Development of Sensitivity Analysis Algorithm for a Straight-Line Beam Structure by the Transfer Stiffness Coefficient Method*

Deok-Hong MOON** and Myung-Soo CHOI***

This paper describes the sensitivity analysis algorithm extending the concept of the transfer stiffness coefficient method (TSCM) which is suitable to analyze accurately vibrations of complex and large structures on a personal computer with small computer memories. This sensitivity analysis algorithm is based on the concept transferring the sensitivities of the stiffness coefficient with respect to design parameters. In this paper, the free vibration analysis algorithm by the TSCM for a straight-line structure is shortly introduced and the free vibration sensitivity analysis algorithm is formulated by the present method. To demonstrate the validity of the present method, a straight-line beam structure with various boundary conditions is proposed as a numerical example. Eigenvalues and eigenvectors and their derivatives for the beam obtained by the present method are compared with those of the Fox's method and the reanalysis on a personal computer.

Key Words: Sensitivity Analysis, Eigenvalue Analysis, Finite Element Method, Free Vibration, Transfer Stiffness Coefficient Method, Straight-Line Beam Structure

1. Introduction

To design a mechanical or a structural system with good performance, the main procedure of a typical design work has been made up of repeated variations of design parameters and the investigation of the performance of the system for the variations of these parameters. But this procedure takes a lot of times, effort and experience.

Sensitivity analysis can provide systematic data for improving performance of the system. These data can be used in the structural modification, optimization and identification as well as the design of the systems. In particular, the sensitivities of eigenvalues and eigenvectors are very useful for estimating the trend of dynamic response with respect to the variations of design parameters. Over the past few decades, a considerable number of studies have been conducted on the sensitivities of eigenvalues and eigenvectors by Fox(1), Nelson(2) and Wang(3) etc. By the way, most of these studies have focused on the sensitivity analysis algorithm extending the concept of the finite element method(4) (FEM) that is the most powerful numerical method in the static and the dynamic analyses of structures. However, the FEM is not always the best method because it takes much computer storage and computation time in the case of conducting accurately dynamic analysis and sensitivity analysis of complex and large structures(5).

Yamagawa(6) suggested the sensitivity analysis algorithm extending the concept of the transfer matrix method(7) (TMM) in order to overcome the drawback of the FEM. But in the application of the TMM, the disadvantages in the numerical calculation have been pointed out, e.g. difficulties in calculating the higher order eigenvalues, for the case that intermediate elastic supports are very stiff, and for the case that connecting part between both members is hinge.
Kondou developed the transfer influence coefficient method (TICM) as a method of vibration analysis with high computation efficiency. However, this method could not be applicable to a beam structure having closed loops (so-called Rahmen structure). And so Kondou suggested the transfer stiffness coefficient method (TSCM) by introducing the concept of the substructure synthesis method into the algorithm of the TICM. Moon (1) have developed the enhanced TSCM adopting the spectral element method and the finite element modeling techniques. The TSCM is based on the concept transferring the stiffness coefficient, which is related to the force vector and the displacement vector at each node, from the left end to the right end of the system. The advantage of the TSCM is conducting accurately vibration analysis for complex and large structures on a personal computer with small computer memories.

The authors developed the sensitivity analysis algorithm extending the concept of the TSCM. This algorithm is based on the concept transferring the sensitivities of the stiffness coefficient with respect to design parameters. In this paper, the free vibration analysis algorithm by the TSCM for a straight-line structure is shortly introduced and the free vibration sensitivity analysis algorithm is formulated by the present method. To demonstrate the validity of the present method, a straight-line beam structure with various boundary conditions is proposed as a numerical example. Eigenvalues and eigenvectors and their derivatives for the beam obtained by the present method are compared with those of the Fox's method and the reanalysis on a personal computer.

2. Application of the Transfer Stiffness Coefficient Method

A straight-line structure is considered as an analytical model in order to describe clearly the formulation of sensitivity analysis of structures using the transfer stiffness coefficient method.

2.1 Modeling

A straight-line structure is modeled as a transfer system that consists of many point elements and field elements, as shown in Fig. 1. In this paper, the field element is a shaft element, a rod element, or a beam element; and the point element is a lumped mass or a point spring supporting field elements from the base. Nodes are the connecting points of elements and both ends of the system. In the case of the system with \( n \) elements, each node is consecutively called node 0, node 1, ..., node \( n \) from the left-hand end to the right-hand end of the system.

The boundary conditions at both ends of the structure are modeled as point elements, which are composed of the point springs. For example, if the boundary condition at the left-hand end is clamped, the spring constants of the first point element are all infinites. If the boundary condition is free, the spring constants are all zeros. Therefore, the first and the last elements of the system are point elements which take into account the boundary conditions of the system.

2.2 Free vibration analysis

The relationship between the force vector \( (\mathbf{f}_i) \) and the displacement vector \( (\mathbf{d}_i) \) at node \( i \) is defined as the stiffness coefficient matrix \( (\mathbf{S}_i) \) as follows:

\[
\mathbf{f}_i = \mathbf{S}_i \mathbf{d}_i, \quad \mathbf{S}_i = \mathbf{S}^T_i \tag{1}
\]

In this paper, the subscript \( i \) denotes the physical quantities of the node \( i \) or the element \( i \), and the superscript \( T \) denotes the transposition of the matrix.

The equilibrium equation of the force vectors at both sides of the point element \( i \), as shown in Fig. 2, and the continuous condition of the displacement vectors can be expressed as

\[
\mathbf{f}_i = \mathbf{f}_{i-1} + \mathbf{P}_i \mathbf{d}_i, \quad \mathbf{d}_i = \mathbf{d}_{i-1} \tag{2}
\]

where the matrix \( \mathbf{P}_i \) is the point stiffness matrix which is considered as the additional lumped-mass and the point spring at node \( i \).

If we take the stiffness coefficient matrix \( \mathbf{S}_{i-1} \) at the left-hand side of the point element \( i \), we can obtain the matrix \( \mathbf{S}_i \) at the right-hand side of the point element \( i \) from Eqs. (1) and (2) as follows:

\[
\mathbf{S}_i = \mathbf{S}_{i-1} + \mathbf{P}_i \tag{3}
\]

and Eq. (3) is called as the point transfer equation of the stiffness coefficient matrix.

The left-hand side of the point element 1 is considered as being free \( (\mathbf{f}_1 = 0) \), because the boundary condition at the left-hand end of the structure is modeled as the point element 1. Therefore, we can find out the matrix \( \mathbf{S}_1 \) from Eqs. (1) and (2) into which \( i = 1 \) has been substituted as follows:

\[
\mathbf{S}_1 = \mathbf{P}_1 \tag{4}
\]

The relationship between the force vectors and the displacement vectors of both sides of the field element, as shown in Fig. 3, can easily be derived from
the FEM as follows:

\[
\begin{bmatrix}
 f_i \\
 f_{i-1}
\end{bmatrix} =
\begin{bmatrix}
 A_i & B_i \\
 C_i & D_i
\end{bmatrix}
\begin{bmatrix}
 d_i \\
 d_{i-1}
\end{bmatrix}
\]

In the case of the free vibration analysis of a beam structure, the point stiffness matrix \( P_i \) and the submatrices \( A_i, B_i, C_i, D_i \) are given in Appendix.

If we take the stiffness coefficient matrix \( S_{i+1} \) at the left-hand side of the field element \( i \), we can obtain the matrix \( S_i \) at the right-hand side of the field element \( i \) from Eqs. (1) and (5) as follows:

\[
S_i = A_i + B_i V_i
\]

where

\[
V_i = H_i^{-1} C_i, \quad H_i = S_{i-1} - D_i
\]

and Eq. (6) is called the field transfer equation of the stiffness coefficient matrix.

After finding out \( S_i \) from Eq. (4), if we apply the field and point transfer equation successively to the transfer system, we can obtain finally the stiffness coefficient matrix \( S_n \) at node \( n \).

Because we consider the boundary condition of the right-hand end of the structure as the point element \( n \), the node \( n \) can be considered analytically as being free, that is, \( f_n = 0 \) and \( d_n \neq 0 \). Therefore, from the boundary condition and Eq. (1) into which \( i = n \) has been substituted, the frequency equation is derived as follows:

\[
\det S_n = 0
\]

Because the stiffness coefficient matrix \( S_n \) is the function of eigenvalue (\( \lambda \)), we can find out eigenvalue from Eq. (8).

From Eqs. (1), (5) and (7), the relationship of displacement vectors at both sides of the field element \( i \) is expressed as

\[
d_{i+1} = V_i d_i
\]

After obtaining the eigenvalue from Eq. (8), the displacement vector of node \( n \), \( d_n \), is calculated from \( f_n = 0 \) and Eq. (1) into which \( i = n \) has been substituted, that is, \( S_n d_n = 0 \). The displacement vectors of the other nodes are obtained by applying Eqs. (2) or (9) successively from the right-hand end to the left-hand end of the system.

2.3 Free vibration sensitivity analysis

From differentiating Eq. (1) with respect to a design variable \( x_i \), we get

\[
f_i^* = S_i^* d_i + S_i^* d_n
\]

where the symbol * denotes \( \partial / \partial x_i \), \( d_i^* \) and \( f_i^* \) are the derivatives of the displacement vector and the force vector at node \( i \) with respect to design parameter \( x_i \).

The matrix \( S_i^* \) is called the sensitivity stiffness coefficient matrix.

Differentiating Eq. (2) yields as follows:

\[
f_i^* = f_{i-1}^* + P_i^* d_i + P_i^* d_{i-1}^*, \quad d_n^* = d_n
\]

If we take the sensitivity stiffness coefficient matrix \( S_i^* \) at the left-hand side of the point element \( i \), we can obtain the sensitivity stiffness coefficient matrix \( S_i^* \) at the right-hand side of the point element \( i \), from Eqs. (10) and (11) as follows:

\[
S_i^* = S_{i+1}^* + P_i^*
\]

and Eq. (12) is the point transfer equation of the sensitivity stiffness coefficient matrix.

Because node 0 is considered as being free, we can find out that the force vector \( f_0 \) and its differentiating \( f_0^* \) are the null vectors. We can derive the matrix \( S_0^* \) from Eqs. (10) and (11) as follows:

\[
S_0^* = P_0^*
\]

From differentiating Eq. (5) with respect to the design variable \( x_i \), we obtain

\[
f_i^* = (A_i d_i + B_i d_{i-1})^* \quad f_i^* = (C_i d_i + D_i d_{i-1})^*
\]

In the case of the free vibration sensitivity analysis of a beam structure, the derivatives of the point stiffness matrix \( P_i \) and the submatrices \( A_i, B_i, C_i, D_i \) are given in Appendix.

If we take the matrix \( S_i^* \) at the left-hand side of the field element \( i \), we can obtain the matrix \( S_i^* \) at the right-hand side from Eqs. (10) and (14).

\[
S_i^* = A_i + B_i V_i + Z_i (H_i^* V_i - C_i^*)
\]

where \( H_i^* \) and \( Z_i \) are given by

\[
H_i^* = S_{i+1}^* - D_i^*, \quad Z_i = -B_i H_i^{-1}
\]

and Eq. (15) is the field transfer equation of the sensitivity stiffness coefficient matrix.

After obtaining eigenvalues and eigenvectors through the free vibration analysis, we can find out \( S_i^* \) from Eq. (13). If we apply successively the transfer equation, Eqs. (12) and (15), to the transfer system, we can obtain finally out the sensitivity stiffness coefficient matrix \( S_i^* \) at node \( n \).

Because the matrix \( S_n \) is the symmetric matrix and the vector \( f_n^* \) is the null vector, after substituting \( i = n \) into Eq. (10), if we premultiply the equation by \( d_n^* \), we can derive the following equation:

\[
d_n^* S_n^* d_n = 0
\]

Because the sensitivity stiffness coefficient matrix \( S_n^* \) is the function of the derivative of eigenvalue, \( \lambda \), we can find out the derivative of eigenvalue from Eq. (17).

Substituting \( i = n \) into Eq. (10) yields

\[
S_n^* d_n = g_n
\]

where a known vector \( g_n \) is

\[
g_n = -S_n^* d_n
\]
Because the matrix $S_s$ is the singular matrix, we cannot find out the vector $d_s^* \equiv \mathbf{d}_s^*$ directly from Eq. (18). The displacement vector at right-hand end has the following property:

$$d_s^* = \text{constant} \quad (20)$$

From differentiating Eq. (20) with respect to a design variable $x_i$, we get

$$d_s^* \equiv \mathbf{d}_s^* = 0 \quad (21)$$

If we eliminate an equation from the simultaneous equations, Eq. (18), the remainder simultaneous equations become

$$\mathbf{S}_s \mathbf{d}_s^* = \mathbf{g}_s \quad (22)$$

From Eqs. (21) and (22), we get

$$\begin{bmatrix} \mathbf{S}_s \\ d_s^* \end{bmatrix} d_s^* = \begin{bmatrix} \mathbf{g}_s \\ 0 \end{bmatrix} \quad (23)$$

Therefore, we can find out the derivative of displacement vector at node $n$ as follows:

$$d_s^* = \begin{bmatrix} \mathbf{S}_s \\ d_s^* \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{g}_s \\ 0 \end{bmatrix} \quad (24)$$

The relationship of the derivatives of displacement vectors at both sides of the field element $i$ is obtained from Eqs. (10), (14) and (16) as follows:

$$d_s^* = V_i d_s^* + H_i^{-1} \left( C_i^* - H_i \mathbf{r}_i \right) \mathbf{d}_i \quad (25)$$

After obtaining the derivatives of the displacement vector of node $n$ from Eq. (24), the derivatives of the displacement vectors at the other nodes can be obtained by applying Eqs. (11) or (25) successively from the right-hand end to the left-hand end of the system.

3. Numerical Computational Results and Discussions

We made computational programs for sensitivity analysis of beam structures by using the TSCM and the Fox’s method. 8) Eigenvalues, eigenvectors and their derivatives for a straight-line beam structure with various boundary conditions were calculated on a personal computer.

The numerical example is the straight-line beam with length $L = 2$ m, modulus of elasticity $E = 206$ GPa/m$^2$ and mass density $\rho = 7860$ kg/m$^3$. The beam was composed of five beam elements, six nodes, and twelve degrees of freedom. The cross sections of the beam are circular with diameter $d = 0.03$ m. In numerical calculation, the boundary conditions of the beam are considered as six cases, that is, clamped-clamped, clamped-simply supported, clamped-free, simply supported-simply supported, simply supported-free, and free-free. Fig. 4 shows the cantilever beam whose boundary condition is clamped-free.

When the design variables are the diameters of each beam element, eigenvalues and their derivatives were calculated by the TSCM and the Fox’s method.

Table 1 indicates the first three eigenvalues and their derivatives for the beam with six boundary conditions. The results calculated by the TSCM coincide with those of Fox’s method. Therefore, we can confirm the accuracy of the TSCM.

In the case of the beam with clamped-clamped boundary condition, modifying diameters of the first and the last beam elements is more effective than the others to change largely the first eigenvalue, as shown in Table 1 (a). The value of $\partial \lambda_1 / \partial d_5$ which is the derivative of the first eigenvalue ($\lambda_1$) with respect to the diameter ($d_5$) of the second beam element, is negative. This means that the first eigenvalue is decreased when the diameter of the second beam element is increased.

In the case of the beam with clamped-free boundary condition, as shown in Fig. 4, modifying diameter of the first beam element is more effective than the others to change largely the first eigenvalue, as shown in Table 1 (c). Because the values of $\partial \lambda_1 / \partial d_5$ and $\partial \lambda_1 / \partial d_4$ are negative, the first eigenvalue is decreased when the diameters of the fourth and the last beam element are increased.

In the case of the beam with simply supported-simply supported boundary condition, modifying diameter of the middle beam element is more effective than the others to change largely the first eigenvalue, as shown in Table 1 (d). Because derivatives of the first three eigenvalues are all positive in Table 1 (d), these eigenvalues may be increased when the diameters of all beam elements are increased.

In the case of the beam with free-free boundary condition, modifying diameter of the third beam element is more effective than the others to change largely the first eigenvalue, as shown in Table 1 (f). Because the values of $\partial \lambda_1 / \partial d_5$ and $\partial \lambda_1 / \partial d_4$ are negative, the first eigenvalue is decreased when the diameters of the first and the last beam element are increased. Two rigid-body modes are eliminated in the Table 1 (f).

When the design variables are the diameters of each beam element, the derivatives of eigenvectors were calculated by the TSCM. Table 2 indicates the derivatives of the first three eigenvectors (transverse displacements ($y_n, y_n, \cdots, y_n$)) for the beam with clamped-free boundary condition. $\partial \mathbf{y}_n^0 / \partial d_5$ in Table 1 (a) means the derivative of the transverse displacement
### Table 1
Eigenvalues and derivatives of eigenvalues with respect to diameters of beam elements

(a) Boundary condition: clamped-clamped

<table>
<thead>
<tr>
<th>Order ((k))</th>
<th>(\Lambda_k)</th>
<th>(\frac{\partial \Lambda_k}{\partial d_1})</th>
<th>(\frac{\partial \Lambda_k}{\partial d_2})</th>
<th>(\frac{\partial \Lambda_k}{\partial d_3})</th>
<th>(\frac{\partial \Lambda_k}{\partial d_4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.5764</td>
<td>1.7266</td>
<td>-2.7825</td>
<td>1.8245</td>
<td>-2.7825</td>
</tr>
<tr>
<td>2</td>
<td>5.3536</td>
<td>4.9131</td>
<td>4.1576</td>
<td>1.3196</td>
<td>6.1426</td>
</tr>
<tr>
<td>3</td>
<td>4.2657</td>
<td>2.1957</td>
<td>2.1147</td>
<td>2.6217</td>
<td>1.1407</td>
</tr>
</tbody>
</table>

### Table 2
Derivatives of eigenvectors for cantilever beam with five beam elements

(a) Design variable: diameter of first beam element

<table>
<thead>
<tr>
<th>Order ((k))</th>
<th>(\frac{\partial y_1^{(1)}}{\partial d_1})</th>
<th>(\frac{\partial y_1^{(1)}}{\partial d_2})</th>
<th>(\frac{\partial y_1^{(1)}}{\partial d_3})</th>
<th>(\frac{\partial y_1^{(1)}}{\partial d_4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.1512</td>
<td>7.5953</td>
<td>-12.269</td>
<td>-8.2384</td>
</tr>
<tr>
<td>3</td>
<td>-16.1633</td>
<td>9.0486</td>
<td>9.5201</td>
<td>-2.8238</td>
</tr>
</tbody>
</table>

(b) Design variable: diameter of second beam element

<table>
<thead>
<tr>
<th>Order ((k))</th>
<th>(\frac{\partial y_2^{(2)}}{\partial d_1})</th>
<th>(\frac{\partial y_2^{(2)}}{\partial d_2})</th>
<th>(\frac{\partial y_2^{(2)}}{\partial d_3})</th>
<th>(\frac{\partial y_2^{(2)}}{\partial d_4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.6149</td>
<td>3.3365</td>
<td>1.472</td>
<td>0.9442</td>
</tr>
<tr>
<td>2</td>
<td>4.7974</td>
<td>10.9608</td>
<td>4.9001</td>
<td>2.8351</td>
</tr>
<tr>
<td>3</td>
<td>-18.0460</td>
<td>-19.621</td>
<td>0.5112</td>
<td>-2.6970</td>
</tr>
</tbody>
</table>

(c) Design variable: diameter of third beam element

<table>
<thead>
<tr>
<th>Order ((k))</th>
<th>(\frac{\partial y_3^{(3)}}{\partial d_1})</th>
<th>(\frac{\partial y_3^{(3)}}{\partial d_2})</th>
<th>(\frac{\partial y_3^{(3)}}{\partial d_3})</th>
<th>(\frac{\partial y_3^{(3)}}{\partial d_4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.5551</td>
<td>4.7385</td>
<td>5.6797</td>
<td>4.0222</td>
</tr>
<tr>
<td>2</td>
<td>0.2886</td>
<td>15.6386</td>
<td>19.4045</td>
<td>7.7541</td>
</tr>
<tr>
<td>3</td>
<td>-3.3113</td>
<td>-9.6026</td>
<td>7.7073</td>
<td>0.4922</td>
</tr>
</tbody>
</table>

(d) Design variable: diameter of forth beam element

<table>
<thead>
<tr>
<th>Order ((k))</th>
<th>(\frac{\partial y_4^{(4)}}{\partial d_1})</th>
<th>(\frac{\partial y_4^{(4)}}{\partial d_2})</th>
<th>(\frac{\partial y_4^{(4)}}{\partial d_3})</th>
<th>(\frac{\partial y_4^{(4)}}{\partial d_4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4666</td>
<td>1.6472</td>
<td>3.1890</td>
<td>3.1890</td>
</tr>
<tr>
<td>3</td>
<td>3.4117</td>
<td>-6.1697</td>
<td>3.9708</td>
<td>22.7656</td>
</tr>
</tbody>
</table>

(e) Design variable: diameter of fifth beam element

<table>
<thead>
<tr>
<th>Order ((k))</th>
<th>(\frac{\partial y_5^{(5)}}{\partial d_1})</th>
<th>(\frac{\partial y_5^{(5)}}{\partial d_2})</th>
<th>(\frac{\partial y_5^{(5)}}{\partial d_3})</th>
<th>(\frac{\partial y_5^{(5)}}{\partial d_4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2344</td>
<td>-0.3680</td>
<td>-0.4088</td>
<td>-0.2088</td>
</tr>
<tr>
<td>2</td>
<td>-6.4098</td>
<td>-7.5344</td>
<td>-20.6803</td>
<td>-15.5680</td>
</tr>
<tr>
<td>3</td>
<td>29.872</td>
<td>26.3775</td>
<td>-21.3788</td>
<td>-17.2442</td>
</tr>
</tbody>
</table>

### Notes
- \(y_2\) at node 2 in the third eigenvector with respect to the diameter \(d_1\) of the first beam element.
- Nodal displacements are normalized by defining a maximum transverse displacement as unit value.
- When the diameter of the fifth beam element was increased by 30%, the change of the first mode shape is very small, but the changes of the second and the third mode shapes are very large, as shown in Fig. 5 (e). Though only the first order derivative is considered, the reanalysis results are quite similar to approximations using the derivatives of eigenvectors obtained by the TSCM. Therefore, we can confirm the trust of sensitivity analysis by the TSCM.

### Conclusions
The authors developed the sensitivity analysis algorithm extending the concept of the transfer stiffness coefficient method which has advantage in conducting accurately vibration analysis for complex and large structures on a personal computer with small computer memories. This sensitivity analysis algorithm is based on the concept transferring the sensitivities of the stiffness coefficient with respect to design parameters.
In this paper, the free vibration analysis algorithm by the TSCM for a straight-line structure was shortly introduced and the free vibration sensitivity analysis algorithm was formulated by the present method. Through sensitivity analysis finding out the derivatives of eigenvalues and eigenvectors for a straight-line beam structure with various boundary conditions by the present method, the Fox's method and the reanalysis, we confirmed the validity of the present method.

In future studies, the authors would like to research about higher order sensitivity analysis, optimization, and identification by using the present method.

Appendix: The Point Stiffness Matrix, The Submatrices, and Their Derivatives

In the case of the free vibration analysis of a beam structure, the point stiffness matrix $P$, and the submatrices $A_i, B_i, C_i, D_i$ are as follows:

$$P = \begin{bmatrix} k_i - m_i \lambda & 0 \\ 0 & K_i - m_i \lambda \end{bmatrix} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = K_i - \lambda M_i,$$

$$K_i = \frac{\pi E d^4}{64 I^2} \begin{bmatrix} 12 & -6I & -12 & -6I \\ -6I & 4I^2 & 6I & 2I^2 \\ 12 & -6I & -12 & -6I \\ 6I & -2I^2 & -6I & -4I^2 \end{bmatrix},$$

$$M_i = \frac{\pi I d^3}{1680} \begin{bmatrix} 156 & -22I & 54 & 13I \\ -22I & 4I^2 & -13I & -3I^2 \\ -54 & 13I & -156 & -22I \\ -13I & 3I^2 & -22I & -4I^2 \end{bmatrix},$$

where $k_i$ and $K_i$ symbolize the spring constants of transversal and rotational springs at node $i$, $m_i$ and $J_i$ symbolize the mass and the mass moment of inertia of additional lumped mass at node $i$, and $\lambda$ is eigenvalue. $l$ is the length of the beam element, $d$ is diameter, $E$ is Young's modulus.

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Fig. 5 Change of mode shapes for cantilever beam with five beam elements

1. Design variable: diameter of first beam element
2. Design variable: diameter of second beam element
3. Design variable: diameter of third beam element
4. Design variable: diameter of forth beam element
5. Design variable: diameter of fifth beam element
is Young’s modulus and \( \rho \) is mass density.

For example, if a design variable is the diameter of the \( j \)-th beam element, the derivative of the point stiffness matrix with respect to design variable, \( P^i_\ast \), is always null matrix. In the case of \( i \neq j \), the derivatives of the submatrices, \( A^i_\ast, B^i_\ast, C^i_\ast, D^i_\ast \), are null matrices. However, in the case of \( i = j \), the matrices \( A^i_\ast, B^i_\ast, C^i_\ast, D^i_\ast \) are as follows:

\[
\begin{bmatrix}
A^i_\ast & B^i_\ast \\
C^i_\ast & D^i_\ast
\end{bmatrix} = K^i_\ast - \lambda^i M^i_\ast - \lambda M^i_\ast,
\]

\[
K^i_\ast = \frac{\pi E d^3}{16 l^3}
\begin{bmatrix}
12 & -6l & -12 & -6l \\
-6l & 4l^2 & 6l & 2l^2 \\
6l & -2l^2 & -6l & -4l^2 \\
156 & -22l & 54 & 13l
\end{bmatrix}
\]

\[
M^i_\ast = \frac{\pi \rho l d}{840}
\begin{bmatrix}
-22l & 4l^2 & -13l & -3l^2 \\
-54 & 13l & -156 & -22l \\
-13l & 3l^2 & -22l & -4l^2
\end{bmatrix}
\]

References


