principal plane of the conical involute gear at the pitch point. These vectors \( e_\infty \) and \( e'_\infty \) are obtained by substituting \( r = \infty \) into Eqs. (18) and (19).

\[
\begin{align*}
e_\infty &= \frac{B_1 I + B_2 J}{\sqrt{B_1^2 + B_2^2}} \\
e'_\infty &= \frac{B_1 B_2 I + B_3 B_2 J - (B_1^2 + B_2^2)K}{\sqrt{B_1^2 + B_2^2}} \\
B_1 &= \cos a_0 \cos \phi \\
B_2 &= \sin a_0 \sin \phi \\
B_3 &= \sin a_0 \sin \phi \\
B_4 &= \cos a_0 \cos \phi \\
B_5 &= \sin a_0 \sin \phi \\
B_6 &= \sin a_0 \cos \phi \\
B_7 &= \cos a_0 \sin \phi \\
B_8 &= \cos a_0 \cos \phi \\
B_9 &= \sin a_0 \sin \phi \\
B_{10} &= \sin a_0 \cos \phi \\
B_{11} &= \cos a_0 \sin \phi
\end{align*}
\]

The unit vectors \( e, e', e_\infty \) and \( e'_\infty \) are in the relative positions shown in Fig. 3. The angle \( \mu \) between vectors \( e_\infty \) and \( e'_\infty \) is obtained from Eq. (25).

\[
e \cdot e'_\infty = \cos \mu
\]

The tooth section profile of Concave conical gear is measured along the tooth surface generating line of the corresponding conical involute gear \( l \). The radius of curvature of this section profile \( R'_\infty \) is obtained by Euler's formula (26).

\[
\left( \frac{1}{R'_\infty} \right) = \cos \mu + \sin \mu \frac{\cos \phi}{R}
\]

3.3 Theoretical tooth section profile

The theoretical tooth section profile of Concave conical gear, which is cut through the plane in contact with the base cylinder of the corresponding conical involute gear, is obtained by using Eq. (27), as presented in the first report.

\[
y_i = \frac{1}{r_b} \left( A_1 V + A_2 W - \sqrt{r_b^2 - r_i^2} \sin \beta_b \right) + A_3 \cos \beta_b
\]

\[
\omega_i = \frac{1}{r_b} \left( A_1 V + A_2 W - \sqrt{r_b^2 - r_i^2} \cos \beta_b \right) + A_3 \sin \beta_b
\]

\[
V = \sqrt{r_b^2 - r_i^2} \cos \phi - r_i \sin \phi
\]

\[
W = \sqrt{r_b^2 - r_i^2} \sin \phi + r_i \cos \phi
\]

4. Left Tooth Surface of Concave Conical Gear

In sections 2 and 3, the right tooth surface was discussed. We can discuss the left tooth surface in the same manner by using \(-\phi\) instead of \(\phi\) in Eqs. (2) to (27).

5. Verification of Principal Normal Radius

Table 1 shows the basic dimensions of test gears made for trial.

<table>
<thead>
<tr>
<th>No.</th>
<th>( m )</th>
<th>( a_0 )</th>
<th>( \psi )</th>
<th>( \delta )</th>
<th>( z )</th>
<th>( r_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>30°</td>
<td>15°</td>
<td>30°</td>
<td>32</td>
<td>72</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>30°</td>
<td>0°</td>
<td>30°</td>
<td>32</td>
<td>72</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>30°</td>
<td>15°</td>
<td>0°</td>
<td>32</td>
<td>72</td>
</tr>
<tr>
<td>D</td>
<td>3</td>
<td>30°</td>
<td>0°</td>
<td>0°</td>
<td>32</td>
<td>72</td>
</tr>
</tbody>
</table>
superposed on the latter. Here, only profiles of test gear A are shown, but the measured profiles for gears B, C and D also agree well with the theoretical ones.

5.2 Verification of the principal normal radius

As shown in Fig. 4, it is proven that the tooth surface of the ground gear is in good agreement with the theoretical one. However, in actual design, the tooth surface is designed using the principal normal radii in Eq. (16). Therefore, we verify Eq. (16). In the verification, theoretical $R_e'$ of Eq. (26) is compared with the measured $R_e'$. Theoretical $R_e'$ is obtained using Eqs. (16), (25) and (26). Next the measured $R_e'$ is obtained from the measured profile. Here the measured profile near the pitch point $P_0$ is approximated by a circle with radius $R_e'$, as shown in Fig. 5. From Fig. 5, the radius $R_e'$ is obtained using Eq. (28).

$$R_e' = \frac{y_z^2 + z^2}{2z_e}$$  \hspace{1cm} (28)

Figure 6 shows both theoretical and measured values of $R_e'$. The measured value at the pitch point obtained by interpolation agrees well with the theoretical one (○). These results verify $R$ and $R'$ in Eq. (16). In the following, we treat the concave tooth surface near the pitch point as a hyperboloid having principal normal radii $R$ and $R'$.

6. Allowable Normal Load of a Pair of Concave Conical Gears

6.1 Basic theory

The contact between tooth surfaces of a pair of conical involute gears is approximated by the contact between two cylinders, as shown in Fig. 7(a). On the other hand, the contact between tooth surfaces of a pair of Concave conical gears can be approximated by the contact between two hyperboloids, as shown in Fig. 7(b). The allowable normal loads in these cases are obtained as follows.

First let us introduce the Hertzian formulas for two elastic bodies in point contact under load $P$. Curved surface 1 is shown in Fig. 8, where planes I and $\Gamma'$ are the principal curvature planes at point $Q$, and the minimum and maximum radii of curvature are $R_1$ and $R_\Gamma$, respectively. The vector $n$ is the normal unit vector of curved surface 1 at point $Q$.

In the same way, curved surface 2 is shown in Fig. 9, where planes $\Pi$ and $\Pi'$ are the principal curvature planes at point $Q$, and the minimum and maximum
radii of curvature are $R_1$ and $R_2$, respectively. The normal unit vector at point $Q$ is also expressed by vector $n$.

Figure 10 shows the contact between the two bodies shown in Fig. 7, as the contact between curved surfaces 1 and 2. Here angle $\theta$ is between principal planes I and II. In this case, the major semiaxis $a$ and minor semiaxis $b$ of the Hertzian contact ellipse, and the Hertzian stress $\sigma_h$ are obtained as follows:

\[
a = M_1 \left( \frac{3}{4} \right) \frac{P}{H} \left( \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right)
\]

\[
b = M_2 \left( \frac{3}{4} \right) \frac{P}{H} \left( \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right)
\]

\[
\sigma_h = 3P/(2\pi ab)
\]

The nomenclature used in this paper is described below. Subscripts 1 and 2 refer to body 1 or curved surface 1, and body 2 or curved surface 2, respectively.

$P$ = Load (N)

$E_1$, $E_2$ = Young’s modulus of bodies 1 and 2

$\nu_1$, $\nu_2$ = Poisson’s ratio of bodies 1 and 2

$R_i$, $R_i'$ = minimum and maximum principal radii of curved surface 1 at point $Q$

$R_1$, $R_2$ = minimum and maximum principal radii of curved surface 2 at point $Q$

$H$ = relative average curvature

$H = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_1'} + \frac{1}{R_2} + \frac{1}{R_2'} \right)$

$T$ = relative principal torsion

$T = \frac{1}{2} \left[ \frac{1}{R_1} \left( \frac{1}{R_1'} \right)^2 + \left( \frac{1}{R_2} \right)^2 \left( \frac{1}{R_2'} \right)^2 \right]

+ \frac{2}{R_1} \left( \frac{1}{R_1'} \right) \left( \frac{1}{R_2} \right) \cos 2\theta \left( \frac{1}{R_2'} \right)

K_1, K_2$ = relative principal curvature

$K_1 = H - T$

$K_2 = H + T$
Fig. 11 $M$, $N$ and $MN$ versus $T/H$

$K = K_1 K_2 = H^2 - T^2$

$M, N = \text{variable parameters}$

Parameters $M$ and $N$ can be expressed in terms of $T/H$, as shown in Fig. 11, from Roark's table(39).

Next, we obtain the allowable normal load $P_a$. Letting the smaller of the allowable surfaces of bodies 1 and 2 be $\sigma_a$, we obtain the following equation by using $P_a$ and $\sigma_a$ instead of $P$ and $\sigma_t$, respectively.

$$\sigma_a = 3 P_a / (2 \pi a b)$$

Substituting Eqs. (29) and (30) into this equation, and rearranging, we obtain(31)-(44)

$$P_a = \pi^2 \sigma_a \left( 1 \right) \left( \frac{1}{E_1} + \frac{1}{E_2} \right) \frac{(MN)^2}{H^2}. \tag{37}$$

Here, we introduce a new parameter $\kappa$ defined by the following equation.

$$(MN)^2 / H^2 = \kappa \tag{38}$$

Then Eq. (37) is reduced to

$$P_a = \pi^2 \sigma_a \left( 1 \right) \left( \frac{1}{E_1} + \frac{1}{E_2} \right) \frac{(MN)^2}{H^2}. \tag{39}$$

In Eq. (39), parameters $\sigma_a$, $E_1$, $E_2$, $\nu_1$ and $\nu_2$ are determined by the material of bodies 1 and 2. Parameter $\kappa$ is determined by curved surfaces 1 and 2. That is to say, parameter $\kappa$ can be obtained if $R_1$, $R_2$, $R_3$, $R_4$ and $\theta$ are given.

In the case of Concave conical gear, $R_1$, $R_2$, $R_3$, and $R_4$ at pitch point $P_0$ are obtained using Eq. (16). The details of obtaining angle $\theta$ are described in section 6.2.

6.2 Angle between principal planes on tooth surfaces of a pair of Concave conical gears

6.2.1 Angle $\theta_{rr}$ between principal planes when right tooth surface of gear 1 engages with right tooth surface of gear 2

Figure 12 shows the contact between tooth surfaces of conical involute gears 1 and 2. Here, subscripts $r$ and $rr$ indicate the right tooth surface, and engagement between two right tooth surfaces, respectively. Angle $\theta_{rr}$ is the angle between the principal planes of conical involute gears 1 and 2. Moreover, the angle $\theta_{rr}$ is obtained from Eq. (40).

$$\cos \theta_{rr} \left( \sin \gamma_{rr1} \sin \gamma_{rr2} \right) = - \cos^2 \sigma_a \sin \delta_1 \sin \delta_2$$

$$+ (\cos^2 \sigma_a \cos^2 \phi_1 + \sin^2 \sigma_a) \cos \delta_1 \cos \delta_2$$

$$- \cos \sigma_a \sin \sigma_a \sin \psi_1 \sin (\delta_1 - \delta_2)$$

$$\cos \gamma_{rr} = \sin \sigma_a \sin \delta_2 - \cos \sigma_a \sin \psi_1 \cos \delta_1$$

$$\left( i = 1, 2 \right) \tag{40}$$

The angle between the principal planes of Concave conical gears $\theta_{rr}$ is obtained using Eq. (41).

$$\theta_{rr} = \theta_{rr0} - \mu_{rr} - \mu_{rr} \tag{41}$$

Here, angles $\mu_{rr}$ and $\mu_{rr}$ are given by Eq. (25).

6.2.2 Angle $\theta_{rl}$ between principal planes when left tooth surface of gear 1 engages with left tooth surface of gear 2

Figure 13 shows the contact between tooth surfaces of the conical involute gears 1 and 2. Here, subscripts $l$ and $ll$ indicate the left tooth surface, and engagement between two left tooth surfaces, respectively. Angle $\theta_{ll}$ is the angle between the principal planes of the conical involute gears 1 and 2. Furthermore, the angle $\theta_{ll}$ is obtained using Eq. (42).

$$\theta_{ll} = \theta_{ll0} - \mu_{ll} - \mu_{ll} \tag{42}$$

Here, angles $\mu_{ll}$ and $\mu_{ll}$ are given by Eq. (25).


JSME International Journal
Fig. 14 Procedure for determining allowable normal load $P_{arr}$

\[
\cos \theta_{追}(\sin \gamma_{追} \sin \gamma_{追'i}) = \cos^2 \alpha_0 \sin \delta_1 \sin \delta_2 \\
-\left( \cos^2 \alpha_0 \cos^2 \phi_1 \sin^2 \alpha_0 \cos \delta_1 \cos \delta_2 \right) + \cos \alpha_0 \sin \phi_1 \sin \delta_1 \sin (\delta_2 - \delta_1) \\
\cos \gamma_{追} = -\sin \alpha_0 \sin \delta_1 \\
+ \cos \alpha_0 \sin \phi_1 \cos \delta_1 \quad (i=1,2)
\]

(42)

The angle between the principal planes of Concave Conical gears $\theta_{追}$ is obtained from

$$\theta_{追} = \theta_{追'i} - \mu_1 - \mu_2.$$

(43)

Here, angles $\mu_1$ and $\mu_2$ are obtained as described in section 4.

6. 3 Allowable normal load of a pair of Concave conical gears

Figure 14 shows the procedure for determining the allowable normal load $P_{arr}$ when the right tooth surface of gear 1 engages with the right tooth surface of gear 2. For the case where the left tooth surface of gear 1 engages with the left tooth surface of gear 2, the allowable normal load can be obtained similarly.

7. Practical Design of Concave Conical Gears

Table 2 shows the basic dimensions of intersecting-axis Concave conical gears designed and made for trial. Table 3 shows calculated results for the case where the right tooth surface of gear 1 engages with the right tooth surface of gear 2. Here, $P_{arr}$ is the allowable normal load of the corresponding conical involute gear. We obtain $P_{arr}/P_{arr}=2.55$. That is to say, the allowable normal load of Concave conical gear increased 155% in comparison with that of the corresponding conical involute gear.

Table 4 shows calculated results for the case of engagement between left tooth surfaces of gears 1 and 2. Since $P_{arr}/P_{arr}=2.25$, the allowable normal load of Concave conical gear increased 125% in comparison with that of the corresponding conical involute gear. The amount of increase can be controlled by the pitch radius of the conical wheel $r_c$, presented in the 1st report. If $r_c=\infty$, then $P_{arr}=P_{arr}$ and $P_{arr}=P_{arr}$. Figure 15 shows the tooth bearing of Concave conical gears and that of the corresponding conical involute gears. The width of the tooth bearing of the former increased to more than twice that of the latter. This proves that the Hertzian contact ellipse of Concave conical gears becomes larger than that of the corresponding conical involute gears.

This design and production system is applied to Concave conical gears for marine transmission. Figure 16 shows the tooth bearing of the pinion gear used.
of the basic dimensions of Concave conical gear. These $R$ and $R'$ were verified by experiments.

(3) In the design of a pair of intersecting-axis Concave conical gears, the allowable normal load is obtained in terms of the basic dimensions and material parameters of a pair of Concave conical gears.

(4) Test gears were designed and made. The theoretical allowable normal load of Concave conical gears increased more than twofold in comparison with the corresponding conical involute gears.

(5) In tooth bearing tests, the width of the tooth bearing of Concave conical gears became about twice the length of the tooth bearing of the corresponding conical involute gears. This confirmed that the Hertzian contact ellipse of Concave conical gears became larger in area and its allowable normal load increased considerably, in comparison with the corresponding conical involute gears.

(6) This research has opened the door to the development of Concave conical gears for heavy-duty marine transmissions.

8. Conclusions

(1) The tooth surface of Concave conical gear generated by a conical wheel was expressed in terms of the basic dimensions of Concave conical gear: $m$, $a$, $\phi$, $\delta$, $z$ and $r_s$, and the parameters $u$ and $v$ of the conical wheel.

(2) Radii of principal curvature $R$ and $R'$ of the tooth surface at the pitch point were obtained in terms

References


(2) Mitome, K., Concave Conical Gear, Int. Conf. Motion Transmissions, 3D2 (1991), Hiroshima, Japan.


