Output Time Delayed Control System Design Subject to the Mixed Sensitivity Problem*

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This paper presents a design methodology for MIMO output time delayed systems satisfying the mixed sensitivity specification. Nevertheless the procedure is on the basis of the state space approach, it provides a controller with tuneable parameters, which makes the technique useful for practical applications. The key features of the proposed method are its capability to handle multiple time delays and to allow the calculations to be performed as in the time delay free system case. A solution for output time delay unstable system is illustrated with a numerical example.

Key Words: Output Time Delayed Control System, Control System Design, Mixed Sensitivity Problem, Robust Control, Process Control, Multiple Time Delays

1. Introduction

The design of time delay systems to have disturbance rejection and robust stability characteristics has been known to be a very difficult problem for a long time^{(1),(2)}. On the basis of the H_∞ control, this problem has been formulated as the mixed sensitivity problem, and solutions have been proposed to solve it⁽³⁾⁻⁽⁷⁾. However, these investigations focused primarily either on the SISO systems or MIMO systems with single time delay component. Also, the expressions that appear along the calculations involve the time delay terms, which require, in many cases, numerical computations.

In this paper, it is proposed a design method for MIMO output time delay systems that (i) provides a solution to the mixed sensitivity problem, (ii) handles multiple output time delay terms, (iii) allows the majority of the calculations to be performed as in the delay free case, and (iv) presents a controller with a tuneable parameter that can be used to adjust the system performance characteristics during the process operation.

In fact, considering that many industrial processes have multiple time delay components⁽⁸⁾ and

they are required to have tuneable parameters to change the system performance during the operation⁽⁹⁾, the proposed method can be useful to this kind of applications.

The paper is organized as follows. Section 2 describes the system in consideration, section 3 formulates the problem, section 4 provides the realisation, section 5 summarizes the design procedure in an algorithm, and section 6 presents a numerical example.

2. Preliminaries

2.1 Plant and controller

Consider the output time delayed plant described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ \begin{bmatrix} y_1(t-L_1) \\ \vdots \\ y_m(t-L_m) \end{bmatrix} &= \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} x(t) \end{aligned} \quad (1)$$

and let the controller be the generalised stabilizing controller⁽¹⁰⁾ which is defined by the system of linear equations

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) - Kv(t) \\ \dot{\xi}(t) &= A_b \xi(t) + B_b v(t) \\ u(t) &= -Fz(t) + Q_a C_b \xi(t) + Q_a D_b v(t) \\ v(t) &= e(t) - \pi(t) + \bar{C}z(t) \\ e(t) &= r(t) - y(t) \end{aligned} \quad (2)$$

where

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$$\begin{aligned} A &\in R^{n \times n}, B \in R^{n \times m}, C \in R^{m \times n}, K \in R^{n \times m} \\ A_b &\in R^{(n_s-m) \times (n_s-m)}, B_b \in R^{(n_s-m) \times m} \\ C_b &\in R^{m \times (n_s-m)}, D_b \in R^{m \times m}, F \in R^{m \times n} \\ Q_a &\in R^{m \times m} \end{aligned} \quad (3)$$

$$\bar{C} = \begin{bmatrix} C_1 e^{-AL_1} \\ \vdots \\ C_m e^{-AL_m} \end{bmatrix} \quad (4)$$

$$\pi(t) = \begin{bmatrix} \int_{t-L_1}^t C_1 e^{A(t-L_1-\tau)} B u(\tau) d\tau \\ \vdots \\ \int_{t-L_m}^t C_m e^{A(t-L_m-\tau)} B u(\tau) d\tau \end{bmatrix} \quad (5)$$

n_s : to be defined in section 2.3.

Note that this controller has been shown to be useful for handling the mixed sensitivity problems⁽¹⁰⁾. In fact, it will be shown later on that the feedback control system is able to use the components F and Q_a of the controller for handling one half of the control system design specification, and the term $Q_b(s)(Q_b(s) = D_b + C_b(sI - A_b)^{-1}B_b)$ of the controller for handling the other part of the specification. Also, it is useful pointing out here that $Q_b(s)$ is responsible for the low sensitivity characteristic of the feedback control system.

Now, the Laplace transform of Eq.(1) renders the plant

$$\begin{aligned} G_p(s) &= \Gamma(s)G(s) \in R^{m \times m}(s) \\ G(s) &= C(sI - A)^{-1}B \\ \Gamma(s) &= \text{diag}[e^{-sL_i}], L_i \geq 0 \end{aligned} \quad (6)$$

whereas the controller $G_c(s)$ for the control system represented in Fig. 1 is given by (appendix A)

$$\begin{aligned} u(s) &= G_c(s)e(s) \\ G_c(s) &= [I - P_a(s)P_c(s)P_b(s)\Gamma(s)G(s)]^{-1} \\ &\quad \times P_a(s)P_c(s)P_b(s) \end{aligned} \quad (7)$$

where

$$\begin{aligned} P_a(s) &= [I + F(sI - A)^{-1}B]^{-1}Q_a \\ P_b(s) &= [I + \bar{C}(sI - A)^{-1}K]^{-1} \\ P_c(s) &= [Q_b(s) + Q_a^{-1}F(sI - A)^{-1}K] \end{aligned} \quad (8)$$

and $\Pi(s)$ is the finite Laplace transform of $\pi(t)$. Furthermore, Q_a is non-singular; and $Q_b(s)$, $A - BF$ and $A - K\bar{C}$ are all asymptotically stable.

2.2 Assumptions

The control system design is accomplished under the following assumptions:

(a1) the triplet $(\bar{C}, A, B)$ is controllable and ob-

Fig. 1 Control system with time delays in the output

servable.

(a2) the nominal model (1) has no purely imaginary zeroes, and has the same number of poles as the real plant.

(a3) decoupling of $G(s)$ is possible by means of the state feedback. In other words, let ν_i ($i=1, \dots, m$) be positive integers satisfying for $i=1, \dots, m$

(i) $C_i A^j B = 0$ for $j=1, 2, \dots, \nu_i - 2$

(ii) $C_i A^{\nu_i - 1} B \neq 0$

and if a matrix Φ is defined as

$$\Phi = \begin{bmatrix} C_1 A^{\nu_1 - 1} B \\ \vdots \\ C_m A^{\nu_m - 1} B \end{bmatrix} \quad (9)$$

it implies

$$\text{rank } \Phi = m. \quad (10)$$

(a4) the control system in Fig. 1 has output multiplicative model uncertainty given by

$$\begin{aligned} W_T(s) &\in R(s) \\ |W_T(j\omega)| &\begin{cases} < 1, & 0 \leq \omega < \omega_T \\ = 1, & \omega = \omega_T \\ > 1, & \omega_T < \omega \end{cases} \end{aligned} \quad (11)$$

for some $\omega_T \geq 0$

and the relative degree of $1/|W_T(s)|$ is less than or equal to the minimum ν amongst $\nu_1, \dots, \nu_m$.

2.3 Some definitions

(d1) Let $W_s(s) \in R(s)$ be

$$\begin{aligned} \text{diag}(W_{s_i}(s)) &= C_s(sI_{n_s} - A_s)^{-1}B_s \\ A_s &= \text{diag}[A_{s1}, \dots, A_{sm}] \in R^{n_s \times n_s} \\ B_s &= \text{diag}[b_{s1}, \dots, b_{sm}] \in R^{n_s \times m} \\ C_s &= \text{diag}[c_{s1}, \dots, c_{sm}] \in R^{m \times n_s} \\ W_{s_i}(s) &= c_{si}(sI - A_{si})^{-1}b_{si} \end{aligned} \quad (12)$$

where I_{n_s} indicates the unitary matrix of size n_s .

(d2) Define $\Lambda(s)$ as

$$\Lambda(s) = \text{diag} \left[\frac{\alpha_{i\nu_i}}{s^{\nu_i} + \alpha_{i1}s^{\nu_i-1} + \dots + \alpha_{i\nu_i}} \right] \quad (13)$$

$$\begin{aligned} s^{\nu_i} + \alpha_{i1}s^{\nu_i-1} + \dots + \alpha_{i\nu_i} &= \\ \begin{cases} (s^2 + 2\zeta\omega_n s + \omega_n^2)^{\frac{\nu_i}{2}}, & \nu_i: \text{even} \\ (s^2 + 2\zeta\omega_n s + \omega_n^2)^{\frac{\nu_i-1}{2}}(s + \omega_n), & \nu_i: \text{odd} \\ \zeta = \frac{1}{\sqrt{2}}, & 0 < \omega_n \leq \omega_T. \end{cases} \end{aligned} \quad (14)$$

(d3) Let Ψ and Ω be

$$\Psi = \begin{bmatrix} C_1 A^{\nu_1} + \alpha_{11} C_1 A^{\nu_1-1} + \dots + C_1 \alpha_{1\nu_1} \\ \vdots \\ C_m A^{\nu_m} + \alpha_{m1} C_m A^{\nu_m-1} + \dots + C_m \alpha_{m\nu_m} \end{bmatrix} \quad (15)$$

$$\Omega = \text{diag}[\alpha_{1\nu_1}, \dots, \alpha_{m\nu_m}]. \quad (16)$$

(d4) Let the non-singular matrix $\hat{T}$ be

$$\hat{T} = \begin{bmatrix} I_{n_s} & T_1 & T_2 & T_3 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_{(n_s-m)} & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}. \quad (17)$$

3. Problem Statement

Assuming that the disturbances act on the output of the control system shown in Fig. 1, the output side sensitivity and complementary sensitivity functions are

$$S(s) = I - \Gamma(s)G(s)P_a(s)P_c(s)P_b(s) \quad (18)$$

$$T(s) = \Gamma(s)G(s)P_a(s)P_c(s)P_b(s) \quad (19)$$

respectively.

Under these considerations, the system is required to be internally stable⁽¹¹⁾; i.e., all the entries of the transfer function from $r(s)$ and $d(s)$ to $y(s)$ and $u(s)$ be stable for a stable $P(s) = P_a(s)P_c(s)P_b(s)$ in the transfer function expressed by

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \Gamma G P & (I - \Gamma G P) \Gamma G \\ P & P \Gamma G \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}. \quad (20)$$

Moreover, the system has to satisfy the mixed sensitivity specification described by

$$\left\| \begin{bmatrix} W_s(s)S(s) \\ W_T(s)T(s) \end{bmatrix} \right\|_{\infty} < 1. \quad (21)$$

4. Realisation

The final purpose of this section is to show that $P_a(s)$ and $P_b(s)$ are primarily concerned with the control system design specification $W_T(s)T(s)$ of Eq. (21), which means that $P_a(s)$ and $P_b(s)$ assure the stability of the feedback control system. In addition, it is shown that $P_c(s)$ is mainly related to the specification $W_s(s)S(s)$ of Eq. (21), which is established by canceling the purely imaginary poles of $W_s(s)$ by means of the component $Q_b(s)$ of $P_c(s)$.

However, before presenting the result above, it is shown how each component $P_a(s)$, $P_b(s)$ and $P_c(s)$ of the controller are realised. Note that the matrices F and Q_a in $P_a(s)$, the matrix K in $P_b(s)$ and the transfer function $Q_b(s)$ in $P_c(s)$ are realised in order to have the internal stability condition satisfied, which is equivalent to the stability of the transfer function expressed in Eq. (20).

4.1 Realisation of $P_a(s)$

In this section, it is demonstrated that $P_a(s)$ in Eq. (8) can be pursued from the feedback decoupling⁽¹²⁾ of $G(s)$. Moreover, it is shown that a such $P_a(s)$ stabilizes $\Gamma(s)G(s)P_a(s)$ in Eq. (20).

To accomplish it, perform the decoupling of $G(s)$ by means of the state feedback⁽¹²⁾. Then

$$C(sI - A + B\Phi^{-1}\Psi)^{-1}B\Phi^{-1}\Omega = \Lambda(s) \quad (22)$$

is yielded. This can be rewritten as

$$\begin{aligned} \Lambda^{-1}(s)C(sI - A)^{-1}B\Phi^{-1}\Omega \\ = I + \Omega^{-1}\Psi(sI - A)^{-1}B\Phi^{-1}\Omega. \end{aligned} \quad (23)$$

Since $G(s)$ is assumed to be free of purely imaginary zeroes, the Riccati equation for the right side of the Eq. (23); i.e.,

$$\begin{aligned} X(A - B\Phi^{-1}\Psi) + (A - B\Phi^{-1}\Psi)^T X \\ - XB\Phi^{-1}\Omega^2(\Phi^{-1})^T B^T X = 0 \end{aligned} \quad (24)$$

has a stabilizing solution $X = X^T \geq 0$. If F and Q_a are defined as

$$F = \Phi^{-1}\Omega[\Omega^{-1}\Psi + \Omega(\Phi^{-1})^T B^T X] \quad (25)$$

$$Q_a = \Phi^{-1}\Omega \quad (26)$$

respectively. Now, the inner factorisation^{(13),(14)} gives

$$\Lambda^{-1}(s)C(sI - A)^{-1}BQ_a = G_I(s)G_O(s)$$

$$G_I(s) = I + (\Omega^{-1}\Psi - Q_a^{-1}F)(sI - A + BF)^{-1}BQ_a$$

$$G_O(s) = I + Q_a^{-1}F(sI - A)^{-1}BQ_a \quad (27)$$

where $G_I(s)$ is an inner matrix, and the matrix $G_O(s)$ is a matrix with no unstable zeroes; $G_O(s)$ is an outer matrix if the plant is stable.

Now, comparing Eq. (27) and (8), one can see that

$$\begin{aligned} P_a(s) &= Q_a G_O^{-1}(s) \\ &= [I + F(sI - A)^{-1}B]^{-1}Q_a \end{aligned} \quad (28)$$

holds. From this expression, it is clear that $P_a(s)$ is a stable matrix and also, it demonstrates the first assertion in the beginning of this subsection. Finally, it is clear from the expression

$$\Gamma(s)G(s)P_a(s) = \Gamma(s)\Lambda(s)G_I(s) \quad (29)$$

that $P_a(s)$ stabilizes $\Gamma(s)G(s)P_a(s)$ in Eq. (20), since $\Gamma(s)$ is a matrix whose entries express the time delay components of the system, and $\Lambda(s)$ and $G_I(s)$ are stable matrices.

4.2 Realisation of $P_b(s)$

$P_b(s)$ is realised in order to guarantee the stability of the term $P_b(s)\Gamma(s)G(s)$ in Eq. (20). However, due to Eq. (64) and the fact that there occur cancellations of poles and zeroes in the finite Laplace transform operations⁽¹⁵⁾, the stabilisation of $P_b(s)\Gamma(s)G(s)$ reduces to the stabilisation of $P_b(s)\bar{C}(sI - A)^{-1}B$. Therefore, K has to be a stabilizing matrix of $A - K\bar{C}$.

To establish the desired matrix K , consider the Riccati equation

$$YA^T + AY - Y\bar{C}^T\bar{C}Y = 0 \quad (30)$$

that has a stabilizing solution $Y = Y^T \geq 0$, and define the matrix K as

$$K = Y\bar{C}^T \quad (31)$$

and the desired result is pursued⁽¹⁴⁾. Moreover, since the eigenvalues of $A - K\bar{C}$ are the stable as well as the mirror values with respect to the imaginary axis of the unstable eigenvalues of A , $P_b(s)$ in Eq. (20) is stable.

4.3 Realisation of $P_c(s)$

The realisation of $P_c(s)$ reduces to the realisation of $Q_b(s)$, which is determined such that the terms $P_a(s)P_c(s)$ and $P_b^{-1}(s) - \Gamma(s)G(s)P_a(s)P_c(s)$ are stable in Eq. (20). Furthermore, it is required here that the purely imaginary zeroes of $Q_b(s)$ cancels the purely imaginary poles of $W_s(s)$ in $W_s(s)S(s)$ of Eq. (21); i.e., $Q_b(s)$ turns the unstable poles of $W_s(s)$ uncontrol-

lable.

It is shown here that the transfer function $Q_b(s)$ determined on the basis of the poles and zeroes canceling requirement presents the desired $Q_b(s)$.

To establish the result above, recall that the sensitivity function can be briefly written from Eq. (18) and (19) as

$$S(s) = I - T(s). \quad (32)$$

Thus, the uncontrollability condition is equivalent to the poles of $W_s(s)$ uncontrollable in

$$\begin{aligned} I^{-1}(s) W_s(s) S(s) &= W_s(s) \{ I^{-1}(s) - C(sI - A \\ &+ BF)^{-1} B [Q_a Q_b(s) [I - \bar{C}(sI - A + K\bar{C})^{-1} K] \\ &+ F(sI - A + K\bar{C})^{-1} K] \}. \end{aligned} \quad (33)$$

For the sensitivity weighting function given by equation

$$W_s(s) = \frac{\rho}{s} \quad (34)$$

$$\rho \in R > 0 \quad (35)$$

it suffices to determine Q_b such that $S(0) = 0$ holds; i.e.,

$$\begin{aligned} Q_b &= Q_a^{-1} \{ [C(-A + BF)^{-1} B]^{-1} - F(-A + K\bar{C})^{-1} \\ &\times K[I - \bar{C}(-A + K\bar{C})^{-1} K]^{-1} \}. \end{aligned} \quad (36)$$

For the sensitivity weighting function $W_s(s)$, $W(s) \in R(s)$, $Q_b(s)$ is established in the following way. Assume without loss of generality that $W_s(s)S(s)$ and $G(s)$ are unequipped with common poles. Then

$$\begin{aligned} \mathcal{L}[\mathcal{L}^{-1}[I^{-1}(s) W_s(s) S(s)] U(t)] &= \hat{C}(sI - \hat{A})^{-1} \hat{B} \\ U(t) &= \begin{cases} I, & t \geq 0 \\ 0, & t < 0 \end{cases} \end{aligned} \quad (37)$$

holds. Here $\mathcal{L}[*]$ means the Laplace transform of $*$, and

$$\hat{A} = \begin{bmatrix} A_s & -B_s C & 0 & 0 \\ 0 & A_F & B_a C_b & B(F - Q_a D_b \bar{C}) \\ 0 & 0 & A_b & -B_b \bar{C} \\ 0 & 0 & 0 & A_K \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} \bar{B}_s \\ B_a D_b \\ B_b \\ K \end{bmatrix}, \quad \begin{aligned} A_F &= A - BF, \\ A_K &= A - K\bar{C}, \\ B_a &= BQ_a, \end{aligned}$$

$$\bar{C} = [C_s, 0, 0, 0],$$

$$\bar{B}_s = [e^{A_s L_1} b_{s1}, \dots, e^{A_s L_m} b_{sm}]. \quad (38)$$

Next, use Eq. (17) to change the coordinates of the matrices $\hat{A}$ and $\hat{B}$ as

$$\hat{T}^{-1} \hat{A} \hat{T} = \begin{bmatrix} A_s & \hat{A}_{12} & \hat{A}_{13} & \hat{A}_{14} \\ 0 & A_F & B_a C_b & B(F - Q_a D_b \bar{C}) \\ 0 & 0 & A_b & -B_b \bar{C} \\ 0 & 0 & 0 & A_K \end{bmatrix} \quad (39)$$

$$\hat{T}^{-1} \hat{B} = \begin{bmatrix} \bar{B}_1 \\ B_a D_b \\ B_b \\ K \end{bmatrix} \quad (40)$$

respectively. Where

$$\begin{aligned} \hat{A}_{12} &= A_s T_1 - T_1 A_F - B_s C \\ \hat{A}_{13} &= A_s T_2 - T_2 A_b - T_1 B_a C_b \\ \hat{A}_{14} &= A_s T_3 - T_3 A_K - T_1 B(F - Q_a D_b \bar{C}) - T_2 B_b \bar{C} \\ \bar{B}_1 &= \bar{B}_s - T_1 B_a D_b - T_2 B_b + T_3 K. \end{aligned} \quad (41)$$

From these, it is clear that in order to make the poles of $W_s(s)$ uncontrollable, it suffices to make sure that the matrices $\hat{A}_{12}$, $\hat{A}_{13}$, $\hat{A}_{14}$, and $\bar{B}_1$ are null matrices.

In fact, since the matrices A_s and A_F have no common eigenvalues, there exists a matrix T_1 such that $\hat{A}_{12} = 0$. Moreover, as shown in appendix B, $T_1 B_a \in R^{n_s \times m}$ is column full rank, so that it is always possible to define a non-singular matrix $\bar{T}$ by defining an appropriate matrix $M \in R^{n_s \times (n_s - m)}$; i.e.,

$$\bar{T} = [T_1 B_a, M]. \quad (42)$$

and apply it on A_s to change the coordinates of A_s as

$$\bar{T}^{-1} A_s \bar{T} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{matrix} m & \\ n_s - m & \end{matrix} \quad (43)$$

Note that the pair (A_{22}, A_{21}) obtained from this factorisation is controllable (appendix C). If A_b is defined as

$$A_b = A_{22} + A_{21} F_b \quad (44)$$

there exists a matrix $F_b \in R^{m \times (n_s - m)}$ that places the poles of A_b at $-\omega_n$. Also, Let the matrix C_b be

$$C_b = A_{11} F_b - F_b A_b + A_{12}. \quad (45)$$

Since A_s and A have no common eigenvalues, there exists a matrix T_3 such that the expression

$$A_s T_3 - T_3 A = T_1 B F - \bar{B}_s \bar{C} \quad (46)$$

holds. Thus define B_b and D_b as

$$B_b = [0 \ I] \bar{T}^{-1} (\bar{B}_s - T_3 K) \quad (47)$$

$$D_b = [I \ 0] \bar{T}^{-1} (\bar{B}_s - T_3 K) - F_b B_b \quad (48)$$

respectively. As shown in appendix D, a such $Q_b(s) = D_b + C_b(sI - A_b)^{-1} B_b$ makes the unstable poles of $W_s(s)$ uncontrollable as desired. It is useful pointing out that due to the matrix F_b as defined in Eq. (44), $Q_b(s)$ is stable.

Now, it remains to show that $P_c(s)$ indeed guarantees that $P_a(s)P_c(s)$, and $P_b^{-1}(s) - \Gamma(s)G(s)P_a(s)P_c(s)$ are stable. In fact,

$$\begin{aligned} P_a(s)P_c(s) &= [I - F(sI - A + BF)^{-1} B] Q(s) \\ &+ F(sI - A + BF)^{-1} K \\ P_b^{-1}(s) - \Gamma(s)G(s)P_a(s)P_c(s) &= I + \bar{C}(sI - A \\ &+ BF)^{-1} [K - BQ(s)] + \Pi(s)P_a(s)P_c(s) \end{aligned} \quad (49)$$

are stable by means of the term $A - BF$.

4.4 Realisation of the controller $G_c(s)$

It is shown here that the controller $G_c(s)$ of Eq. (7) can provide a solution to the problem stated in section 3. First, let us check the internal stability condition. In fact, considering the $P_a(s)$, $P_b(s)$ and $P_c(s)$ as in the previous sections, it is clear that $P(s) = P_a(s)P_c(s)P_b(s)$, which is also the entry (2, 1) of the transfer function in Eq. (20), is stable. From this, it is

easy to see that $P(s)\Gamma(s)G(s)=P_a(s)P_c(s)P_b(s)\Gamma(s)\times G(s)$ (entry (2, 2)) is a stable matrix. Also, since $P_b^{-1}(s)$ has no unstable zeroes, $\Gamma(s)G(s)P(s)$ (entry (1, 1)) is stable. Moreover, due to the stability of $P_b^{-1}(s) - \Gamma(s)G(s)P_a(s)P_c(s) = [I - \Gamma(s)G(s)P(s)]P_b^{-1}(s)$, $[I - \Gamma(s)G(s)P(s)]\Gamma(s)G(s) = [P_b^{-1}(s) - \Gamma(s)G(s)P_a(s)P_c(s)]P_b(s)\Gamma(s)G(s)$ (entry (1, 2)) is stable. Consequently, the overall feedback control system is internally stable as claimed before.

Now, note that by substituting Eq.(29) into Eqs. (18) and (19), the sensitivity and the complementary sensitivity functions can be rewritten as

$$S(s) = I - \Gamma(s)\Lambda(s)G_I(s)P_c(s)P_b(s) \quad (50)$$

$$T(s) = \Gamma(s)\Lambda(s)G_I(s)P_c(s)P_b(s) \quad (51)$$

where $\bar{\sigma}[\Gamma(j\omega)] = 1$ and $\bar{\sigma}[G_I(j\omega)] = 1$ hold. Further on, from Eq.(8) and K as in section 4.2, $\bar{\sigma}[P_b(j\omega)] = 1$ holds.

Thus, substitution of Eq.(50) and (51) into Eq. (21) and the fact that the influence of $\bar{\sigma}[P_c(j\omega)]$ is defined by $W_s(s)S(s)$ lead to the conclusion that the control system design specification (21) depends on the existence of ω_n in Eq.(14). Hence, the assertion that $G_c(s)$ as defined here can indeed provide a solution to the problem is proved. In the following section it is given an algorithm that summarizes the control system design procedure.

5. Design Algorithm

The design of the control system is summarized in the algorithm below.

Step 1: Solve the Riccati Eq.(24), and calculate F in Eq.(25) as well as Q_a in Eq.(26).

Step 2: Determine K in Eq.(31) by solving the Riccati equation given by Eq.(30).

Step 3: Compute the matrix Q_b from either the Eq. (36) or Eq.(44), (45), (47) and (48).

Step 4: Fix ζ as $\zeta = 1/\sqrt{2}$ (see discussion in Ref. (17)) and let the initial value of ω_n be $\omega_n = \omega_r$.

Step 5: Calculate $S(s)$ and $T(s)$ in Eqs.(18) and (19). Plot $\bar{\sigma}[S(j\omega)]$, $\bar{\sigma}[T(j\omega)]$, $1/|W_s(j\omega)|$ and $1/|W_r(j\omega)|$. On this plot, determine the frequency range for ω_n in which both $\bar{\sigma}[S(j\omega)] < 1/|W_s(j\omega)|$, and $\bar{\sigma}[T(j\omega)] < 1/|W_r(j\omega)|$ hold. If there is no that fulfils both the conditions, a solution by this procedure cannot be pursued. If a range is figured out, draw the gain plot of expression (21), and decrease ω_n from the maximum value in the range until the specification is met. If no ω_n in the range leads to the desired result, a solution by this procedure is not found. An ω_n satisfying the demand is a solution to the problem.

6. Numerical Example

In this section, a solution to an unstable output time delayed plant is presented. Let $\Gamma(s)G(s)$, $W_s(s)$ and $W_r(s)$ be given by

$$\begin{bmatrix} \frac{(s+1)e^{-s}}{(s-1+j)(s-1-j)} & \frac{2(s-1)e^{-s}}{(s+1)(s+3)} \\ \frac{e^{-0.5s}}{(s+4)} & \frac{(s-1)e^{-0.5s}}{(s+2)(s+1)} \end{bmatrix} \quad (52)$$

$$W_s(s) = \frac{0.05}{s^2}, \quad W_r(s) = \frac{(s+3)}{100} \quad (53)$$

respectively.

Letting all the poles of A_b be placed at $-\omega_n$ on the occasion of the realisation of $P_c(s)$, it is possible to determine the controller $G_c(s)$, which is a solution to the problem proposed in section 3, as long as the parameter ω_n in Eq.(14) is in the frequency range defined by $1.10 \text{ rad/s} < \omega_n < 2.45 \text{ rad/s}$. Note that according to the definitions of sensitivity and complementary sensitivity functions ω_n , close to 1.10 rad/s means that the system is quite stable, but presents sluggish step response characteristic. On the other hand, ω_n close to 2.45 rad/s leads to a system on the verge of instability, but with fast step response characteristic. Fig. 2 shows the plot of the mixed sensitivity specification result for $\omega_n = 2.44 \text{ rad/s}$. For this case, the execution of the algorithm leads to the following result:

$$F = \begin{bmatrix} 9.000 & 43.968 & 46.678 & 11.711 \\ -3.546 & -15.867 & -21.272 & -8.951 \\ -5.511 & -29.023 & -48.071 & -24.556 \\ 1.431 & 7.675 & 12.600 & 6.360 \end{bmatrix} \quad (54)$$

$$Q_a = \begin{bmatrix} -2.440 & 4.880 \\ 2.440 & -2.440 \end{bmatrix} \quad (55)$$

Fig. 2 Solution to mixed sensitivity problem

$$A = \begin{bmatrix} 0.212 & 0.000 \\ 0.952 & 0.000 \\ 0.846 & 0.000 \\ 0.370 & 0.000 \\ 0.000 & 0.000 \\ 0.000 & 0.000 \\ 0.000 & 0.000 \\ 0.000 & 0.000 \end{bmatrix} \quad (56)$$

$$A_b = \begin{bmatrix} -2.440 & 0.000 \\ 0.000 & -2.440 \end{bmatrix} \quad (57)$$

$$B_b = \begin{bmatrix} -4.245 & -23.120 \\ 11.484 & 10.391 \end{bmatrix} \quad (58)$$

$$C_b = \begin{bmatrix} -2.234 & -3.509 \\ -1.725 & -4.810 \end{bmatrix} \quad (59)$$

$$D_b = \begin{bmatrix} 18.967 & -7.028 \\ 36.111 & 4.728 \end{bmatrix} \quad (60)$$

Finally, the step response of the feedback control system accomplished by means of the controller above is shown in Fig. 3.

Appendix A : Outline of the calculation of $G_c(s)$

Equations (1)-(5) yield

$$G_c(s) = [I + (F - Q(s)\bar{C})(sI - A + K\bar{C})^{-1} \times (B + K\Pi(s)) + Q(s)\Pi(s)]^{-1} [Q(s) + (F - Q(s)\bar{C})(sI - A + K\bar{C})^{-1}K] \quad (61)$$

$$Q(s) = Q_a Q_b(s)$$

$$Q_b(s) = D_b + C_b(sI - A_b)^{-1}B_b$$

where $\Pi(s)$ given by

$$\Pi(s) = \begin{bmatrix} \int_{-L_1}^0 C_1 e^{A(-L_1-\tau)} B e^{s\tau} d\tau \\ \vdots \\ \int_{-L_m}^0 C_m e^{A(-L_m-\tau)} B e^{s\tau} d\tau \end{bmatrix} \quad (62)$$

is the finite Laplace transform of $\pi(t)$. Here note that

$$\begin{aligned} & \int_{-L_i}^0 C_i e^{A(-L_i-\tau)} B e^{s\tau} d\tau \\ &= \int_0^\infty C_i e^{-AL_i} e^{A\tau} B e^{-s\tau} d\tau - \int_{L_i}^\infty C_i e^{A(\tau-L_i)} B e^{-s\tau} d\tau \\ &= C_i e^{-AL_i} (sI - A)^{-1} B - e^{-sL_i} C_i (sI - A)^{-1} B \end{aligned} \quad (63)$$

is equivalent to the equation

$$\Gamma(s)C(sI - A)^{-1}B = \bar{C}(sI - A)^{-1}B - \Pi(s) \quad (64)$$

Now, substituting $\Pi(s)$ of Eq.(64) into Eq.(61)

Fig. 3 Step response of the system

and making some algebraic manipulations, $G_c(s)$ rewritten as in Eq.(7) is established.

Appendix B : Column full-rankness of $T_1 B_a$

To show that $T_1 B_a$ is column full-rank, write A_s in the Jordan form; i.e.,

$$T_s^{-1} A_s T_s = \text{diag} [J_1, \dots, J_q] \quad (65)$$

where J_i stands for

$$J_i = \text{diag} [J_{i1} \dots J_{im}] \quad (66)$$

and J_{ij} is a matrix composed by eigenvalues p_i

$$J_{ij} = \begin{bmatrix} p_i & 1 & 0 \\ & \ddots & \vdots \\ 0 & & p_i \end{bmatrix} \quad (67)$$

And

$$T_s^{-1} B_s = \begin{bmatrix} B_1 \\ \vdots \\ B_q \end{bmatrix} \quad (68)$$

where B_i means

$$B_i = \begin{bmatrix} B_{i1} \\ \vdots \\ B_{im} \end{bmatrix} \quad (69)$$

and B_{ij} represents the matrix

$$B_{ij} = \begin{bmatrix} B_{ij1} \\ \vdots \\ B_{ijl} \end{bmatrix} \quad (70)$$

with l indicating either the last block or the last row of the matrix. Since (A_s, B_s) is controllable, the matrix

$$\tilde{B}_i = \begin{bmatrix} B_{i1l} \\ \vdots \\ B_{iml} \end{bmatrix} \quad (71)$$

is non-singular.

Now, pre-multiplication of T_s^{-1} with

$$A_s T_1 - T_1 A_F = B_s C \quad (72)$$

yields

$$T_s^{-1} A_s T_s T_s^{-1} T_1 - T_s^{-1} T_1 A_F = T_s^{-1} B_s C \quad (73)$$

and the extraction of the last row from the block $J_{i1}, \dots, J_{iu}$ gives

$$(T_s^{-1} T_1)_i (p_i I - A_F) = \tilde{B}_i C \quad (74)$$

where $(T_s^{-1} T_1)_i$ is a matrix composed by the row corresponding to $T_s^{-1} T_1$.

Thus,

$$(T_s^{-1} T_1)_i B = \tilde{B}_i C (p_i I - A_F)^{-1} B \quad (75)$$

is pursued. Since the right side of the expression (75) is row full-rank, $(T_s^{-1} T_1)_i B$ is row full-rank, and the pair $(T_s^{-1} A_s T_s, T_s^{-1} T_1 B)$ is controllable. Adding to these facts the non-singularity of Q_a , leads to the conclusion that the pair $(A_s, T_1 B Q_a)$ is controllable⁽¹⁶⁾.

Appendix C : Controllability of the pair (A_{22}, A_{21})

As shown in appendix B, the pair $(A_s, T_1 B_a)$ is controllable. Hence, the pair $(T^{-1} A_s T, T^{-1} T_1 B_a)$ is controllable. Furthermore, since

$$T^{-1}T_1B_a = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad (76)$$

holds, the expression

$$\begin{aligned} n_s &= \text{rank} [T^{-1}T_1B_a, T^{-1}A_sTT^{-1}T_1B_a, \dots, \\ &\quad (T^{-1}A_sT)^{n_s-1}T^{-1}T_1B_a] \\ &= \text{rank} \begin{bmatrix} I & A_{11} & A_{11}^2 + A_{12}A_{21} & * \\ 0 & A_{21} & A_{21}A_{11} + A_{22}A_{21} & * \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} I_m & A_{11} & \dots & * & * \\ * & A_{12} & \dots & A_{22}^{n_s-m-1}A_{21} & * \end{bmatrix} \end{aligned} \quad (77)$$

is also true, where * means any value. This implies that the pair (A_{22}, A_{21}) is controllable.

Appendix D: Proof that $Q_b(s)$ is a solution to the problem

Here it is shown that, in fact, the matrix $Q_b(s) = D_b + C_b(sI - A_b)^{-1}B_b$ turns the poles of $S(s)$ uncontrollable in $W_s(s)S(s)$. For this, let T_2 be

$$T_2 = \bar{T} \begin{bmatrix} F_b \\ 1 \end{bmatrix} \quad (78)$$

thus, from

$$\begin{aligned} A_sT_2 - T_2A_b &= \bar{T}[\bar{T}^{-1}A_s\bar{T}\bar{T}^{-1}T_2 - \bar{T}^{-1}T_2A_b] \\ &= \bar{T} \left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} F_b \\ I \end{bmatrix} - \begin{bmatrix} F_b \\ I \end{bmatrix} A_b \right\} \\ &= \bar{T} \begin{bmatrix} C_b \\ 0 \end{bmatrix} \end{aligned} \quad (79)$$

and

$$T_1B_aC_b = \bar{T}\bar{T}^{-1}T_1B_aC_b = \bar{T} \begin{bmatrix} C_b \\ 0 \end{bmatrix} \quad (80)$$

the equality $\hat{A}_{13} = 0$ holds. Also, from

$$\begin{aligned} T_2B_b + T_1B_aD_b &= \bar{T}[\bar{T}^{-1}T_2B_b + \bar{T}^{-1}T_1B_aD_b] \\ &= \bar{T} \left\{ \begin{bmatrix} F_b \\ I \end{bmatrix} B_b + \begin{bmatrix} I \\ 0 \end{bmatrix} D_b \right\} \\ &= \bar{T} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \bar{T}^{-1}[\bar{B}_s - T_3K] \\ &= \bar{B}_s - T_3K \end{aligned} \quad (81)$$

the expression $\hat{B}_1 = 0$ holds.

Furthermore, post-multiplication of the Eq.(81) by matrix C , and addition of this result to the expression (46) yield $\hat{A}_{14} = 0$. In other words, the poles of $W_s(s)$ are uncontrollable as expected.

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