Output Time Delayed Control System Design Subject to the Mixed Sensitivity Problem

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This paper presents a design methodology for MIMO output time delayed systems satisfying the mixed sensitivity specification. Nevertheless the procedure is on the basis of the state space approach, it provides a controller with tuneable parameters, which makes the technique useful for practical applications. The key features of the proposed method are its capability to handle multiple time delays and to allow the calculations to be performed as in the time delay free system case. A solution for output time delay unstable system is illustrated with a numerical example.

Key Words: Output Time Delayed Control System, Control System Design, Mixed Sensitivity Problem, Robust Control, Process Control, Multiple Time Delays

1. Introduction

The design of time delay systems to have disturbance rejection and robust stability characteristics has been known to be a very difficult problem for a long time[13,14]. On the basis of the \( H_\infty \) control, this problem has been formulated as the mixed sensitivity problem, and solutions have been proposed to solve it[15-17]. However, these investigations focused primarily either on the SISO systems or MIMO systems with single time delay component. Also, the expressions that appear along the calculations involve the time delay terms, which require, in many cases, numerical computations.

In this paper, it is proposed a design method for MIMO output time delay systems that (i) provides a solution to the mixed sensitivity problem, (ii) handles multiple output time delay terms, (iii) allows the majority of the calculations to be performed as in the delay free case, and (iv) presents a controller with a tuneable parameter that can be used to adjust the system performance characteristics during the process operation.

In fact, considering that many industrial processes have multiple time delay components[18] and they are required to have tuneable parameters to change the system performance during the operation[19], the proposed method can be useful to this kind of applications.

The paper is organized as follows. Section 2 describes the system in consideration, section 3 formulates the problem, section 4 provides the realization, section 5 summarizes the design procedure in an algorithm, and section 6 presents a numerical example.

2. Preliminaries

2.1 Plant and controller

Consider the output time delayed plant described by

\[
\begin{bmatrix}
    y_1(t-L) \\
    \vdots \\
    y_m(t-L_m)
\end{bmatrix} =
\begin{bmatrix}
    C_1 \\
    \vdots \\
    C_m
\end{bmatrix}
\begin{bmatrix}
    x(t) \\
    \vdots \\
    x(t)
\end{bmatrix}
\]

(1)

and let the controller be the generalised stabilizing controller[10] which is defined by the system of linear equations

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) - Kv(t) \\
\dot{z}(t) &= A_\delta z(t) + B_\delta v(t) \\
u(t) &= -Fz(t) + Q_\delta z(t) + Q_\delta D_\delta v(t) \\
v(t) &= z(t) - y(t) \\
e(t) &= r(t) - y(t)
\end{align*}
\]

where
**(a2)** the nominal model (1) has no purely imaginary zeroes, and has the same number of poles as the real plant.

**(a3)** Decoupling of \( G(s) \) is possible by means of the state feedback. In other words, let \( \nu_i (i = 1, \ldots, m) \) be positive integers satisfying for \( i = 1, \ldots, m \)

\[(i) \quad C_i A_i B = 0 \quad \text{for} \quad j = 1, 2, \ldots, \nu_i - 2 \]
\[(ii) \quad C_i A_i^{\nu_i - 1} B + 0\]

and if a matrix \( \Phi \) is defined as

\[
\Phi = \begin{bmatrix}
C_1 A_1^{\nu_1 - 1} B \\
\vdots \\
C_m A_m^{\nu_m - 1} B
\end{bmatrix}
\]

it implies

\[
\text{rank } \Phi = m.
\]

**(a4)** the control system in Fig. 1 has output multi-plicative model uncertainty given by

\[
W_r(s) = R(s) \begin{cases}
< 1, & 0 \leq \omega < \omega_r \\
= 1, & \omega = \omega_r \\
> 1, & \omega_r < \omega
\end{cases}
\]

and for some \( \omega_r \geq 0 \)

and the relative degree of \( 1/|W_r(s)| \) is less than or equal to the minimum \( \nu \) amongst \( \nu_1, \ldots, \nu_m \).

### 2.3 Some definitions

**(d1)** Let \( W_r(s) \in R(s) \) be

\[
\text{diag} \{ W_1(s), \ldots, W_m(s) \} = \text{diag} \{ C_1 A_1 - A_2 \}^{-1} B_2
\]

\[
A_3 = \text{diag} \{ A_1, \ldots, A_m \} \in \text{R}^{n \times n}
\]

\[
B_3 = \text{diag} \{ b_1, \ldots, b_m \} \in \text{R}^{n \times n}
\]

\[
C_3 = \text{diag} \{ c_1, \ldots, c_m \} \in \text{R}^{n \times n}
\]

\[
W_3(s) = C_3(sI - A_3)^{-1} B_3
\]

where \( I_n \) indicates the unitary matrix of size \( n \).

**(d2)** Define \( A(s) \) as

\[
A(s) = \text{diag} \left[ s^{\nu_1} + a_1 s^{\nu_1 - 1} + \cdots + a_{\nu_1} \right]
\]

\[
s^{\nu_1} + a_1 s^{\nu_1 - 1} + \cdots + a_{\nu_1} = 0,
\]

\[
v_1: \text{even}
\]

\[
v_2: \text{odd}
\]

\[
\frac{s^2 + 2\omega_s s + \omega_s^2}{\frac{1}{s^2}} = \frac{1}{s^2}
\]

\[
0 \leq \omega_s \leq \omega_r.
\]

**(d3)** Let \( \Phi \) and \( \Omega \) be

\[
\Phi = \begin{bmatrix}
C_1 A_1^{\nu_1} + a_1 C_1 A_1^{\nu_1 - 1} + \cdots + C_1 A_1 a_{\nu_1} \\
\vdots \\
C_m A_m^{\nu_m} + a_1 C_m A_m^{\nu_m - 1} + \cdots + C_m a_{\nu_m}
\end{bmatrix}
\]

\[
\Omega = \text{diag} \{ a_{\nu_1}, \ldots, a_{\nu_m} \}
\]

**(d4)** Let the non-singular matrix \( \tilde{T} \) be

\[
\tilde{T} = \begin{bmatrix}
I_n & T_1 & T_2 & T_3 \\
0 & I_n & 0 & 0 \\
0 & 0 & I_{(n-1)} & 0
\end{bmatrix}
\]

Fig. 1 Control system with time delays in the output.
3. Problem Statement

Assuming that the disturbances act on the output of the control system shown in Fig. 1, the output side sensitivity and complementary sensitivity functions are

\[
S(s) = I - \Gamma(s)G(s)P_d(s)P_c(s)P(s) \quad (18)
\]

\[
T(s) = \Gamma(s)G(s)P_d(s)P_c(s)P(s) \quad (19)
\]

respectively.

Under these considerations, the system is required to be internally stable\(^{111}\); i.e., all the entries of the transfer function from \(r(s)\) and \(d(s)\) to \(y(s)\) and \(u(s)\) be stable for a stabilizing \(P(s) = P_d(s)P_c(s)P(s)\) in the transfer function expressed by

\[
\begin{bmatrix}
    y \\
    u
\end{bmatrix} = \begin{bmatrix}
    I & G & P \\
    P & PFG & P \\
    P & PFG & r \\
    d & d & d
\end{bmatrix}.
\]

Moreover, the system has to satisfy the mixed sensitivity specification described by

\[
\begin{bmatrix}
    W_1(s)S(s) \\
    W_2(s)T(s)
\end{bmatrix} < 1.
\]

4. Realisation

The final purpose of this section is to show that \(P_d(s)\) and \(P_c(s)\) are primarily concerned with the control system design specification \(W_1(s)T(s)\) of Eq. (21), which means that \(P_d(s)\) and \(P_c(s)\) assure the stability of the feedback control system. In addition, it is shown that \(P_d(s)\) is mainly related to the specification \(W_1(s)S(s)\) of Eq. (21), which is established by canceling the purely imaginary poles of \(W_1(s)\) by means of the component \(Q_0(s)\) of \(P_c(s)\).

However, before presenting the result above, it is shown how each component \(P_d(s), P_c(s)\) and \(P(s)\) of the controller are realised. Note that the matrices \(F\) and \(Q_0\) in \(P_d(s)\), the matrix \(K\) in \(P(s)\) and the transfer function \(Q_0(s)\) in \(P(s)\) are realised in order to have the internal stability condition satisfied, which is equivalent to the stability of the transfer function expressed in Eq. (20).

4.1 Realisation of \(P_d(s)\)

In this section, it is demonstrated that \(P_d(s)\) in Eq. (8) can be pursued from the feedback decoupling\(^{110}\) of \(G(s)\). Moreover, it is shown that a such \(P_d(s)\) stabilizes \(\Gamma(s)G(s)P_d(s)\) in Eq. (20).

To accomplish it, perform the decoupling of \(G(s)\) by means of the state feedback\(^{110}\). Then

\[
C(sI - A + B\Phi^{-1}\Psi)^{-1}B\Phi^{-1}\Omega = A(s) \quad (22)
\]

is yielded. This can be rewritten as

\[
A^{-1}(s)C(sI - A)^{-1}B\Phi^{-1}\Omega
= I + \Omega^{-1}(sI - A)^{-1}B\Phi^{-1}\Omega. \quad (23)
\]

Since \(G(s)\) is assumed to be free of purely imaginary zeroes, the Riccati equation for the right side of the Eq. (23); i.e.,

\[
X(A - B\Phi^{-1}\Psi)^{-1}X + (A - B\Phi^{-1}\Psi)^{-1}B\Phi^{-1}\Omega + XB\Phi^{-1}\Omega^{-1}(sI - A)^{-1}B\Phi^{-1}X = 0 \quad (24)
\]

has a stabilizing solution \(X = X^T \geq 0\). If \(F\) and \(Q_0\) are defined as

\[
F = \Phi^{-1}\Omega[\Omega^{-1}F + \Omega(\Phi^{-1})^{-1}B\Phi^{-1}X] \quad (25)
\]

\[
Q_0 = \Phi^{-1}\Omega \quad (26)
\]

respectively. Now, the inner factorisation\(^{110}\) gives

\[
A^{-1}(s)C(sI - A)^{-1}B\Phi^{-1}\Omega = G(s)G(s)
\]

\[
G(s) = [I + (Q^{-1}F - Q_0^{-1}F)(sI - A + BF)^{-1}B\Phi^{-1}]
\]

\[
G(s) = [I + Q_0^{-1}F(sI - A)^{-1}B\Phi^{-1}]
\]

(27)

where \(G(s)\) is an inner matrix, and the matrix \(G(s)\) is a matrix with no unstable zeroes; \(G(s)\) is an outer matrix if the plant is stable.

Now, comparing Eq. (27) and (8), one can see that

\[
P_d(s) = Q_0G(s)
\]

\[
[I + F(sI - A)^{-1}B]^{-1}Q_0
\]

holds. From this expression, it is clear that \(P_d(s)\) is a stable matrix and also, it demonstrates the first assertion in the beginning of this subsection. Finally, it is clear from the expression

\[
\Gamma(s)G(s)P_d(s) = \Gamma(s)A(s)G_1(s)
\]

that \(P_d(s)\) stabilizes \(\Gamma(s)G(s)P_d(s)\) in Eq. (20), since \(\Gamma(s)\) is a matrix whose entries express the time delay components of the system, and \(A(s)\) and \(G_1(s)\) are stable matrices.

4.2 Realisation of \(P_c(s)\)

\(P_c(s)\) is realised in order to guarantee the stability of the term \(P_c(s)\Gamma(s)G(s)\) in Eq. (20). However, due to Eq. (64) and the fact that there occur cancellations of poles and zeroes in the finite Laplace transform operations\(^{110}\), the stabilisation of \(P_c(s)\Gamma(s)G(s)\) reduces to the stabilisation of \(P_c(s)C(sI - A)^{-1}B\Phi^{-1}\). Therefore, \(K\) has to be a stabilizing matrix of \(A - KC\).

To establish the desired matrix \(K\), consider the Riccati equation

\[
YA + YC - YBCY = 0 \quad (30)
\]

that has a stabilizing solution \(Y = Y^T \geq 0\), and define the matrix \(K\) as

\[
K = YC^T \quad (31)
\]

and the desired result is pursued\(^{110}\). Moreover, since the eigenvalues of \(A - KC\) are the stable as well as the mirror values with respect to the imaginary axis of the unstable eigenvalues of \(A, P_c(s)\) in Eq. (20) is stable.

4.3 Realisation of \(P_s(s)\)

The realisation of \(P_s(s)\) reduces to the realisation of \(Q_0(s)\), which is determined such that the terms \(P_d(s)P_c(s)\) and \(P_s(s)\Gamma(s)\) are stabilizing. Furthermore, it is required here that the purely imaginary zeroes of \(Q_0(s)\) cancels the purely imaginary poles of \(W(s)\) in \(W(s)S(s)\) of Eq. (21); i.e., \(Q_0(s)\) turns the unstable poles of \(W(s)\) uncontroll-
lable.

It is shown here that the transfer function \(Q_0(s)\) determined on the basis of the poles and zeroes canceling requirement presents the desired \(Q_0(s)\).

To establish the result above, recall that the sensitivity function can be briefly written from Eq. (18) and (19) as
\[
S(s) = I - T(s).
\]
(32)

Thus, the uncontrollability condition is equivalent to the poles of \(W(s)\) uncontrollable in
\[
\Gamma^{-1}(s)W(s)S(s) = W(s)[\Gamma^{-1}(s) - C(sI - A + BF)^{-1}B][Q_0Q_0(s)[I - C(sI - A + K\bar{C})^{-1}K] + F(sI - A + K\bar{C})^{-1}K].
\]
(33)

For the sensitivity weighting function given by equation
\[
W(s) = \frac{\rho}{s},
\]
(34)
\[
\rho \in R > 0
\]
(35)
it suffices to determine \(Q_0\) such that \(S(0) = 0\) holds; i.e.,
\[
Q_0 = Q_0^{-1}[C(-A + BF)^{-1}B]^{-1} - F(-A + K\bar{C})^{-1}K^{-1}I + C(sI - A + K\bar{C})^{-1}K-1.
\]
(36)

For the sensitivity weighting function \(W(s)\), \(W(s)\) is established in the following way. Assume without lost of generality that \(W(s)\) and \(G(s)\) are unequipped with common poles. Then
\[
\mathcal{L}[\mathcal{L}^{-1}(s)W(s)S(s)] = \mathcal{L}(sI - A)^{-1}B
\]
(37)
holds. Here \(\mathcal{L}[\bullet]\) means the Laplace transform of \(\bullet\), and
\[
\bar{A} = \begin{bmatrix}
A_2 & -B_2 \bar{C} & 0 & 0 \\
0 & A_r & B_2 \bar{C} & B(F - Q_0D_2\bar{C}) \\
0 & 0 & A_b & -B_2 \bar{C} \\
0 & 0 & 0 & A_k
\end{bmatrix},
\]
\[
\bar{B} = \begin{bmatrix}
B_2 \\
B_2D_2 \\
B_2b_2 \\
K
\end{bmatrix}
\]
\[
\bar{C} = \begin{bmatrix}
C_2, 0, 0, 0
\end{bmatrix},
\]
\[
\bar{B}_s = [e^{s\lambda_1}b_{s1}, \ldots, e^{s\lambda_n}b_{sn}].
\]

Next, use Eq. (17) to change the coordinates of the matrices \(\bar{A}\) and \(\bar{B}\) as
\[
\bar{T}^{-1}\bar{A}\bar{T} = \begin{bmatrix}
A_2 & A_{12} & A_{13} & A_{14} \\
A_F & A_2\bar{C} & B(F - Q_0D_2\bar{C}) \\
0 & A_b & -B_2\bar{C} \\
0 & 0 & 0 & A_k
\end{bmatrix},
\]
(39)
\[
\bar{T}^{-1}\bar{B} = \begin{bmatrix}
\bar{B}_2 \\
B_2D_2 \\
B_2b_2 \\
K
\end{bmatrix}
\]
(40)
respectively. Where
\[
\bar{A}_{12} = A_2T_1 - T_1A_2 - T_1A_F - B_2\bar{C}
\]
\[
\bar{A}_{13} = A_2T_1 - T_1A_2 - T_1B_2\bar{C}
\]
\[
\bar{A}_{14} = A_2T_1 - T_1A_2 - T_1B(F - Q_0D_2\bar{C}) - T_1B_2C
\]
\[
\bar{B}_2 = \bar{B}_2 - T_1B_2D_2 - T_1B_2b - T_1K.
\]
(41)

From these, it is clear that in order to make the poles of \(W(s)\) uncontrollable, it suffices to make sure that the matrices \(\bar{A}_{12}, \bar{A}_{13}, \bar{A}_{14}\), and \(\bar{B}_2\) are null matrices.

In fact, since the matrices \(A_2\) and \(A_F\) have no common eigenvalues, there exists a matrix \(T_1\) such that \(\bar{A}_{13} = 0\). Moreover, as shown in appendix B, \(T_1B_2 \in R_{(n_x,n_y)}\) is column full rank, so that it is always possible to define a non-singular matrix \(\bar{T}\) by defining an appropriate matrix \(M \in R_{(n_x,n_y)}\); i.e.,
\[
\bar{T} = [T_1B_2 M].
\]
(42)
and apply it on \(\bar{A}_2\) to change the coordinates of \(\bar{A}_2\) as
\[
T^{-1}\bar{A}_2 T = \begin{bmatrix}
A_1 & A_{12} \\
A_{13} & A_{14}
\end{bmatrix}
\]
(43)

Note that the pair \((A_{21}, A_{22})\) obtained from this factorisation is controllable (appendix C). If \(A_s\) is defined as
\[
A_s = A_{22} + A_{21}F_0
\]
(44)
there exists a matrix \(F_0 \in R_{(n_y,n_y)}\) that places the poles of \(A_s\) at \(-\omega_c\). Also, Let the matrix \(C_s\) be
\[
C_s = A_{21}F_0 - F_0A_{22} + A_{23}.
\]
(45)
Since \(A_s\) and \(A\) have no common eigenvalues, there exists a matrix \(T\) such that the expression
\[
A_sT_3 - T_3A = T_1BF - B\bar{C}
\]
(46)
holds. Thus define \(B_3\) and \(D_3\) as
\[
B_3 = [I 0] T^{-1}(T_3 - T_3K)
\]
(47)
\[
D_3 = [I 0] T^{-1}(T_3 - T_3K) - F_3B_3
\]
(48)
respectively. As shown in appendix D, a such \(Q_k(s) = D_3 + C_k(sI - A_k)^{-1}B_k\) makes the unstable poles of \(W(s)\) uncontrollable as desired. It is useful pointing out that due to the matrix \(F_0\) as defined in Eq. (44), \(Q_k(s)\) is stable.

Now, it remains to show that \(P(s)\) indeed guarantees that \(P_0(s)P_0(s)\) and \(P(s) - P(s)G(s)P_0(s)P_0(s)\) are stable. In fact,
\[
P_0(s)P_0(s) = [I - F(sI - A + BF)^{-1}B]Q(s)
\]
\[
+ F(sI - A + BF)^{-1}K
\]
\[
P_0(s) - P(s)G(s)P_0(s)P_0(s) = I + C(sI - A) - B\bar{C}
\]
\[
+ BF^{-1}K - BQ(s) + IF(s)P_0(s)P_0(s)
\]
(49)
are stable by means of the term \(A - BF\).

4.4 Realisation of the controller \(G_c(s)\)

It is shown here that the controller \(G_c(s)\) of Eq. (7) can provide a solution to the problem stated in section 3. First, let us check the internal stability condition. In fact, considering the \(P_0(s), P_0(s)\) and \(P(s)\) as in the previous sections, it is clear that \(P(s) = P_0(s)P_0(s)\) is also the entry (2, 1) of the transfer function in Eq. (20), is stable. From this, it is
easy to see that \( P(s)\Gamma(s)G(s) = \bar{P}_s(s)P_s(s)\bar{P}_s(s)\Gamma(s) \times G(s) \) (entry (2,2)) is a stable matrix. Also, since \( P_\infty(s) \) has no unstable zeroes, \( \Gamma(s)G(s)P(s) \) (entry (1,1)) is stable. Moreover, due to the stability of \( P_s^{-1}(s) = [I & -I\Gamma(s)G(s)P(s)]P_s^{-1}(s) \), \( [I & -I\Gamma(s)G(s)P(s)]P(s) = [P_s^{-1}(s) - \Gamma(s)G(s)P(s)]P(s) = P_s^{-1}(s) \Gamma(s)G(s) \) (entry (1,2)) is stable. Consequently, the overall feedback control system is internally stable as claimed before.

Now, note that by substituting Eq. (29) into Eqs. (18) and (19), the sensitivity and the complementary sensitivity functions can be rewritten as

\[
S(s) = I - \Gamma(s)A(s)G(s)P(s)P(s) \tag{50}
\]

\[
T(s) = A(s)G(s)\Gamma(s)P(s)P(s) \tag{51}
\]

where \( \sigma[\Gamma(j\omega)] = 1 \) and \( \sigma[G(j\omega)] = 1 \) hold. Further on, from Eq. (8) and \( K \) as in section 4.2, \( \sigma[P(j\omega)] = 1 \) holds.

Thus, substitution of Eq. (50) and (51) into Eq. (21) and the fact that the influence of \( \sigma[P(j\omega)] \) is defined by \( W_s(s)S(s) \) lead to the conclusion that the control system design specification (21) depends on the existence of \( \omega_s \) in Eq. (14). Hence, the assertion that \( G_c(s) \) as defined here can indeed provide a solution to the problem is proved. In the following section it is given an algorithm that summarizes the control system design procedure.

5. Design Algorithm

The design of the control system is summarized in the algorithm below.

**Step 1**: Solve the Riccati Eq. (24), and calculate \( F \) in Eq. (25) as well as \( Q_e \) in Eq. (26).

**Step 2**: Determine \( K \) in Eq. (31) by solving the Riccati equation given by Eq. (30).

**Step 3**: Compute the matrix \( Q_e \) from either the Eq. (36) or Eq. (44), (45), (47) and (48).

**Step 4**: Fix \( \zeta = 1/\sqrt{2} \) (see discussion in Ref. (17)) and let the initial value of \( \omega_s \) be \( \omega_s = \omega_r \).

**Step 5**: Calculate \( S(s) \) and \( T(s) \) in Eqs. (18) and (19). Plot \( \sigma[S(j\omega)], \sigma[T(j\omega)], 1/|W_s(j\omega)| \) and \( 1/|W_r(j\omega)| \). On this plot, determine the frequency range for \( \omega_s \) in which both \( \sigma[S(j\omega)] < 1/|W_s(j\omega)| \), and \( \sigma[T(j\omega)] < 1/|W_r(j\omega)| \) hold. If there is no that fulfills both the conditions, a solution by this procedure cannot be pursued. If a range is figured out, draw the gain plot of expression (21), and decrease \( \omega_s \) from the maximum value in the range until the specification is met. If no \( \omega_s \) in the range leads to the desired result, a solution by this procedure is not found. An \( \omega_s \) satisfying the demand is a solution to the problem.

6. Numerical Example

In this section, a solution to an unstable output time delayed plant is presented. Let \( \Gamma(s)G(s), W_s(s) \) and \( W_r(s) \) be given by

\[
\begin{bmatrix}
(s+1)e^{-s} \\
(s-1)(s-1-j) \\
e^{0.5s} \\
(s+4) \\
(s+2)(s+1)
\end{bmatrix}
\]

\[
\begin{bmatrix}
2(s-1)e^{-s} \\
(s-1)(s+3) \\
(s-1)e^{0.5s} \\
(s+3)
\end{bmatrix}
\]

\[
\begin{bmatrix}
(s+3) \\
100
\end{bmatrix}
\]

respectively.

Letting all the poles of \( A \) be placed at \( -\omega_s \) on the occasion of the realisation of \( P_\infty(s) \), it is possible to determine the controller \( G_c(s) \), which is a solution to the problem proposed in section 3, as long as the parameter \( \omega_s \) in Eq. (14) is in the frequency range defined by \( 1.10 \text{ rad/s} < \omega_s < 2.45 \text{ rad/s} \). Note that according to the definitions of sensitivity and complementary sensitivity functions \( \omega_s \), close to \( 1.10 \text{ rad/s} \) means that the system is quite stable, but presents sluggish step response characteristic. On the other hand, \( \omega_s \) close to \( 2.45 \text{ rad/s} \) leads to a system on the verge of instability, but with fast step response characteristic. Fig. 2 shows the plot of the mixed sensitivity specification result for \( \omega_s = 2.44 \) rad/s. For this case, the execution of the algorithm leads to the following result:

\[
F = \begin{bmatrix}
9.000 \\
-3.546 \\
-5.511 \\
1.431
\end{bmatrix} \begin{bmatrix}
43.968 \\
-15.867 \\
-48.071 \\
7.675
\end{bmatrix} \begin{bmatrix}
46.678 \\
-21.272 \\
-24.556 \\
12.600
\end{bmatrix} \begin{bmatrix}
11.711 \\
-8.951 \\
-24.556 \\
6.360
\end{bmatrix}
\]

\[
Q_e = \begin{bmatrix}
-2.440 \\
2.440 \\
4.880 \\
-2.440
\end{bmatrix}
\]

![Fig. 2 Solution to mixed sensitivity problem](image-url)
Finally, the step response of the feedback control system accomplished by means of the controller above is shown in Fig. 3.

Appendix A : Outline of the calculation of $G_c(s)$

Equations (1) - (5) yield

\[
G_c(s) = \left[I + (F - Q(s)\overline{C})(sI - A + K\overline{C})^{-1}
\times (B + Ks I(s)) + Q(s)\Pi(s)\right]^{-1}[Q(s) + (F - Q(s)\overline{C})(sI - A + K\overline{C})^{-1}K] \tag{61}
\]

where $\Pi(s)$ given by

\[
\Pi(s) = \left[\int_{-\infty}^{0} C_v e^{(s)\overline{A}(r)}B_v \, dr \right]
\]

is the finite Laplace transform of $\pi(t)$. Here note that

\[
\int_{-\infty}^{0} C_v e^{(s)\overline{A}(r)}B_v \, dr = \int_{0}^{\infty} C_v e^{(s)\overline{A}(r)}B_v \, dr
\]

\[
= C_v e^{(s)\overline{A}(0)}B_v = C_v e^{(s)\overline{A}(sI - A)^{-1}B}
\]

\[
= C_v e^{(s)\overline{A}(-sI - A)^{-1}B} = C_v e^{(s)\overline{A}(sI - A)^{-1}B}
\]

is equivalent to the equation

\[
\Pi(s)C\overline{B}(sI - A)^{-1}B = \overline{C}(sI - A)^{-1}B - II(s). \tag{64}
\]

Now, substituting $II(s)$ of Eq.(64) into Eq.(61)

and making some algebraic manipulations, $G_c(s)$ rewritten as in Eq.(7) is established.

Appendix B : Column full-rankness of $T_cB_c$

To show that $T_cB_c$ is column full-rank, write $A_c$ in the Jordan form; i.e.,

\[
T_c^{-1}A_cT_c = \text{diag} [J_{i_1}, \cdots, J_{i_n}] \tag{65}
\]

where $J_i$ stands for

\[
J_i = \text{diag} [J_{i_{k_1}}, \cdots, J_{i_{k_m}}] \tag{66}
\]

and $J_{i_k}$ is a matrix composed by eigenvalues $p_{i_k}$

\[
J_{i_k} = \begin{bmatrix}
0 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}
\]

\[
And T_c^{-1}B_c = \begin{bmatrix}
B_{i_1} \\
\vdots \\
B_{i_m}
\end{bmatrix} \tag{68}
\]

where $B_{i_l}$ means

\[
B_{i_l} = \begin{bmatrix}
B_{i_{l1}} \\
\vdots \\
B_{i_{ln}}
\end{bmatrix} \tag{70}
\]

with $l$ indicating either the last block or the last row of the matrix. Since $(A_c, B_c)$ is controllable, the matrix

\[
\tilde{B}_c = \begin{bmatrix}
\tilde{B}_{c1} \\
\vdots \\
\tilde{B}_{cm}
\end{bmatrix} \tag{71}
\]

is non-singular.

Now, pre-multiplication of $T_c^{-1}$ with

\[
A_cT_c - T_cA_c = B_cC \tag{72}
\]

yields

\[
T_c^{-1}A_cT_c - T_c^{-1}T_cA_c = T_c^{-1}B_cC \tag{73}
\]

and the extraction of the last row from the block $J_{i_l}$ gives

\[
(T_c^{-1}T_c)(p_{i_l} - A_{i_l}) = \tilde{B}_cC \tag{74}
\]

where $(T_c^{-1}T_c)$, is a matrix composed by the row corresponding to $T_c^{-1}T_c$.

Thus,

\[
(T_c^{-1}T_c)B_c = \tilde{B}_cC(p_{i_l} - A_{i_l})^{-1}B \tag{75}
\]

is pursued. Since the right side of the expression (75) is row full-rank, $(T_c^{-1}T_c)B_c$ is row full-rank, and the pair $(T_c^{-1}A_cT_c, T_c^{-1}T_cB_c)$ is controllable. Adding to these facts the non-singularity of $Q_a$, leads to the conclusion that the pair $(A_c, T_cB_c)$ is controllable.

Appendix C : Controllability of the pair $(A_{22}, A_{21})$

As shown in appendix B, the pair $(A_{22}, T_cB_{2})$ is controllable. Hence, the pair $(T_c^{-1}A_{22}, T_c^{-1}T_cB_{2})$ is controllable. Furthermore, since
\[ T^{-1}T_{1}B_{0} = \begin{bmatrix} I \\ 0 \end{bmatrix} \]  \hspace{1cm} (76) \\
holds, the expression 
\[
\begin{aligned}
\nu_{n} &= \text{rank} \left[ \begin{array}{cc}
T^{-1}T_{1}B_{0}, & T^{-1}A_{2}TT^{-1}T_{1}B_{0}, \ldots,
\end{array}
\right] \\
&= \text{rank} \left[ \begin{array}{cc}
I & A_{11}^{\dagger}A_{12}\!A_{22} \\
0 & A_{21}^{\dagger}A_{11}\!A_{12}\!A_{22} \\
\end{array}
\right] \\
&= \text{rank} \left[ \begin{array}{cc}
I & A_{11}^{\dagger} \\
0 & A_{12}^{\dagger} \ldots A_{2}^{\dagger-n_{1}}A_{11}^{\dagger} \\
\end{array}
\right] \\
& \text{is also true, where } \ast \text{ means any value. This implies that }
\text{the pair } (A_{2n}, A_{2n}) \text{ is controllable.}
\end{aligned}
\]  \hspace{1cm} (77)

Appendix D: Proof that \( Q_{0}(s) \) is a solution to the problem

Here it is shown that, in fact, the matrix 
\[ Q_{0}(s) = D_{0} + C_{0}(sI - A_{0})^{-1}B_{0} \]
turns the poles of \( S(s) \) uncontrollable in \( W_{0}(s)S(s) \). For this, let \( T_{z} \) be 
\[ T_{z} = \begin{bmatrix} F_{b} \\ I \end{bmatrix} \]  \hspace{1cm} (78) \\
thus, from 
\[ A_{5}T_{z} - T_{z}A_{5} = T_{z}[T^{-1}A_{5}TT^{-1}T_{z} - T^{-1}T_{z}A_{5}] \\
= T_{z}\begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{32}
\end{bmatrix} \begin{bmatrix} F_{b} \\ I \end{bmatrix} - \begin{bmatrix} F_{b} \\ I \end{bmatrix} A_{5} \\
= T_{z}\begin{bmatrix} C_{0} \\ 0 \end{bmatrix} \]  \hspace{1cm} (79) \\
and 
\[ T_{1}B_{a}C_{0} = TT^{-1}T_{1}B_{a}C_{0} = T_{z}\begin{bmatrix} C_{0} \\ 0 \end{bmatrix} \]  \hspace{1cm} (80)
the equality \( \bar{A}_{1n} = 0 \) holds. Also, from 
\[ T_{1}B_{a} + T_{1}B_{a}D_{a} = T_{z}\begin{bmatrix} F_{b} \\ I \end{bmatrix} B_{a} + \begin{bmatrix} I \\ 0 \end{bmatrix} D_{a} \]
\[ = T_{z}\begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} F_{b} \\ I \end{bmatrix} \]
\[ = T_{z}\begin{bmatrix} I \\ 0 \end{bmatrix} TT^{-1}[\bar{B}_{b} - T_{3}K] \\
= B_{5} - T_{4}K \]  \hspace{1cm} (81)
the expression \( B_{5} = 0 \) holds.

Furthermore, post-multiplication of the Eq.(81) by matrix \( C \), and addition of this result to the expression (46) yield \( \bar{A}_{1n} = 0 \). In other words, the poles of \( W_{0}(s) \) are uncontrollable as expected.

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