Analysis and Synthesis of General-Type Goldberg Five-Revolute Mechanism*

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Geometric constraints and formation of general-type Goldberg five-revolute mechanism are outlined through the analytical geometry and its general analytical kinematic closed-form solutions are derived based on matrix algebra. Applying the concept of relative motion and using the coordinate-transformation matrices, we not only present the parametric formulas of its coupler link and coupler point but also provide a procedure for the higher-order analysis of coupler curve. Moreover, this mechanism is used for the dimensional synthesis, function, path and motion generation by the nonlinear programming algorithm. Finally, three numerical examples are given for illustrations and their results of synthesis are separately verified by computer animation with 3D computer graphics.

Key Words: General-Type Goldberg Five-Revolute Mechanism, Formation of Mechanism, Kinematic Closed-Form Solutions, Higher-Order Analysis, Dimensional Synthesis, Nonlinear Programming Algorithm

1. Introduction

Various investigations⁽¹⁾⁻⁽¹¹⁾, up to now, have demonstrated that general-type Goldberg five-revolute mechanism disobeying the Kutzbach-Grubler constraint criterion has constrained motion. This mechanism was first synthesized by combining two Bennett four-revolute chains⁽¹⁾. Actually, Myard five-revolute mechanism⁽¹²⁾, which is symmetrical about a plane, is a special case of the general-type Goldberg five-revolute one because it is a combination of two Bennett 4R chains with the normal link-twists⁽⁷⁾.

The studies of general-type Goldberg five-revolute mechanism are still received the attention of several researchers for a long time. Among them, Goldberg⁽¹⁾ merely formed the mechanism instead of analyzing it. Baker⁽⁴⁾ delineated the mechanism by a complete set of closure equations without deriving closed-form solutions. As for the dimensional synthe-

sis of this mechanism, none of the researchers was ever engaged in it. However, to generate new linkages from Goldberg and Bennett chains, Bennett⁽¹³⁾ introduced the pioneering work. Goldberg⁽¹⁾ formed four types of spatial 6R linkages using three Bennett linkages. In addition, Waldron⁽¹⁴⁾ made the hybrid linkages but did not quantitatively consider the dimensions. Yu and Baker⁽⁵⁾, based on the Waldron's concepts, generated the 6R linkages just using Bennett 4R loops. Baker⁽¹⁵⁾ further made a comparative survey of the Bennett-based 6R kinematic loops. Lee⁽¹⁶⁾ systematically synthesized six types of movable spatial 6R linkages through the analytical coordinate geometry and computer graphics. It is worth mentioning that Wohlhart⁽¹⁷⁾ presented an approach to generate 6R linkages from two Goldberg 5R chains. Now, in this work, from the viewpoint of analytical coordinate geometry and computer graphics, we not only utilize general-type Goldberg 5R and Bennett 4R chains to synthesize the movable spatial 5R or 6R mechanisms systematically but also confirm their constrained motions by the Newton-Raphson iteration^{(18),(20)} and Davidon-Fletcher-Powell optimization methods^{(19),(20)}. So far as we know, there are very few practical

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applications for general-type Goldberg five-revolute mechanism except one potential spatial mixing machine developed by Baker, Min & Bhati⁽²¹⁾. Their paper also exhibited the actual application image of this mechanism. But, its further investigations for the industrial application are still fascinating and challenging.

The purpose of this work is focused on the analysis and synthesis of general-type Goldberg five-revolute mechanism. First, the geometric constraints and the way to form the general type Goldberg five-revolute or other movable six-revolute mechanisms from basic building block, such as Bennett 4R or Goldberg 5R loop are outlined from the standpoint of the analytical coordinate geometry. And based on matrix algebra, we derive the general analytical kinematic closed-form solutions for the preliminary investigations. After that, applying the concept of relative motion and using the transformation matrices, we develop the parametric formulas of the coupler curve & link as well as provide a procedure for the higher-order analysis of coupler curve. Moreover, this mechanism is used for the dimensional synthesis, function, path and motion generation synthesis by the nonlinear programming. Finally, three numerical examples are taken to illustrate the synthesis algorithm accompanied with 3D computer graphics.

2. Geometry and Formation of Mechanism

General-type Goldberg five-revolute mechanism is a famous movable spatial 5R mechanism characterized by constrained motion and is supposed to have the potential practical application in the future. In fact, Myard 5R mechanism⁽¹²⁾ is just a special one of this general-type mechanism⁽⁷⁾. In the following, we apply the frequently used symbolic sign, Denavit and Hartenbergs' notation⁽²²⁾, as shown in Fig. 1, in which a_i , α_i , and S_i are link parameters and θ_i is a joint variable ($i=1, 2, 3, \dots, 5$). This general-type mechanism can be derived by means of addition or subtraction of two Bennett 4R chains⁽¹⁾, which are arranged

Fig. 1 Link parameters and joint variable

with one pair of common normal links and two pairs of their adjoint revolute joint axes together. In addition, we replace two adjacent normal links by one common normal link with non-zero link offsets, remove one pair of collinear revolute joints, and then delete the original common normal links. The derived mechanism must also satisfy the following dimensional constraints as

$$a_1 = a_4, \quad \alpha_1 = \alpha_4, \quad S_1 = S_5, \quad S_2 = S_3 = S_4 = 0, \\ \text{and} \quad a_1/\sin \alpha_1 = \pm a_2/\sin \alpha_2 = \pm a_3/\sin \alpha_3 \quad (1)$$

When the relationships in Eq.(1) are taken into consideration and are satisfied as well, the mobility of mechanism is verified by the correction and validity of its constrained motion. In the following, s is an abbreviation for $\sin$ and c for $\cos$.

The systematic formation of general-type mechanism is shown in Fig. 2. First, take two Bennett 4R loops $1'2'3'4'$ and $1''2''3''4''$ respectively, and put the origin o' of a local coordinate system $x-y-z$ at joint $4'$ (or $1''$). Axis z is coincident with axis $z_{4'}$, axis x directed along axis $x_{1'}$ (or a_4) and axis y is determined by the right-hand rule. Without loss of generality, we take the assemble angle Ψ as the angle between two adjacent links $a_{4'}$ and $a_{4''}$, and then from the viewpoint of analytical coordinate geometry, we have vector loop equation as follows

$$a_{4'} \mathbf{u}_{a_{4'}} - S_1 \mathbf{u}_{S_1} - a_5 \mathbf{u}_{a_5} - S_5 \mathbf{u}_{S_5} - a_{4''} \mathbf{u}_{a_{4''}} = \mathbf{0} \quad (2)$$

where $\mathbf{u}_{a_{4'}} = [1, 0, 0]$, $\mathbf{u}_{S_1} = [0, -s\alpha_2, c\alpha_2]$, $\mathbf{u}_{a_5} = [(s\alpha_2 c\alpha_3 - c\alpha_2 s\alpha_3 c\Psi)/\kappa - c\alpha_2 s\alpha_3 s\Psi/\kappa, -s\alpha_2 s\alpha_3 s\Psi/\kappa]$, $\mathbf{u}_{S_5} = [s\alpha_3 s\Psi, -s\alpha_3 c\Psi, c\alpha_3]$, $\mathbf{u}_{a_{4''}} = [c\Psi, s\Psi, 0]$, and $\mathbf{k} = [1 - (c\alpha_2 c\alpha_3 + s\alpha_2 s\alpha_3 c\Psi)^2]^{1/2}$.

Based on the Cramer's rule, we establish link-offsets S_1 and S_5 , link-length a_5 , and link-twist α_5 as follows

$$a_5 = \sin^{-1}(\mathbf{k}) \quad (3.a)$$

$$S_1 = S_5 = [a_2 s\alpha_3 (c\alpha_2 c\alpha_3 - 1) s\Psi + a_2 s\alpha_2 s^2 \alpha_3 s\Psi c\Psi] / \mathbf{k} \quad (3.b)$$

Fig. 2 Formation of general-type Goldberg five-revolute mechanism

$$a_5 = [a_3 c \alpha_2 (s \alpha_3 - s \alpha_2 c \Psi) + a_2 c \alpha_3 (s \alpha_2 - s \alpha_3 c \Psi)] / k \quad (3.c)$$

If we take $a_1 = a_4 = a$, $\alpha_1 = \alpha_4 = \alpha$, $a_5 = a_2 + a_3$, $\alpha_5 = \alpha_2 + \alpha_3$, and $a_1/\sin \alpha_1 = \pm a_2/\sin \alpha_2 = \pm a_3/\sin \alpha_3$, we obtain the basic Goldberg 5R mechanism, as shown in Fig. 3 which are arranged with two coincident and collinear mating adjacent common normal links. Furthermore, if we take $a_1 = a_4 = a$, $a_2 = a_3$, $\alpha_1 = 90^\circ$, $\alpha_4 = -90^\circ$, $a_5 = a_2 - a_3 = 0$, $\alpha_3 = -\alpha_2$, $\alpha_5 = \pi - 2\alpha_2$, and $a_1/\sin \alpha_1 = \pm a_2/\sin \alpha_2 = \pm a_3/\sin \alpha_3$, it is a Myard 5R mechanism which is symmetrical about a plane as shown in Fig. 4.

By the way, combining one general-type Goldberg 5R and one Bennett 4R chains with two pairs of the adjacent common normal link and removing these original common ones, we construct one movable spatial 5R mechanism, Goldberg-Bennett 5R mechanism. For clear explanation, referring to Fig. 5, let link lengths $a_{2'}$ & $a_{1''}$, $a_{3'}$ & $a_{2''}$ and three pairs of joint axes $Z_{2'}$ & $Z_{1''}$, $Z_{3'}$ & $Z_{2''}$, $Z_{4'}$ & $Z_{3''}$ between general-type Goldberg 5R chain $1'-2'-3'-4'-5'$ and Bennett 4R chain $1''-2''-3''-4''$ be collinear together. Then, the symbols

Fig. 3 A basic Goldberg 5R mechanism

Fig. 4 Myard 5R mechanism

$1', 2', 3', 4', 5', 1'', 2'', 3'',$ and $4''$ indicate revolute joints. Then, the generated mechanism is 1-2-3-4-5, which belongs to an isomeric configuration of the original Goldberg 5R-mechanism⁽²³⁾. For a numerical illustration, take link parameters of general-type Goldberg 5R and Bennett 4R chains respectively as: (A) $a_{1'} = a_{4'} = 10$, $a_{2'} = 6$, $a_{3'} = 8.4853$, $a_{5'} = 12.9331$, $S_{1'} = S_{5'} = 2.1931$, $S_{2'} = S_{3'} = S_{4'} = 0$, $\alpha_{1'} = \alpha_{4'} = -56.44^\circ$, $\alpha_{2'} = 30^\circ$, $\alpha_{3'} = 45^\circ$; $\alpha_{5'} = 68.83^\circ$, and (B) $a_{1''} = a_{3''} = 8.4853$, $a_{2''} = a_{4''} = 6$, $S_{1''} = S_{2''} = S_{3''} = S_{4''} = 0$, $\alpha_{1''} = \alpha_{3''} = 45^\circ$, $\alpha_{2''} = \alpha_{4''} = 30^\circ$, with the common normal link lengths $a_{2'}$ & $a_{1''}$, $a_{3'}$ & $a_{2''}$, $Z_{2'}$ & $Z_{1''}$, $Z_{3'}$ & $Z_{2''}$, and $Z_{4'}$ & $Z_{3''}$. Hence, the technique cited above leads us to obtain a Goldberg-Bennett 5R mechanism, whose link parameters are $a_1 = a_4 = 10.0$, $a_2 = 8.4853$, $a_3 = 6.0$, $a_5 = 12.9331$, $\alpha_1 = \alpha_4 = -56.44^\circ$, $\alpha_2 = 45^\circ$, $\alpha_3 = 30^\circ$, $\alpha_5 = 68.83^\circ$, $S_1 = S_5 = 2.19306$, and $S_2 = S_3 = S_4 = 0$. The constrained motion of this mechanism confirmed by DFP optimization and Newton-Raphson methods⁽¹⁶⁾ is shown in Fig. 6.

Fig. 5 Hybrid 5R mechanism from Goldberg 5R and Bennett 4R chains

Fig. 6 Displacement diagram of hybrid 5R mechanism

Furthermore, by merging any two general-type Goldberg 5R chains with one pair of common coincident link length and two pairs of collinear joint axes as well as by using a developed technique and derived mathematical formula above, we construct one movable spatial Goldberg-Goldberg 6R mechanism. For illustration, let link lengths $a_{5'}$ & $a_{5''}$ and two pairs of joint axes $z_{1'}$ & $z_{1''}$ and $z_{5'}$ & $z_{5''}$ of two general-type Goldberg 5R chains $1'-2'-3'-4'-5'$ and $1''-2''-3''-4''-5''$ be collinear together, as shown in Fig. 7. The symbols of $1', 2', 3', 4', 5', 1'', 2'', 3'', 4''$ and $5''$ represent revolute axes and joints. Then, based on Eqs. (3. a) to (3. c), we can determine the common normal and link twist between them. Besides, the coordinates of feet on the common normal for computer graphics can also be derived by utilization of the approach adopted in the author's referred paper⁽¹⁶⁾. Therefore, the synthesized 6R mechanism is 1-2-3-4-5-6. This generating process is slightly different from Wohlhart's approach,

Fig. 7 Hybrid 6R mechanism from two general-type Goldberg 5R chains

Fig. 8 Displacement diagram of hybrid 6R mechanism

which uses the corresponding pairs of "roof-links" equal links in common⁽¹⁷⁾. Furthermore, for a numerical example, take the link parameters of two general-type Goldberg 5R chains respectively as: (A) $a_{1'}=a_{4'}=10$, $a_{2'}=6$, $a_{3'}=8.4853$, $a_{5'}=12.9331$, $S_{1'}=S_{5'}=2.1931$, $S_{2'}=S_{3'}=S_{4'}=0$, $\alpha_{1'}=\alpha_{4'}=-56.44^\circ$, $\alpha_{2'}=30^\circ$, $\alpha_{3'}=45^\circ$, $\alpha_{5'}=68.83^\circ$, and (B) $a_{1''}=a_{4''}=5$, $a_{2''}=6$, $a_{3''}=8.4853$, $a_{5''}=12.9331$, $S_{1''}=S_{5''}=2.1931$, $S_{2''}=S_{3''}=S_{4''}=0$, $\alpha_{1''}=\alpha_{4''}=-24.62^\circ$, $\alpha_{2''}=30^\circ$, $\alpha_{3''}=45^\circ$; $a_{5''}=68.83^\circ$, with the common link lengths $a_{5'}$ and $a_{5''}$. Due to the above technique, link parameters of the derived Goldberg-Goldberg 6R mechanism are $a_1=5.3675$, $a_2=a_6=6.0$, $a_3=a_5=8.4853$, $a_4=6.5642$, $\alpha_1=-34.03^\circ$, $\alpha_2=\alpha_6=30^\circ$, $\alpha_3=\alpha_5=45^\circ$, $\alpha_4=-41.03^\circ$, $S_1=S_2=0.7793$, $S_3=S_6=0.0$, and $S_4=S_5=-1.6350$. The constrained motion of this mechanism confirmed by DFP optimization and Newton-Raphson methods⁽¹⁶⁾ is shown in Fig. 8.

3. Kinematic Closed-Form Solutions

Several methods are available for the analysis of spatial mechanisms. A method of analysis with the four by four coordinate-transformation matrix is used in this work. First, we derive the general displacement closed-form solutions through the matrix method for further investigations and synthesis. Moreover, by the differentiation of them, we analyze the angular velocity and acceleration of this mechanism. These derived equations are also easily coded into the Fortran and Lisp programs on a personal computer.

Based on the 4×4 coordinate-transformation-matrix method⁽²²⁾, a general-type Goldberg five-revolute mechanism characterized by the link parameters a_i , α_i , and S_i and joint variables θ_i ($i=1, 2, 3, 4, 5$) has the following matrix loop equation

$$T_4 T_5 T_1 = T_3^{-1} T_2^{-1} \quad (4.a)$$

or

$$T_5 T_1 T_2 = T_4^{-1} T_3^{-1} \quad (4.b)$$

where T_i^{-1} is the inverse matrix of T_i and

$$T_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_i c \theta_i & c \theta_i & -s \theta_i c \alpha_i & s \theta_i s \alpha_i \\ a_i s \theta_i & s \theta_i & c \theta_i c \alpha_i & -c \theta_i s \alpha_i \\ S_i & 0 & s \alpha_i & c \alpha_i \end{bmatrix}$$

is the coordinate-transformation-matrix in which $c \theta_i \equiv \cos \theta_i$, $s \theta_i \equiv \sin \theta_i$, $c \alpha_i \equiv \cos \alpha_i$, and $s \alpha_i \equiv \sin \alpha_i$ ($i=1, 2, 3, 4, 5$).

Without loss of generality, let us assign $S_2=S_3=S_4=0$ and the other non-zeros link parameters a_i and α_i ($i=1, 2, 3, 4, 5$) satisfy mobility criteria in Eq. (1). Thus, adding the squares of expansion elements a_{21} , a_{31} & a_{41} and eliminating θ_3 with element a_{44} in matrix Eq. (4. a) yield

$$A_0 s \theta_5 + B_0 c \theta_5 + C_0 s \theta_5 s \theta_1 + F_0 c \theta_5 c \theta_1 + G_0 s \theta_1 + H_0 c \theta_1 + Q_0 = 0 \quad (5)$$

where

$$\begin{aligned} A_0 &= 2S_1a_4s\alpha_2s\alpha_3s\alpha_5 \\ B_0 &= 2(a_4a_5s\alpha_2s\alpha_3 - a_2a_3c\alpha_1s\alpha_4s\alpha_5) \\ C_0 &= 2(a_2a_3s\alpha_1s\alpha_4 - a_1a_4c\alpha_5s\alpha_2s\alpha_3) \\ F_0 &= 2(a_1a_4s\alpha_2s\alpha_3 - a_2a_3c\alpha_5s\alpha_1s\alpha_4) \\ G_0 &= 2S_5a_1s\alpha_2s\alpha_3s\alpha_5 \\ H_0 &= 2(a_1a_5s\alpha_2s\alpha_3 - a_2a_3c\alpha_4s\alpha_1s\alpha_5) \\ Q_0 &= s\alpha_2s\alpha_3(a_4^2 + a_5^2 + a_1^2 - a_3^2 - a_2^2 + S_1^2 + S_5^2 \\ &\quad + 2S_5S_1c\alpha_5) + 2a_2a_3(c\alpha_1c\alpha_4c\alpha_5 - c\alpha_2c\alpha_3) \end{aligned}$$

This equation is called an input-output displacement equation of general-type Goldberg five-revolute mechanism. Furthermore, let $t_1 = \tan \theta_1$, $t_2 = \tan \theta_2$, $t_3 = \tan \theta_5$, and based on the half-angle relations, Eq. (5) becomes

$$At_5^2 + 2Bt_5 + C = 0 \quad (6)$$

where

$$\begin{aligned} A &= (-B_0 + F_0 - H_0 + Q_0)t_1^2 + 2G_0t_1 \\ &\quad + (-B_0 - F_0 + H_0 + Q_0) \\ B &= A_0t_1^2 + 2C_0t_1 + A_0 \\ C &= (B_0 - F_0 - H_0 + Q_0)t_1^2 + 2G_0t_1 \\ &\quad + (B_0 + F_0 + H_0 + Q_0) \end{aligned}$$

Then, Eq. (6) offers two solutions of the output-angular displacement θ_5 as

$$\theta_5 = 2 \tan^{-1} [(-B \pm \sqrt{B^2 - AC})/A] \quad (7)$$

In the same manner, by the utilization of expansion elements a_{21} , a_{31} , a_{41} and a_{44} in matrix Eq. (4. b), we have

$$\begin{aligned} \theta_2 &= 2 \tan^{-1} \{ [-(G_1 - E_1)t_1^2 - 2C_1t_1 - (G_1 + E_1) \\ &\quad \pm \sqrt{Q_1}] / [(B_1 - F_1 - H_1 + Q_1) + 2(A_1 - D_1)t_1 \\ &\quad + (-B_1 + F_1 - H_1 + Q_1)t_1^2] \} \quad (8) \end{aligned}$$

where

$$\begin{aligned} A_1 &= 2S_5a_1s\alpha_3s\alpha_4s\alpha_5 \\ B_1 &= 2(a_5a_1s\alpha_3s\alpha_4 - a_3a_4c\alpha_2s\alpha_1s\alpha_5) \\ C_1 &= 2(a_3a_4s\alpha_2s\alpha_5 - a_2a_5c\alpha_1s\alpha_3s\alpha_4) \\ D_1 &= 2S_5a_2s\alpha_3s\alpha_4s\alpha_5 \\ E_1 &= 2S_5a_2c\alpha_1s\alpha_3s\alpha_4s\alpha_5 \\ F_1 &= 2(a_2a_5s\alpha_3s\alpha_4 - a_3a_4c\alpha_1s\alpha_2s\alpha_5) \\ G_1 &= 2a_2s\alpha_1s\alpha_3s\alpha_4(S_5c\alpha_5 + S_1) \\ H_1 &= 2(a_1a_2s\alpha_3s\alpha_4 - a_3a_4c\alpha_5s\alpha_1s\alpha_2) \\ Q_1 &= s\alpha_3s\alpha_4(a_5^2 + a_1^2 + a_2^2 - a_4^2 - a_3^2 + S_1^2 + S_5^2) \\ &\quad + 2a_3a_4(c\alpha_1c\alpha_2c\alpha_5 - c\alpha_3c\alpha_4) + 2S_1S_5c\alpha_5s\alpha_3s\alpha_4 \\ Q_1 &= [(G_1 - E_1)t_1^2 + 2C_1t_1 + (G_1 + E_1)]^2 \\ &\quad - [(B_1 - F_1 - H_1 + Q_1) + 2(A_1 - D_1)t_1 \\ &\quad + (-B_1 + F_1 - H_1 + Q_1)t_1^2][(B_1 + F_1 + H_1 + Q_1) \\ &\quad + 2(A_1 + D_1)t_1 + (-B_1 - F_1 + H_1 + Q_1)t_1^2] \end{aligned}$$

Again, from elements a_{41} and a_{44} in matrix Eq. (4. b), the joint angle θ_4 can also be derived as

$$\theta_4 = m\pi + \tan^{-1}(N_1/M_1) \quad (m \text{ is an integer}) \quad (9)$$

where

$$\begin{aligned} M_1 &= -[s\alpha_2s\alpha_5s\theta_2s\theta_1 - s\alpha_2c\theta_2(c\alpha_1s\alpha_5c\theta_1 + c\alpha_5s\alpha_1) \\ &\quad + c\alpha_2(-s\alpha_1s\alpha_5c\theta_1 + c\alpha_5c\alpha_1) - c\alpha_3c\alpha_4]/(s\alpha_3s\alpha_4) \\ N_1 &= -[(S_5 + a_1s\alpha_5s\theta_1 + S_1c\alpha_5) + a_2s\alpha_5c\theta_2s\theta_1 \\ &\quad + a_2s\theta_2(c\alpha_1s\alpha_5c\theta_1 + c\alpha_5s\alpha_1)]/(a_3s\alpha_4) \end{aligned}$$

In addition, from elements a_{41} and a_{44} in matrix Eq.

(4. a), we develop

$$\theta_3 = m\pi + \tan^{-1}(N_2/M_2) \quad (m \text{ is an integer}) \quad (10)$$

where

$$\begin{aligned} M_2 &= -[s\alpha_1s\alpha_4s\theta_1s\theta_5 - s\alpha_1c\theta_1(c\alpha_5s\alpha_4c\theta_5 + c\alpha_4s\alpha_5) \\ &\quad + c\alpha_1(-s\alpha_4s\alpha_5c\theta_5 + c\alpha_4c\alpha_5) - c\alpha_2c\alpha_3]/(s\alpha_2s\alpha_3) \\ N_2 &= -[(a_5s\alpha_4s\theta_5 + S_5c\alpha_4) + a_1s\alpha_4c\theta_1s\theta_5 \\ &\quad + a_1s\theta_1(s\alpha_4c\alpha_5c\theta_5 + c\alpha_4s\alpha_5) \\ &\quad + S_1(-s\alpha_4s\alpha_5c\theta_5 + c\alpha_4c\alpha_5)]/(a_2s\alpha_3) \end{aligned}$$

To confirm the correctness and validity of the derived solutions, let us take the link parameters of $a_1 = a_4 = 10.0$, $a_2 = 6.0$, $a_3 = 8.4853$, $a_5 = 12.9331$, $\alpha_1 = \alpha_4 = -56.44^\circ$, $\alpha_2 = 30^\circ$, $\alpha_3 = 45^\circ$, $\alpha_5 = 68.83^\circ$, $S_1 = S_5 = 2.1931$, and $S_2 = S_3 = S_4 = 0$ as an example. And its displacement diagram of joint angles is shown in Fig. 9.

Differentiate angular displacement closed-form solutions above with respect to time, and then the angular velocities of joint variables denoted by $\dot{\theta}_i$ ($i = 1, 2, 3, \dots, 5$) can be derived as

$$\dot{\theta}_5 = -(D_{11}/D_{10})\dot{\theta}_1 \quad (11. a)$$

$$\dot{\theta}_2 = (D_{21}/D_{20})\dot{\theta}_1 \quad (11. b)$$

$$\dot{\theta}_4 = (\dot{N}_1M_1 - N_1\dot{M}_1)(M_1^2 + N_1^2)^{-1} \quad (11. c)$$

$$\dot{\theta}_3 = (\dot{N}_2M_2 - N_2\dot{M}_2)(M_2^2 + N_2^2)^{-1} \quad (11. d)$$

where

$$\begin{aligned} D_{10} &= -A_0c\theta_5 + B_0s\theta_5 - C_0s\theta_1c\theta_5 + F_0c\theta_1s\theta_5 \\ D_{11} &= -G_0c\theta_1 + H_0s\theta_1 - C_0c\theta_1s\theta_5 + F_0s\theta_1c\theta_5 \\ D_{20} &= -C_1s\theta_1c\theta_2 + D_1s\theta_1s\theta_2 - E_1c\theta_1c\theta_2 \\ &\quad + F_1c\theta_1s\theta_2 - G_1c\theta_2 + H_1s\theta_2 \\ D_{21} &= A_1c\theta_1 - B_1s\theta_1 + C_1c\theta_1s\theta_2 + D_1c\theta_1c\theta_2 \\ &\quad - E_1s\theta_1s\theta_2 - F_1s\theta_1c\theta_2 \\ \dot{M}_1 &= -(M_{11}\dot{\theta}_1 + M_{12}\dot{\theta}_2)/(s\alpha_3s\alpha_4) \\ \dot{N}_1 &= -(N_{11}\dot{\theta}_1 + N_{12}\dot{\theta}_2)/(a_3s\alpha_4) \\ M_{11} &= s\alpha_2s\alpha_5c\theta_1s\theta_2 + c\alpha_1s\alpha_2s\alpha_5s\theta_1c\theta_2 \\ &\quad + s\alpha_1c\alpha_2s\alpha_5s\theta_1 \\ M_{12} &= s\alpha_2s\alpha_5s\theta_1c\theta_2 + c\alpha_1s\alpha_2s\alpha_5c\theta_1s\theta_2 \\ &\quad + s\alpha_1s\alpha_2c\alpha_5s\theta_2 \end{aligned}$$

Fig. 9 Displacement diagram of general-type Goldberg 5R mechanism

$$\begin{aligned}
N_{11} &= a_1 s a_5 c \theta_1 + a_2 s a_5 c \theta_1 c \theta_2 \\
&\quad - a_2 c a_1 s a_5 s \theta_1 s \theta_2 \\
N_{12} &= -a_2 s a_5 s \theta_1 s \theta_2 + a_2 c a_1 s a_5 c \theta_1 c \theta_2 \\
&\quad + a_2 s a_1 c a_5 c \theta_2 \\
\dot{M}_2 &= -(M_{21} \dot{\theta}_1 + M_{22} \dot{\theta}_5) / (s a_2 s a_3) \\
\dot{N}^2 &= -(N_{21} \dot{\theta}_1 + N_{22} \dot{\theta}_5) / (a_2 s a_3) \\
M_{21} &= s a_1 s a_4 c \theta_1 s \theta_5 + s a_1 s a_4 c a_5 s \theta_1 c \theta_5 \\
&\quad + s a_1 c a_4 s a_5 s \theta_1 \\
M_{22} &= s a_1 s a_4 s \theta_1 c \theta_5 + s a_1 s a_4 c a_5 c \theta_1 s \theta_5 \\
&\quad + c a_1 s a_4 s a_5 s \theta_5 \\
N_{21} &= -a_1 s a_4 s \theta_1 s \theta_5 + a_1 c a_5 s a_4 c \theta_1 c \theta_5 \\
&\quad + a_1 c a_4 s a_5 c \theta_1 \\
N_{22} &= a_5 s a_4 c \theta_5 + a_1 s a_4 c \theta_1 c \theta_5 - a_1 c a_5 s a_4 s \theta_1 s \theta_5 \\
&\quad + S_1 s a_4 s a_5 s \theta_5
\end{aligned}$$

Once more, differentiate Eqs. (11. a) to (11. d) with respect to time. After the heavy algebraic manipulations, we obtain the angular accelerations of joints denoted by $\ddot{\theta}_i$ ($i=1, 2, \dots, 5$) as follows

$$\ddot{\theta}_5 = [-D_{11} \ddot{\theta}_1 - D_{12} (\dot{\theta}_1)^2 - D_{13} (\dot{\theta}_5)^2 + 2D_{14} \dot{\theta}_1 \dot{\theta}_5] / D_{10} \quad (12. a)$$

$$\ddot{\theta}_2 = [D_{21} \ddot{\theta}_1 - D_{22} (\dot{\theta}_1)^2 + 2D_{23} \dot{\theta}_1 \dot{\theta}_2 - D_{24} (\dot{\theta}_2)^2] / D_{20} \quad (12. b)$$

$$\ddot{\theta}_4 = (\dot{N}_1 M_1 - N_1 \dot{M}_1) (M_1^2 + N_1^2)^{-1} - 2(\dot{N}_1 M_1 - N_1 \dot{M}_1) (M_1 \dot{M}_1 + N_1 \dot{N}_1) \times (M_1^2 + N_1^2)^{-2} \quad (12. c)$$

$$\ddot{\theta}_3 = (\dot{N}_2 M_2 - N_2 \dot{M}_2) (M_2^2 + N_2^2)^{-1} - 2(\dot{N}_2 M_2 - N_2 \dot{M}_2) (M_2 \dot{M}_2 + N_2 \dot{N}_2) \times (M_2^2 + N_2^2)^{-2} \quad (12. d)$$

The parameters of D_{12} , D_{13} , D_{14} , $\dots$ etc, are expressed in the appendix for the reason of simplicity.

For the higher order analysis of coupler curve, the Jerk of $\ddot{\theta}_2$ is determined by differentiating Eq. (12. b) with respect to time once again as follows

$$\begin{aligned}
\ddot{\theta}_2 &= [D_{20} (\dot{\theta}_2)^3 + D_{21} \ddot{\theta}_1 - D_{21} (\dot{\theta}_1)^3 - 3D_{22} \dot{\theta}_1 \dot{\theta}_1 \\
&\quad + 3D_{23} \ddot{\theta}_1 \dot{\theta}_2 + 3D_{23} \dot{\theta}_1 \ddot{\theta}_2 + 3D_{24} \dot{\theta}_2 \ddot{\theta}_2 \\
&\quad - 3D_{25} (\dot{\theta}_1)^2 \dot{\theta}_2 + 3D_{25} \dot{\theta}_1 (\dot{\theta}_2)^2] / D_{20}
\end{aligned} \quad (13)$$

where

$$\begin{aligned}
D_{25} &= C_1 s \theta_1 c \theta_2 - D_1 s \theta_1 s \theta_2 + E_1 c \theta_1 c \theta_2 - F_1 c \theta_1 s \theta_2 \\
D_{26} &= -C_1 c \theta_1 s \theta_2 - D_1 c \theta_1 c \theta_2 + E_1 s \theta_1 s \theta_2 + F_1 s \theta_1 c \theta_2
\end{aligned}$$

To verify the correctness of the derived results, we take the same illustration as that in the displacement diagram. The angular velocity and acceleration diagrams of the mechanism are shown in Figs. 10 and 11. So far, the general analytical kinematic closed-form solutions of general-type Goldberg five-revolute mechanism have been determined and they permit an analytical analysis to any aspect of kinematic problems. For the convenience of computer programming, all derived closed-form solutions are expressed by trigonometric tangent function. These equations have been easily coded into a Fortran or Lisp program on a personal computer.

In order to understand the range of motion for the joint angles, based on the differential matrix loop

Fig. 10 Velocity diagram of general-type Goldberg 5R mechanism

Fig. 11 Acceleration diagram of general-type Goldberg 5R mechanism

Fig. 12 Displacement diagram of basic Goldberg 5R mechanism

equations, we can utilize a systematic compact six-by-four matrix criterion⁽²⁴⁾ to confirm that general-type Goldberg five-revolute mechanism has full-cycle mobility with no absence of stationary configurations about the specified input angle θ_1 . This matrix criterion is obtained by excluding the 3rd column of six-by-five Jacobian matrix whose elements are not dominant in determining the limit-position configurations. A basic Goldberg 5R mechanism in Fig. 3 has $a_1 = a_4$, $a_1 = a_4$, $a_1/sa_1 = a_2/sa_2 = a_3/sa_3$, $a_2 + a_3 = a_5$, $a_2 + a_3 = a_5$ and $S_i = 0$ ($i = 1, 2, 3, 4, 5$). The displacement closed-form solutions derived above can be applied in the analysis of this mechanism. The result, shown in Fig. 12, further confirms that the Goldberg 5R mechanism has full-cycle mobility with no absence of the singular configurations about the input angle.

4. Coupler Curve and Coupler Motion

The paths generated by points on the coupler link of general-type Goldberg five-revolute mechanism are termed coupler curves and the generating point is called a coupler point. This terminology will be used in the following. In Fig. 13, a local coordinate system is considered at point B. X -axis is along link-length a_2 , z -axis is coincident with joint axis z_2 , and y -axis is determined by the right-hand rule. A coupler point P on coupler link can be defined with respect to point B by specifying the parameters d , e and δ . d is a distance from point B along x -axis, e is a perpendicular distance projected from P onto coupler link BC, and δ is a rotated angle from x - z plane. Based on the 4×4 coordinate-transformation matrix, the coordinates of a coupler point $P(X_p, Y_p, Z_p)$ with respect to a global coordinate system X - Y - Z , which has the origin located at O, X -axis along link a_5 , and Z -axis coincident with joint z_1 , are

$$[1, X_p, Y_p, Z_p]^t$$

Fig. 13 Coupler point and coupler curve

$$= T_M(a_i, \theta_i, a_i, S_i)[1, d, es\delta, ec\delta]^t \quad (14)$$

where $T_M(a_i, \theta_i, a_i, S_i)$ is a combined coordinate-transformation matrix between the local and global systems. It is determined as follows

$$T_M(a_i, \theta_i, a_i, S_i) = \begin{bmatrix} 1 & 0 \\ a_1 c\phi & c\phi c\theta_2 - ca_1 s\phi s\theta_2 \\ a_1 s\phi & s\phi c\theta_2 + ca_1 c\phi s\theta_2 \\ S_1 & sa_1 s\theta_2 \\ 0 & 0 \\ -c\phi s\theta_2 - ca_1 s\phi c\theta_2 & sa_1 s\phi \\ -s\phi s\theta_2 + ca_1 c\phi c\theta_2 & -sa_1 c\phi \\ sa_1 c\theta_2 & ca_1 \end{bmatrix}$$

where we take $\phi + \pi = \theta_1$ for simplicity. Expanding Eq.(14) yields the parametric formulas of coupler curve as

$$\begin{aligned} X_p &= a_1 c\phi + d(c\phi c\theta_2 - ca_1 s\phi s\theta_2) \\ &\quad - es\delta(c\phi s\theta_2 + ca_1 s\phi c\theta_2) + es a_1 c\delta s\phi \\ Y_p &= a_1 s\phi + d(s\phi c\theta_2 + ca_1 c\phi s\theta_2) \\ &\quad - es\delta(s\phi s\theta_2 - ca_1 c\phi c\theta_2) - es a_1 c\delta c\phi \\ Z_p &= S_1 + d sa_1 s\theta_2 + es a_1 s\delta c\theta_2 + ec a_1 c\delta \end{aligned} \quad (15)$$

Differentiating Eq.(15) with respect to time once and twice respectively, we have the velocity ($\dot{X}_p, \dot{Y}_p, \dot{Z}_p$) and acceleration ($\ddot{X}_p, \ddot{Y}_p, \ddot{Z}_p$) of the coupler point P as

$$\begin{aligned} \dot{X}_p &= -Y_p \dot{\theta}_1 + U_1 \dot{\theta}_2 \\ \dot{Y}_p &= X_p \dot{\theta}_1 + U_2 \dot{\theta}_2 \\ \dot{Z}_p &= U_3 \dot{\theta}_2 \end{aligned} \quad (16)$$

$$\begin{aligned} \ddot{X}_p &= -Y_p \ddot{\theta}_1 + U_1 \ddot{\theta}_2 - X_p (\dot{\theta}_1)^2 \\ &\quad + V_1 (\dot{\theta}_2)^2 - 2U_2 \dot{\theta}_1 \dot{\theta}_2 \\ \ddot{Y}_p &= X_p \ddot{\theta}_1 + U_2 \ddot{\theta}_2 - Y_p (\dot{\theta}_1)^2 \\ &\quad + V_2 (\dot{\theta}_2)^2 + 2U_1 \dot{\theta}_1 \dot{\theta}_2 \\ \ddot{Z}_p &= U_3 \ddot{\theta}_2 + V_3 (\dot{\theta}_2)^2 \end{aligned} \quad (17)$$

where

$$\begin{aligned} U_1 &= -d(c\theta_1 s\theta_2 + ca_1 s\theta_1 c\theta_2) \\ &\quad - es\delta(c\theta_1 c\theta_2 - ca_1 s\theta_1 s\theta_2) \\ U_2 &= -d(s\theta_1 s\theta_2 - ca_1 c\theta_1 c\theta_2) \\ &\quad - es\delta(s\theta_1 c\theta_2 + ca_1 c\theta_2 s\theta_2) \\ U_3 &= d sa_1 c\theta_2 - es a_1 s\delta s\theta_2 \\ V_1 &= -d(c\theta_1 c\theta_2 - ca_1 s\theta_1 s\theta_2) \\ &\quad + es\delta(c\theta_1 s\theta_2 + ca_1 s\theta_1 c\theta_2) \\ V_2 &= -d(s\theta_1 c\theta_2 + ca_1 c\theta_1 s\theta_2) \\ &\quad + es\delta(s\theta_1 s\theta_2 - ca_1 c\theta_1 c\theta_2) \\ V_3 &= -d sa_1 s\theta_2 - es a_1 s\delta c\theta_2 \end{aligned}$$

Assume that the jerk $\ddot{\theta}_1$ for the input angle is given, the rate of change of acceleration ($\ddot{X}_p, \ddot{Y}_p, \ddot{Z}_p$) for coupler point P is

$$\begin{aligned} \ddot{X}_p &= -Y_p \ddot{\theta}_1 + U_1 \ddot{\theta}_2 - 3X_p \dot{\theta}_1 \ddot{\theta}_1 + 3V_1 \dot{\theta}_2 \ddot{\theta}_2 \\ &\quad - 3U_2 \dot{\theta}_1 \ddot{\theta}_2 - 3U_2 \dot{\theta}_1 \ddot{\theta}_2 + Y_p (\dot{\theta}_1)^3 - U_1 (\dot{\theta}_2)^3 \\ &\quad - 3U_1 \dot{\theta}_2 (\dot{\theta}_1)^2 - 3V_2 \dot{\theta}_1 (\dot{\theta}_2)^2 \\ \ddot{Y}_p &= X_p \ddot{\theta}_1 + U_2 \ddot{\theta}_2 - 3Y_p \dot{\theta}_1 \ddot{\theta}_1 + 3V_2 \dot{\theta}_2 \ddot{\theta}_2 \\ &\quad + 3U_1 \dot{\theta}_1 \ddot{\theta}_2 + 3U_1 \dot{\theta}_1 \ddot{\theta}_2 - X_p (\dot{\theta}_1)^3 - U_2 (\dot{\theta}_2)^3 \\ &\quad - 3U_2 \dot{\theta}_2 (\dot{\theta}_1)^2 + 3V_1 \dot{\theta}_1 (\dot{\theta}_2)^2 \\ \ddot{Z}_p &= U_3 \ddot{\theta}_2 + 3V_3 \dot{\theta}_2 \ddot{\theta}_2 - U_3 (\dot{\theta}_2)^3 \end{aligned} \quad (18)$$

These equations will offer a higher-order analysis

of the spatial coupler curve for general-type Goldberg five-revolute mechanism. Consider a regular spatial curve, a coupler curve, traced by a coupler point $P(X_P, Y_P, Z_P)$ as shown in Fig. 13. And then, based on the differential geometry of space curve⁽²⁵⁾, its radius of curvature ρ_k in terms of the point velocity and acceleration can be expressed as

$$\rho_k = \sigma^3 / \eta \quad (19)$$

where

$$\begin{aligned} \sigma &= (\dot{X}_P^2 + \dot{Y}_P^2 + \dot{Z}_P^2)^{1/2} \\ \eta &= \{(\dot{Y}_P \ddot{Z}_P - \dot{Z}_P \ddot{Y}_P)^2 + (\dot{Z}_P \ddot{X}_P - \dot{X}_P \ddot{Z}_P)^2 \\ &\quad + (\dot{X}_P \ddot{Y}_P - \dot{Y}_P \ddot{X}_P)^2\}^{1/2} \end{aligned}$$

Let us define the unit vectors directed along the positive tangent, binormal and principal normal of the coupler curve as $\mathbf{t}[t_x, t_y, t_z]$, $\mathbf{b}[b_x, b_y, b_z]$ and $\mathbf{n}[n_x, n_y, n_z]$ respectively, and then

$$\mathbf{t} = [\dot{X}_P / \sigma, \dot{Y}_P / \sigma, \dot{Z}_P / \sigma] \quad (20. a)$$

$$\mathbf{b} = [(\dot{Y}_P \ddot{Z}_P - \dot{Z}_P \ddot{Y}_P) / \eta, (\dot{Z}_P \ddot{X}_P - \dot{X}_P \ddot{Z}_P) / \eta, (\dot{X}_P \ddot{Y}_P - \dot{Y}_P \ddot{X}_P) / \eta] \quad (20. b)$$

$$\mathbf{n} = [b_y t_z - b_z t_y, b_z t_x - b_x t_z, b_x t_y - b_y t_x] \quad (20. c)$$

Therefore, the center of the curvature $C_k(C_x, C_y, C_z)$ is

$$(C_x, C_y, C_z) = (X_P + \rho_k n_x, Y_P + \rho_k n_y, Z_P + \rho_k n_z) \quad (21)$$

Similarly, on the basis of differential geometry fundamental, the radius of torsion ρ_t can be determined by

$$\rho_t = \eta^2 / \xi \quad (22)$$

where

$$\begin{aligned} \xi &= \dot{X}_P \ddot{Y}_P \ddot{Z}_P + \dot{Y}_P \ddot{Z}_P \ddot{X}_P + \dot{Z}_P \ddot{X}_P \ddot{Y}_P \\ &\quad - \dot{Z}_P \ddot{Y}_P \ddot{X}_P - \dot{Y}_P \ddot{X}_P \ddot{Z}_P - \dot{X}_P \ddot{Z}_P \ddot{Y}_P \end{aligned}$$

Equations (19) to (22) can be further provided to complete the higher-order analysis of coupler curves of a general-type Goldberg five-revolute mechanism. Moreover, based on the concepts of relative motion, the angular velocity w_2 of coupler link 2 (BC) can be expressed as

$$\mathbf{w}_2 = \mathbf{w}_{15} \mathbf{u}_{z1} + \mathbf{w}_{21} \mathbf{u}_{z2} \quad (23)$$

where $\mathbf{w}_2$ is the absolute angular velocity of link 2, $\mathbf{w}_{15}$ is the angular velocity of link 1 relative to the ground (link 5), and $\mathbf{w}_{21}$ is the angular velocity of link 2 relative to link 1. Since $\mathbf{u}_{z1} = \mathbf{k}$, $\mathbf{u}_{z2} = s\alpha_1 s\phi \mathbf{i} - s\alpha_1 c\phi \mathbf{j} + c\alpha_1 \mathbf{k}$, $\mathbf{w}_{15} = \dot{\theta}_1$, and $\mathbf{w}_{21} = \dot{\theta}_2$, Eq. (23) becomes

$$\mathbf{w}_2 = s\alpha_1 s\phi \dot{\theta}_2 \mathbf{i} - s\alpha_1 c\phi \dot{\theta}_2 \mathbf{j} + (c\alpha_1 \dot{\theta}_2 + \dot{\theta}_1) \mathbf{k} \quad (24)$$

Differentiating Eq. (24) with respect to time, we offer the absolute angular acceleration ζ_2 of the coupler link 2 (BC) as

$$\begin{aligned} \zeta_2 &= s\alpha_1 (c\phi \dot{\theta}_1 \dot{\theta}_2 + s\phi \ddot{\theta}_2) \mathbf{i} + s\alpha_1 (s\phi \dot{\theta}_1 \dot{\theta}_2 - c\phi \ddot{\theta}_2) \mathbf{j} \\ &\quad + (c\alpha_1 \ddot{\theta}_2 + \dot{\theta}_1) \mathbf{k} \end{aligned} \quad (25)$$

Based on Eqs. (23) to (25), we can also provide an analysis on the motion of coupler link of general-type Goldberg five-revolute mechanism.

5. Dimensional Synthesis

A general-type Goldberg five-revolute mecha-

nism is used for the dimensional synthesis, function, path and motion generation system in what follows, by utilizing the general constrained augmented Lagrange multiplier optimization method⁽¹⁹⁾.

5.1 Function generation synthesis

A general-type Goldberg five-revolute mechanism is designed to produce a desired output rotation for a given input rotation, and to convert rotation in one plane to another one. The input-output equation derived in the above section is applied to investigate the function generating synthesis by means of the optimization technique, and for illustration we also take a numerical example to demonstrate this design algorithm. A schematic representation of general-type Goldberg five-revolute mechanism for function generation is shown in Fig. 14. Equation (5) is an input-output angular displacement equation of this mechanism. When we simply replace $\theta_1 - \pi$ with ϕ , and $2\pi - \theta_5$ with ψ , the closed-form solutions of Eq. (7) presented above will be further applied to study the function-generation synthesis by means of the optimization technique.

Figure 14 shows the basic arrangement of the function generation problem. A global coordinate system XYZ with origin located at point O_1 has X -axis along link-length a_5 from O_1 to O_5 , Z -axis coincident with joint axis z_1 , and Y -axis determined by the right-hand rule. Angles ϕ_i and ψ_i define respectively the input and output rotations in the i -th position as measured counterclockwise from X -axis. For brevity, S_2, S_3 , and S_4 are assigned to zero link-offsets. A general-type Goldberg five-revolute mechanism is designed to satisfy a pair of related values (ϕ_i, ψ_i) for $i=1, \dots, n$. However, it is necessary to avoid a developed mechanism that meets all the position requirements but has the driven link on both branches of the output link motion or is not in the correct order. To determine the mechanism that generates angle posi-

Fig. 14 Basic arrangement of function-generating mechanism

tions closest to those required, we take the sum of squares of the deviations at the desired angles over all accuracy neighborhoods as the objective function and solve it by the optimization method. Therefore, the objective function is defined as

$$F(\mathbf{X}) = \sum (\phi_i - \phi_{di})^2 \quad i=1, \dots, n \quad (26)$$

where $\mathbf{X} = [X_1, X_2, X_3, X_4, X_5] = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \alpha_1, \Psi]$ are design variables, n is the number of design positions, ϕ_{di} are the user-specified output angles, and ϕ_i can be calculated from Eq. (7).

For all positions, the magnitude of the difference between the actual angle ϕ_i and the specified angle ϕ_{di} is constrained to lie within the prescribed allowable deviation e_i . Due to inherent limitations in manufacturing process and the need for pin-joint clearances, these constraints are really reasonable. Thus, we impose inequality constraints as follows:

$$G_i(\mathbf{X}) = |\phi_i - \phi_{di}| < e_i \quad \text{for } i=1, 2, \dots, n \quad (27)$$

5.2 Path generation synthesis

In practical applications, it is often desired to have a mechanism guide a point along a specified path in space. General-type Goldberg five-revolute mechanism may be a useful and economical one for this spatial path motion. Now, we design this type of mechanism to produce a desired path curve for the given input rotations by the optimization approach. Refer to Fig. 15, and let us arrange a path generation system for the design of a general type Goldberg 5R mechanism satisfying the prescribed path within the specified accuracy limits. To avoid a developed mechanism that meets all the position requirements but is not in the correct order and to determine the mechanism which generates the required positions most closely, we use the sum of squares of the deviations at the desired positions over all accuracy neighborhoods as the objective function and deal it by the optimization method. Thus, the objective function is defined as

$$F(\mathbf{X}) = \sum [(X_{Pi} - X_{di})^2 + (Y_{Pi} - Y_{di})^2 + (Z_{Pi} - Z_{di})^2] \quad i=1, 2, \dots, n \quad (28)$$

where $\mathbf{X} = [X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8] = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \alpha_1, \mathbf{d}, \mathbf{e}, \delta, \Psi]$ are design variables, n is the number of design positions, $\mathbf{R}_{di}(X_{di}, Y_{di}, Z_{di})$ are the user specified points, and $\mathbf{R}_{Pi}(X_{Pi}, Y_{Pi}, Z_{Pi})$ can be directly calculated from Eq. (15).

For all positions, the magnitudes of the vector difference between the actual tracer path points $\mathbf{R}_{Pi}$ and the user specified path points $\mathbf{R}_{di}$ are constrained to lie within the prescribed allowable deviation e_i . Thus, we impose inequality constraints as follows

$$G_i(\mathbf{X}) = |\mathbf{R}_{Pi} - \mathbf{R}_{di}| < e_i \quad \text{for } i=1, 2, \dots, m \quad (29)$$

5.3 Motion generation synthesis

Motion-generation or rigid-body guidance synthesis requires that an entire body be guided through a series of prescribed positions and orientations, and the body to be guided is usually one part of a coupler link of mechanism. By the optimization method, we will study the motion-generation problem that involves the guidance of a rigid body through more than the exact precision specified configurations. Referring to Fig. 16, let a rigid body G occupy reference configuration G_1 initially and assume it intends to guide this rigid body through n successive configurations G_i ($i=1, 2, \dots, n$), all of which are finitely separated. Then, we choose any three noncollinear points of the body, G_i, F_i & H_i , and $\mathbf{u}_F$ & $\mathbf{u}_H$ are the unit vectors from points G_i to F_i , and G_i to H_i , respectively. In the vector form, let $\mathbf{R}_{Gi}$ be the position vector of point G_i ; then the position vectors of points F_i and H_i denoted by $\mathbf{R}_{Fi}$ and $\mathbf{R}_{Hi}$ become

$$\mathbf{R}_{Fi} = \mathbf{R}_{Gi} + \mathbf{u}_F \quad (30.a)$$

$$\mathbf{R}_{Hi} = \mathbf{R}_{Gi} + \mathbf{u}_H \quad (30.b)$$

where $\mathbf{R}_{Gi} = (X_{Gi}, Y_{Gi}, Z_{Gi})$ determined by a combined coordinate-transformation matrix $\mathbf{T}_M(a_1, \theta_i, \alpha_1, S_1)$ between the local and global coordinate systems given above is expressed as

Fig. 15 Basic arrangement of path-generating mechanism

Fig. 16 Basic arrangement of motion-generating mechanism

$$[1, X_{Gi}, Y_{Gi}, Z_{Gi}]^t \\ = \mathbf{T}_M(a_1, \theta_i, a_1, S_1)[1, \mathbf{d}, e\mathbf{s}\delta, e\mathbf{c}\delta]^t \quad (31)$$

As has been noted, to obtain positions closest to those required, we use the sum of squares of the deviations at the desired positions over all accuracy neighborhoods as the objective function. Thus, the objective function is defined as follows

$$F(\mathbf{X}) = \sum (\mathbf{R}_{Gi} - \mathbf{R}_{Gdi})^2 + (\mathbf{R}_{Fi} - \mathbf{R}_{Fdi})^2 \\ + (\mathbf{R}_{Hi} - \mathbf{R}_{Hdi})^2 \quad i=1, \dots, n \quad (32)$$

where $\mathbf{X} = [X_1, X_2, X_3, \dots, X_8] = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, a_1, \mathbf{d}, \mathbf{e}, \delta, \Psi]$ are design variables, n is the number of design positions, $\mathbf{R}_{Gdi}$, $\mathbf{R}_{Fdi}$ and $\mathbf{R}_{Hdi}$ are the user-specified positions, and $\mathbf{R}_{Gi}$, $\mathbf{R}_{Fi}$ and $\mathbf{R}_{Hi}$ can be directly calculated from Eqs.(30) and (31).

For all positions, the magnitudes of the differences between the actual positions R_{Gi} , R_{Fi} and R_{Hi} , and the specified positions R_{Gdi} , R_{Fdi} and R_{Hdi} are constrained to lie within the prescribed allowable deviation e_i . Thus, the inequality constraints are set up as follows

$$G_i(\mathbf{X}) = |\mathbf{R}_{Gi} - \mathbf{R}_{Gdi}| < e_i \\ G_{i+6}(\mathbf{X}) = |\mathbf{R}_{Fi} - \mathbf{R}_{Fdi}| < e_{i+6} \quad \text{for } i=1, 2, \dots, n. \\ G_{i+12}(\mathbf{X}) = |\mathbf{R}_{Hi} - \mathbf{R}_{Hdi}| < e_{i+12} \quad (33)$$

5.4 Method of optimization

On the whole, the nonlinear optimization problem for the dimensional synthesis of general-type Goldberg five-revolute mechanism may be stated as follows

$$\text{Minimize } F(\mathbf{X}); \mathbf{X} = [X_1, X_2, \dots, X_n] \quad (34) \\ \text{subject to } G_i(\mathbf{X}) < 0 \quad i=1, 2, 3, \dots, m \\ H_j(\mathbf{X}) = 0 \quad j=1, 2, 3, \dots, l \\ X_{\min} < X_i < X_{\max} \quad i=1, 2, 3, \dots, n$$

where

$\mathbf{X}$: a column vector of design variables;
for function generation

$$\mathbf{X} = [X_1, X_2, \dots, X_5] = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, a_1, \Psi],$$

for path and motion generations

$$\mathbf{X} = [X_1, X_2, \dots, X_8] = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, a_1, \mathbf{d}, \mathbf{e}, \delta, \Psi],$$

$F(\mathbf{X})$: the objective function,

$G_i(\mathbf{X})$: the inequality constraint functions,

$H_j(\mathbf{X})$: the equality constraint functions,

n : the number of design variables,

m : the number of inequality constraints,

l : the number of equality constraints.

The lower limit $X_{\min}$ and the upper limit $X_{\max}$ chosen so as to be restricted to the reasonable range can be included in the inequality constraints. We can select the plus or minus sign for link-twist a_i defined according to its assembly without any change throughout the optimum process, so the branch defects can be got rid off. Moreover, during the optimization process, it is desirable to normalize the directions of searching for the optimum, which can be attained by dividing all

components of search vectors by a scaling factor of the maximum absolute component of all search vectors.

In general, any constrained optimization technique can deal with this problem of the optimal synthesis. Here, the general augmented Lagrange multiplier (ALM) optimization method is applied to solve this constrained nonlinear programming problem. This general algorithm is given by Vanderplaats⁽¹⁹⁾. It has been coded into a Fortran program implemented on a personal computer⁽²⁶⁾ and is used here. In the following, we have three numerical examples to demonstrate this dimensional synthesis process.

6. Design Examples

In the following, three numerical examples of dimensional synthesis, ten-position optimal function & path generators and eight-position motion generator will be developed for the illustration of applicability of the derived synthesis algorithms. In order to overcome the fact that the link lengths are negative (or $-\pi$), we specify the bounds as $X_{\min}$ equal to zero and $X_{\max}$ equal to ten (or π). Furthermore, for the purpose of efficient and rapid convergence, we normalize the design variables.

Example 1: Consider the design of a general-type Goldberg five-revolute mechanism function generator to meet the following conditions specified by the input angles ϕ_i , output angles ψ_i and deviation allowable tolerances e_i as:

$$\begin{array}{lll} \phi_1 = 30^\circ, & \phi_1 = 39.5^\circ, & e_1 = 0.002; \\ \phi_2 = 45^\circ, & \phi_2 = 64^\circ, & e_2 = 0.004; \\ \phi_3 = 60^\circ, & \phi_3 = 84.5^\circ, & e_3 = 0.004; \\ \phi_4 = 75^\circ, & \phi_4 = 101.5^\circ, & e_4 = 0.003; \\ \phi_5 = 95^\circ, & \phi_5 = 120^\circ, & e_5 = 0.005; \\ \phi_6 = 110^\circ, & \phi_6 = 131^\circ, & e_6 = 0.002; \\ \phi_7 = 125^\circ, & \phi_7 = 141^\circ, & e_7 = 0.004; \\ \phi_8 = 140^\circ, & \phi_8 = 150^\circ, & e_8 = 0.006; \\ \phi_9 = 150^\circ, & \phi_9 = 155^\circ, & e_9 = 0.005; \\ \phi_{10} = 165^\circ, & \phi_{10} = 163^\circ, & e_{10} = 0.002. \end{array}$$

The initial design approximations for this example are specified as $X_1=0.9$, $X_2=0.7$, $X_3=0.8$, $X_4=-1.0$, and $X_5=2.0$. By the method of optimization, the synthesized final mechanism is given as $X_1=1.09138$, $X_2=0.57822$, $X_3=0.74804$, $X_4=-1.13846$, $X_5=2.14296$ and $F(\mathbf{X})=8.43E-05$ as shown in Table 1, in which $G(1) \sim G(36)$ are inequality constraint functions. If we multiply these values by 10 and adopt Eqs.(1) and (3), we have $\mathbf{a}_1=10.9138$, $\mathbf{a}_2=5.7822$, $\mathbf{a}_3=7.4804$, $\mathbf{a}_4=10.9138$, $\mathbf{a}_5=11.1453$, $a_1=-65.23^\circ$, $a_2=28.75^\circ$, $a_3=38.49^\circ$, $a_4=-65.23^\circ$, $a_5=58.39^\circ$, $S_1=1.9850$, $S_2=S_3=S_4=0$, $S_5=1.9850$ and $\Psi=122.78^\circ$. Figure 17 shows the schematic of this mechanism. It is obvious that the generated values deviated from the user-specified

Table 1 Results of ten-position optimal function synthesis

***** THE FINAL SOLUTION *****			
X(1)=	1.0913761015	X(2)=	.5782153747
X(3)=	.7480429417	X(4)=	-1.1384629177
X(5)=	2.1429566035		
RP=	.0040000002		
THE OBJECTIVE FUNCTION=	.0000843457		.0000843457
ERROR=	.0000000002		
THE CONSTRAINT FUNCTION :			
G(1)=	-.00156	G(2)=	-.00174
G(3)=	-.00036	G(4)=	-.00102
G(5)=	-.00139	G(6)=	-.00199
G(7)=	-.00096	G(8)=	-.00145
G(9)=	-.00096	G(10)=	-.00096
G(11)=	-.00096	G(12)=	-.00096
G(13)=	-.00096	G(14)=	-.00096
G(15)=	-.00096	G(16)=	-.00096
G(17)=	-.00096	G(18)=	-.00096
G(19)=	-.00096	G(20)=	-.00096
G(21)=	-.00096	G(22)=	-.00096
G(23)=	-.00096	G(24)=	-.00096
G(25)=	-.00096	G(26)=	-.00096
G(27)=	-.00096	G(28)=	-.00096
G(29)=	-.00096	G(30)=	-.00096
G(31)=	-.00096	G(32)=	-.00096
G(33)=	-.00096	G(34)=	-.00096
G(35)=	-.00096	G(36)=	-.00096

Fig. 17 Ten-position optimal function-generating mechanism

precision points are within the prescribed maximum allowable deviation.

Example 2: A general-type Goldberg five-revolute path generation mechanism is designed to meet the following conditions specified by the input angles ϕ_i , ten user specified position requirements $R_{di}(X_{di}, Y_{di}, Z_{di})$ and deviation allowable tolerances e_i as

- $\phi_1=30^\circ, (X_{d1}, Y_{d1}, Z_{d1})=(0.745, 0.900, 0.491), e_1=0.001;$
 $\phi_2=40^\circ, (X_{d2}, Y_{d2}, Z_{d2})=(0.660, 0.960, 0.550), e_2=0.001;$
 $\phi_3=55^\circ, (X_{d3}, Y_{d3}, Z_{d3})=(0.504, 1.012, 0.640), e_3=0.001;$
 $\phi_4=70^\circ, (X_{d4}, Y_{d4}, Z_{d4})=(0.327, 1.012, 0.721), e_4=0.001;$
 $\phi_5=85^\circ, (X_{d5}, Y_{d5}, Z_{d5})=(0.150, 0.961, 0.786), e_5=0.001;$
 $\phi_6=100^\circ, (X_{d6}, Y_{d6}, Z_{d6})=(-0.012, 0.866, 0.832), e_6=0.001;$
 $\phi_7=115^\circ, (X_{d7}, Y_{d7}, Z_{d7})=(-0.144, 0.737, 0.858), e_7=0.001;$
 $\phi_8=125^\circ, (X_{d8}, Y_{d8}, Z_{d8})=(-0.211, 0.638, 0.866), e_8=0.001;$

Table 2 Results of ten-position optimal path synthesis

***** THE FINAL SOLUTION *****			
X(1)=	.5990297276	X(2)=	.7937091213
X(3)=	.5151097373	X(4)=	-.5592584288
X(5)=	.4491073564	X(6)=	.5520745012
X(7)=	.5279034099	X(8)=	2.3542651139
RP=	.0004000000		
THE OBJECTIVE FUNCTION=	.0000022519		.0000022519
ERROR=	.0000000079		
THE CONSTRAINT FUNCTION :			
G(1)=	-.00030	G(2)=	-.00072
G(3)=	-.00049	G(4)=	-.00078
G(5)=	-.00071	G(6)=	-.00037
G(7)=	-.00071	G(8)=	-.00037
G(9)=	-.00071	G(10)=	-.00037
G(11)=	-.00071	G(12)=	-.00037
G(13)=	-.00071	G(14)=	-.00037
G(15)=	-.00071	G(16)=	-.00037
G(17)=	-.00071	G(18)=	-.00037
G(19)=	-.00071	G(20)=	-.00037
G(21)=	-.00071	G(22)=	-.00037
G(23)=	-.00071	G(24)=	-.00037
G(25)=	-.00071	G(26)=	-.00037
G(27)=	-.00071	G(28)=	-.00037
G(29)=	-.00071	G(30)=	-.00037
G(31)=	-.00071	G(32)=	-.00037
G(33)=	-.00071	G(34)=	-.00037
G(35)=	-.00071	G(36)=	-.00037
G(37)=	-.00071	G(38)=	-.00037
G(39)=	-.00071	G(40)=	-.00037

Fig. 18 Ten-position optimal path-generating mechanism

- $\phi_9=135^\circ, (X_{d9}, Y_{d9}, Z_{d9})=(-0.259, 0.533, 0.866), e_9=0.001;$
 $\phi_{10}=150^\circ, (X_{d10}, Y_{d10}, Z_{d10})=(-0.296, 0.372, 0.854), e_{10}=0.001.$

The initial design guesses for this example are specified as $X_1=0.7, X_2=0.7, X_3=0.7, X_4=-0.5, X_5=0.5, X_6=0.5, X_7=0.5$, and $X_8=2.5$. By the method of optimization cited above, the synthesized final mechanism is given to $X_1=0.59903, X_2=0.79371, X_3=0.51511, X_4=-0.55926, X_5=0.44911, X_6=0.55207, X_7=0.52790, X_8=2.35427$, and $F(X)=2.252E-06$ shown in Table 2, in which $G(1) \sim G(42)$ are inequality constraint functions. If we multiply these values by 10 and use Eqs. (1) and (3), we have $a_1=5.9903, a_2=7.9371, a_3=5.1511, a_4=5.9903, a_5=11.7423, a_6=11.7423, a_7=-32.04^\circ, a_8=44.67^\circ, a_9=27.14^\circ, a_{10}=-32.04^\circ, a_{11}=66.01^\circ, S_1=1.8240, S_2=S_3=S_4=0, S_5=1.8240, d=4.4911, e=5.5207, \delta=30.25^\circ$, and $\Psi=134.89^\circ$. The animation of this mechanism is sketched in Fig. 18. It shows that the generated values deviated from user specified precision positions are always within the prescribed maximum allowable deviation. In addition, let us take the synthesized mechanism as an example for illustrating higher-

order analysis of the coupler curve. Based on Eqs. (19), (21) and (22), both the radii of curvature & torsion and the center of curvature of coupler curve in this mechanism can be easily found out.

Example 3: Synthesize a general type Goldberg five-revolute mechanism to guide a rigid body through the eight finitely separated positions which are described by using three noncolinear points G, F and H embedded in the moving body. The input angles ϕ_i , eight user-specified position requirements G_i , H_i and F_i and deviation allowable tolerances e_i are

$$\begin{aligned} \phi_1 &= 35^\circ, G_1(X_{G1}, Y_{G1}, Z_{G1}) \\ &= (0.8829, 1.2106, 0.2286), \\ F_1(X_{F1}, Y_{F1}, Z_{F1}) \\ &= (0.9666, 1.2650, 0.2340), \\ H_1(X_{H1}, Y_{H1}, Z_{H1}) \\ &= (0.8286, 1.2946, 0.2262), \quad e_1 = 0.001; \\ \phi_2 &= 50^\circ, G_2(X_{G2}, Y_{G2}, Z_{G2}) \\ &= (0.6471, 1.3907, 0.3600), \\ F_2(X_{F2}, Y_{F2}, Z_{F2}) \\ &= (0.7230, 1.4467, 0.3933), \\ H_2(X_{H2}, Y_{H2}, Z_{H2}) \\ &= (0.5873, 1.4708, 0.3614), \quad e_2 = 0.001; \\ \phi_3 &= 65^\circ, G_3(X_{G3}, Y_{G3}, Z_{G3}) \\ &= (0.3507, 1.4786, 0.4974), \\ F_3(X_{F3}, Y_{F3}, Z_{F3}) \\ &= (0.4157, 1.5291, 0.5541), \\ H_3(X_{H3}, Y_{H3}, Z_{H3}) \\ &= (0.2842, 1.5526, 0.5076), \quad e_3 = 0.001; \\ \phi_4 &= 80^\circ, G_4(X_{G4}, Y_{G4}, Z_{G4}) \\ &= (0.0302, 1.4610, 0.6227), \\ F_4(X_{F4}, Y_{F4}, Z_{F4}) \\ &= (0.0863, 1.5000, 0.6957), \\ H_4(X_{H4}, Y_{H4}, Z_{H4}) \\ &= (-0.0433, 1.5250, 0.6449), \\ &e_4 = 0.001; \\ \phi_5 &= 95^\circ, G_5(X_{G5}, Y_{G5}, Z_{G5}) \\ &= (-0.2749, 1.3428, 0.7239), \\ F_5(X_{F5}, Y_{F5}, Z_{F5}) \\ &= (-0.2227, 1.3677, 0.8055), \\ H_5(X_{H5}, Y_{H5}, Z_{H5}) \\ &= (-0.3542, 1.3923, 0.7595), \\ &e_5 = 0.001; \\ \phi_6 &= 110^\circ, G_6(X_{G6}, Y_{G6}, Z_{G6}) \\ &= (-0.5322, 1.1433, 0.7958), \\ F_6(X_{F6}, Y_{F6}, Z_{F6}) \\ &= (-0.4778, 1.1552, 0.8789), \\ H_6(X_{H6}, Y_{H6}, Z_{H6}) \\ &= (-0.6137, 1.1745, 0.8447), \\ &e_6 = 0.001; \\ \phi_7 &= 125^\circ, G_7(X_{G7}, Y_{G7}, Z_{G7}) \\ &= (-0.7205, 0.8884, 0.8387), \\ F_7(X_{F7}, Y_{F7}, Z_{F7}) \\ &= (-0.6590, 0.8917, 0.9175), \end{aligned}$$

$$\begin{aligned} H_7(X_{H7}, Y_{H7}, Z_{H7}) \\ &= (-0.7991, 0.8996, 0.8995), \\ &e_7 = 0.001; \\ \phi_8 &= 150^\circ, G_8(X_{G8}, Y_{G8}, Z_{G8}) \\ &= (-0.8588, 0.4143, 0.8538), \\ F_8(X_{F8}, Y_{F8}, Z_{F8}) \\ &= (-0.7809, 0.4172, 0.9164), \\ H_8(X_{H8}, Y_{H8}, Z_{H8}) \\ &= (-0.9198, 0.3949, 0.9307), \\ &e_8 = 0.001. \end{aligned}$$

The initial design approximations are specified as $X_1 = 0.8$, $X_2 = 0.8$, $X_3 = 0.8$, $X_4 = -0.8$, $X_5 = 0.5$, $X_6 = 0.5$, $X_7 = 0.5$, and $X_8 = 2.5$. Based on the method of optimization, the synthesized final mechanism is given as $X_1 = 1.00014$, $X_2 = 0.60098$, $X_3 = 0.84641$, $X_4 = -0.98502$, $X_5 = 0.40012$, $X_6 = 0.49976$, $X_7 = 0.61031$, $X_8 = 2.35798$, and $F(X) = 2.77E-07$ shown in Table 3, where $G(1) \sim G(54)$ are inequality constraint functions. If we multiply the derived values by 10 and use Eqs. (1) and (3), we obtain $\alpha_1 = 10.0014$, $\alpha_2 = 6.0098$, $\alpha_3 = 8.4641$, $\alpha_4 = 10.0014$, $\alpha_5 = 12.9112$, $\alpha_1 = -56.44^\circ$, $\alpha_2 = 30.05^\circ$, $\alpha_3 =$

Table 3 Results of eight-position optimal motion synthesis

***** THE FINAL SOLUTION *****			
X(1)=	1.0001360948	X(2)=	.6009786406
X(3)=	.8464085740	X(4)=	-.9850232237
X(5)=	.4001155275	X(6)=	.4997565242
X(7)=	.6103050381	X(8)=	2.3579816451
RP=	.0004000000		
THE OBJECTIVE FUNCTION=	.0000002774		.0000002774
ERROR=	.0000000254		
THE CONSTRAINT FUNCTION .			
G(1)=	-.00084	G(2)=	-.00095
G(3)=	-.00090	G(4)=	-.00095
G(5)=	-.00080	G(6)=	-.00089
G(7)=	-.00086	G(8)=	-.00092
G(9)=	-.00092	G(10)=	-.00084
G(11)=	-.00093	G(12)=	-.00091
G(13)=	-.00092	G(14)=	-.00094
G(15)=	-.00097	G(16)=	-.00090
G(17)=	-.00091	G(18)=	-.00091
G(19)=	-.00091	G(20)=	-.00091
G(21)=	-.00091	G(22)=	-.00091
G(23)=	-.00091	G(24)=	-.00091
G(25)=	-.899986	G(26)=	-1.00014
G(27)=	-9.399902	G(28)=	-6.0098
G(29)=	-9.15359	G(30)=	-.84641
G(31)=	-4.12662	G(32)=	-2.15657
G(33)=	-9.59988	G(34)=	-.40012
G(35)=	-9.50024	G(36)=	-4.9976
G(37)=	-2.53129	G(38)=	-3.75190
G(39)=	-.78361	G(40)=	-5.49957
G(41)=	-17.99997	G(42)=	-5.38805
G(43)=	-2.46513	G(44)=	-1.44718
G(45)=	-.99766	G(46)=	-.76743
G(47)=	-.63838	G(48)=	-.53159
G(49)=	-1.50072	G(50)=	-.49928
G(51)=	-1.70520	G(52)=	-.29480
G(53)=	-.06844	G(54)=	-1.93156

Fig. 19 Eight-position optimal motion-generating mechanism

44.85°, $\alpha_4 = -56.44^\circ$, $\alpha_5 = 68.68^\circ$, $S_1 = 2.1938$, $S_2 = S_3 = S_4 = 0$, $S_5 = 2.1938$, $d = 4.0012$, $e = 4.9976$, $\delta = 34.97^\circ$, and $\Psi = 135.10^\circ$. Figure 19 shows the schematic of this mechanism and the generated values deviated from the user-specified precision positions are within the prescribed maximum allowable deviation. Moreover, based on Eqs. (24) and (25), we can make a complete analysis of coupler link motion of this synthesized mechanism.

7. Conclusions

The geometric characteristics, dimensional constraints and the general analytical kinematic closed-form solutions of the mechanism developed in this work will serve as a useful tool in the preliminary investigation of the general Goldberg five-revolute mechanism. There is no denying that no singular configuration offers a good continuous rotation for input driver in mechanism design and application. Here, a systematic generation approach to synthesizing movable spatial 5R & 6R mechanisms using Bennett four-revolute and general-type Goldberg five-revolute chains elucidates the process of formation. These equations and algorithm proposed in this paper are easily verified by one potential application of a spatial mixing machine, which is a special form of basic Goldberg 5R mechanism with one pair of adjacent parallel axes. The parametric formulations of the coupler link and coupler point provide a procedure for the higher-order analysis of coupler curve and link. In addition, the nonlinear programming—the general ALM optimization technique, completes the dimensional synthesis for function, path and motion generators. It is worth mentioning that there is no limit to the number of design positions for this optimal algorithm and that inequality constraints imposed in optimal process are really reasonable due to limitation in manufacture revolute-joint clearance. Through the process of this work, a user-friendly computer aided program by Fortran and Lisp languages accompanied with 3D computer graphics has been easily coded and will be offered for the reference of further investigation and application.

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Appendix

The expressions of D_{12} , D_{13} , D_{22} , D_{23} , D_{24} , M_{13} , M_{14} , M_{15} , N_{13} , N_{14} , N_{15} , M_{23} , M_{24} , M_{25} , N_{23} , N_{24} , N_{25} , $\dot{M}_1$, $\dot{N}_1$, $\dot{M}_2$, and $\dot{N}_2$ can be written as

$$D_{12} = H_0 c\theta_1 + G_0 s\theta_1 + F_0 c\theta_1 c\theta_5 + C_0 s\theta_1 s\theta_5$$

$$\begin{aligned} D_{13} &= B_0 c\theta_5 + A_0 s\theta_5 + F_0 c\theta_1 c\theta_5 + C_0 s\theta_1 s\theta_5 \\ D_{14} &= F_0 s\theta_1 s\theta_5 + C_0 c\theta_1 c\theta_5 \\ D_{22} &= A_1 s\theta_1 + B_1 c\theta_1 + C_1 s\theta_1 s\theta_2 + D_1 s\theta_1 c\theta_2 + E_1 c\theta_1 s\theta_2 \\ &\quad + F_1 c\theta_1 c\theta_2 \\ D_{23} &= C_1 c\theta_1 c\theta_2 - D_1 c\theta_1 s\theta_2 - E_1 s\theta_1 c\theta_2 + F_1 s\theta_1 s\theta_2 \\ D_{24} &= C_1 s\theta_1 s\theta_2 + D_1 s\theta_1 c\theta_2 + E_1 c\theta_1 s\theta_2 + F_1 c\theta_1 c\theta_2 \\ &\quad + G_1 s\theta_2 + H_1 c\theta_2 \\ M_{13} &= -s\alpha_2 s\alpha_5 s\theta_1 s\theta_2 + c\alpha_1 s\alpha_2 s\alpha_5 c\theta_1 c\theta_2 + s\alpha_1 c\alpha_2 s\alpha_5 c\theta_1 \\ M_{14} &= s\alpha_2 s\alpha_5 c\theta_1 c\theta_2 - c\alpha_1 s\alpha_2 s\alpha_5 s\theta_1 s\theta_2 \\ M_{15} &= -s\alpha_2 s\alpha_5 s\theta_1 s\theta_2 + c\alpha_1 s\alpha_2 s\alpha_5 c\theta_1 c\theta_2 + s\alpha_1 s\alpha_2 c\alpha_5 c\theta_2 \\ N_{13} &= -a_1 s\alpha_5 s\theta_1 - a_2 s\alpha_5 s\theta_1 c\theta_2 - a_2 c\alpha_1 s\alpha_5 c\theta_1 s\theta_2 \\ N_{14} &= -a_2 s\alpha_5 c\theta_1 s\theta_2 - a_2 c\alpha_1 s\alpha_5 s\theta_1 c\theta_2 \\ N_{15} &= -a_2 s\alpha_5 s\theta_1 c\theta_2 - a_2 c\alpha_1 s\alpha_5 c\theta_1 s\theta_2 - a_2 s\alpha_1 c\alpha_5 s\theta_2 \\ \dot{M}_1 &= -[M_{11} \ddot{\theta}_1 + M_{12} \ddot{\theta}_2 + M_{13} (\dot{\theta}_1)^2 + 2M_{14} \dot{\theta}_1 \dot{\theta}_2 \\ &\quad + M_{15} (\dot{\theta}_2)^2] / (s\alpha_3 s\alpha_4) \\ \dot{N}_1 &= -[N_{11} \ddot{\theta}_1 + N_{12} \ddot{\theta}_2 + N_{13} (\dot{\theta}_1)^2 + 2N_{14} \dot{\theta}_1 \dot{\theta}_2 \\ &\quad + N_{15} (\dot{\theta}_2)^2] / (a_3 s\alpha_4) \\ M_{23} &= -s\alpha_1 s\alpha_4 s\theta_1 s\theta_5 + s\alpha_1 s\alpha_4 c\alpha_5 c\theta_1 c\theta_5 + s\alpha_1 c\alpha_4 s\alpha_5 c\theta_1 \\ M_{24} &= s\alpha_1 s\alpha_4 c\theta_1 c\theta_5 - s\alpha_1 s\alpha_4 c\alpha_5 s\theta_1 s\theta_5 \\ M_{25} &= -s\alpha_1 s\alpha_4 s\theta_1 s\theta_5 + s\alpha_1 s\alpha_4 c\alpha_5 c\theta_1 c\theta_5 + c\alpha_1 s\alpha_4 s\alpha_5 c\theta_5 \\ N_{23} &= -a_1 s\alpha_4 c\theta_1 s\theta_5 - a_1 c\alpha_5 s\alpha_4 s\theta_1 c\theta_5 - a_1 c\alpha_4 s\alpha_5 s\theta_1 \\ N_{24} &= -a_1 s\alpha_4 s\theta_1 c\theta_5 - a_1 c\alpha_5 s\alpha_4 c\theta_1 s\theta_5 \\ N_{25} &= -a_5 s\alpha_4 s\theta_5 - a_1 s\alpha_4 c\theta_1 s\theta_5 - a_1 c\alpha_5 s\alpha_4 s\theta_1 c\theta_5 \\ &\quad + S_1 s\alpha_4 s\alpha_5 c\theta_5 \\ \dot{M}_2 &= -[M_{21} \ddot{\theta}_1 + M_{22} \ddot{\theta}_5 + M_{23} (\dot{\theta}_1)^2 + 2M_{24} \dot{\theta}_1 \dot{\theta}_5 \\ &\quad + M_{25} (\dot{\theta}_5)^2] / (s\alpha_2 s\alpha_3) \\ \dot{N}_2 &= -[N_{21} \ddot{\theta}_1 + N_{22} \ddot{\theta}_5 + N_{23} (\dot{\theta}_1)^2 + 2N_{24} \dot{\theta}_1 \dot{\theta}_5 \\ &\quad + N_{25} (\dot{\theta}_5)^2] / (a_2 s\alpha_3) \end{aligned}$$

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