Robust Sliding-Mode Control with Disturbance Attenuation Using Only Output Feedback*

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For a linear system with mismatched disturbance, a robust sliding mode control, using output feedback only, is developed in this paper. Although the system has an uncertain term, a novel state estimator is used which, using only measured outputs, can asymptotically estimate the system states. A control law is then designed with respect to the estimated signals. In addition, based on the H_∞ theory, the designed sliding surface can achieve robust stabilization and guarantee disturbance attenuation during sliding mode. Finally, a numerical example is demonstrated for showing the applicability of the proposed scheme.

Key Words: Sliding Mode Control, Mismatched Disturbance, State Estimator, H_∞ Approach

1. Introduction

Earlier results for the sliding mode control (SMC), which is a useful tool for dealing with system uncertainty and external disturbance, were based on the assumption that all the system states are available⁽¹⁾⁻⁽⁴⁾. However, controller designers often encounter systems for which only output information is obtainable. To deal with such situation, several output feedback sliding mode control (OFSMC) algorithms have recently been proposed⁽⁵⁾⁻⁽⁹⁾. In these papers, the uncertain terms must satisfy the so-called matching condition⁽²⁾. In fact, however mismatched uncertainty or disturbance exists in many practical systems. When a SMC system suffers from mismatched disturbance, a component of the disturbance will affect system performance while in sliding mode⁽⁴⁾. Reference (4) considered only the state feedback SMC for systems with mismatched uncertain terms. In this paper, we will study the OFSMC problem and replace the matching condition.

Recently, Young and Ozguner⁽¹⁰⁾ proposed a new sliding surface design with frequency shaping. More-

over, Nonami and Hirasawa⁽¹¹⁾ designed the sliding surface using H_∞ control theory. The main feature in their method is that the compensator is combined to design the sliding surface. In this paper, we apply the integral sliding mode⁽¹²⁾ and H_∞ control theory to design the control algorithm where it is determined by solving the so-called algebraic Riccati equation arising in H_∞ control^{(13),(14)}. However, the system states are included in the sliding surface. Since the system states are not available and the system is subject to an unknown disturbance, robust state estimation is needed. Such a problem is known in the literature as one of robust fault detection or unknown input observers⁽¹⁵⁾⁻⁽¹⁷⁾. For a system with unknown inputs, Yang and Wilde⁽¹⁵⁾ gave a design procedure of full-order observer, and Hou and Müller⁽¹⁶⁾ showed a algorithm of reduced-order observer. In this paper, we use a practical transformation matrix method to design another form of reduced-order observer, which can effectively estimate the true states. Using the estimated system states, the controller is then designed to satisfy the reaching and sliding condition. Note that the estimator and controller design can be taken as two separate steps.

In the next section, a class of systems to be controlled is first introduced with some important assumptions related to the system matrices. Section 3 describes the state estimator with it's a novel transfor-

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mation matrix. The sliding mode controller including the sliding surface design and the control algorithm is proposed Section 4. To demonstrate the developed controller, a numeric example is shown in Section 5. Finally, Section 6 gives the concluding remarks.

2. Problem Formulation

Consider a linear MIMO system expressed by the following state-space equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}\mathbf{d}(t) \quad (1.a)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \quad (1.b)$$

where $\mathbf{x} \in \mathcal{R}^n$ are states of the system, $\mathbf{u} \in \mathcal{R}^m$ are control inputs, $\mathbf{y} \in \mathcal{R}^p$ are measurable outputs of the system. The disturbance $\mathbf{d}(t) \in \mathcal{R}^p$ is defined in the L_2 -norm space, i.e., $\mathbf{d} \in L_2$, and the system (1) is controllable and observable. In addition, the system matrices satisfy $\mathbf{A} \in \mathcal{R}^{n \times n}$, $\mathbf{B} \in \mathcal{R}^{n \times m}$, $\mathbf{C} \in \mathcal{R}^{p \times n}$, and $\mathbf{G} \in \mathcal{R}^{n \times p}$, with $\text{rank}(\mathbf{B}) = m$, $\text{rank}(\mathbf{C}) = l$ and $\text{rank}(\mathbf{G}) = p$. For the plant (1), we make the following assumptions:

(A1) The disturbance term $\mathbf{d}(t)$ satisfies

$$\|\mathbf{d}(t)\| \leq \beta \quad (2)$$

where β is a known constant and $\|\cdot\|$ denotes the Euclidian norm of $\cdot$.

(A2) $\text{rank}(\mathbf{C}\mathbf{G}) = \text{rank}(\mathbf{G}) = p$.

(A3) The invariant zeros of $(\mathbf{C}, \mathbf{A}, \mathbf{G})$ must lie in the left-half complex plane.

Assumption (A1) is standard in sliding mode controller design⁽¹⁾⁻⁽⁹⁾. In addition, assumptions (A2) and (A3) are necessary to design the state estimator⁽¹⁵⁾⁻⁽¹⁷⁾.

The problem of robust disturbance attenuation is design a controller such that there exists $0 \leq \gamma < \infty$ for which the performance bound⁽¹⁴⁾

$$\int_0^t (\mathbf{y}^T \mathbf{y} + \mathbf{u}^T \mathbf{W}_u \mathbf{u}) dt \leq \gamma^2 \int_0^t (\mathbf{d}^T \mathbf{W}_d \mathbf{d}) dt \quad \forall t \geq 0 \quad (3)$$

is satisfied for all $\mathbf{d}$, where the weighting matrices are assumed to be symmetric and satisfy $\mathbf{W}_u > 0$ and $\mathbf{W}_d > 0$. In this paper, we combine the sliding-mode control and the H_∞ theory with using output feedback such that the robust disturbance attenuation can be achieved.

3. State Estimator

Consider the plant given in Eq. (1). The standard Luenberger observer is not suitable for this system due to the unknown input term $\mathbf{d}(t)$. Robust estimation of states in the presence of unknown input is discussed in many papers⁽¹⁵⁾⁻⁽¹⁷⁾. In this section, we shall propose a practical state estimator method that uses the transformation method such that $\hat{\mathbf{x}}(t) \rightarrow \mathbf{x}(t)$ for $t \rightarrow \infty$ where $\hat{\mathbf{x}}(t)$ is the estimated states.

Lemma 1⁽⁵⁾: If assumptions (A2) and (A3) are made,

then a matrix $\mathbf{F} \in \mathcal{R}^{p \times l}$ could be found such that the $n-p$ non-zero eigenvalues for $\mathbf{A} - \mathbf{G}(\mathbf{FCG})^{-1}\mathbf{FCA}$ are located in the left-half complex plane. Moreover, Refs. (4), (5), and (7) provide methods for selecting the matrix $\mathbf{F}$.

Lemma 2: The eigenvalues for two matrices $\mathbf{A} - \mathbf{G}(\mathbf{FCG})^{-1}\mathbf{FCA}$ and $\mathbf{A} - \mathbf{AG}(\mathbf{FCG})^{-1}\mathbf{FC}$ are the same.

Proof: By calculations, we can obtain the following equations:

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \frac{1}{\lambda}\mathbf{FC} & \mathbf{I}_p \end{bmatrix} \begin{bmatrix} \lambda\mathbf{I}_n - \mathbf{A} & \mathbf{G} \\ -\mathbf{FC} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ (\mathbf{FCG})^{-1}\mathbf{FCA} & \lambda\mathbf{I}_p \end{bmatrix} \\ = \begin{bmatrix} \lambda\mathbf{I}_n - \mathbf{A} + \mathbf{G}(\mathbf{FCG})^{-1}\mathbf{FCA} & \lambda\mathbf{G} \\ \mathbf{0} & \mathbf{FCG} \end{bmatrix} \quad (4)$$

and

$$\begin{bmatrix} \mathbf{I}_n & -\mathbf{AG}(\mathbf{FCG})^{-1} \\ \mathbf{0} & \lambda\mathbf{I}_p \end{bmatrix} \begin{bmatrix} \lambda\mathbf{I}_n - \mathbf{A} & \mathbf{G} \\ -\mathbf{FC} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & -\frac{1}{\lambda}\mathbf{G} \\ \mathbf{0} & \mathbf{I}_p \end{bmatrix} \\ = \begin{bmatrix} \lambda\mathbf{I}_n - \mathbf{A} + \mathbf{AG}(\mathbf{FCG})^{-1}\mathbf{FC} & \mathbf{0} \\ -\lambda\mathbf{FC} & \mathbf{FCG} \end{bmatrix} \quad (5)$$

Interestingly, if the determinant on both sides from the above Eqs. (4) and (5) is calculated, then it follows that

$$\det(\lambda\mathbf{I} - \mathbf{A} + \mathbf{G}(\mathbf{FCG})^{-1}\mathbf{FCA}) \\ = \det(\lambda\mathbf{I} - \mathbf{A} + \mathbf{AG}(\mathbf{FCG})^{-1}\mathbf{FC})$$

Hence, we can conclude that the eigenvalues for two matrices $\mathbf{A} - \mathbf{G}(\mathbf{FCG})^{-1}\mathbf{FCA}$ and $\mathbf{A} - \mathbf{AG}(\mathbf{FCG})^{-1}\mathbf{FC}$ are the same. $\square$

From lemma 1, and lemma 2, we know $\mathbf{A} - \mathbf{AG}(\mathbf{FCG})^{-1}\mathbf{FC}$ has $n-p$ non-zero eigenvalues λ_i , $i = 1, 2, \dots, n-p$, satisfying $\text{Re}\{\lambda_i\} < 0$. Since $(\mathbf{A} - \mathbf{AG}(\mathbf{FCG})^{-1}\mathbf{FC})\mathbf{G} = \mathbf{0}$ and $\mathbf{G}$ is full rank, the eigenstructure for $\mathbf{A} - \mathbf{AG}(\mathbf{FCG})^{-1}\mathbf{FC}$ can be written as.

$$[\mathbf{A} - \mathbf{AG}(\mathbf{FCG})^{-1}\mathbf{FC}][\mathbf{W} \quad \mathbf{G}] \\ = [\mathbf{W} \quad \mathbf{G}] \begin{bmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (6)$$

or

$$(\mathbf{A} - \mathbf{AG}(\mathbf{FCG})^{-1}\mathbf{FC})\mathbf{W} = \mathbf{W}\mathbf{J} \\ \text{and } (\mathbf{A} - \mathbf{AG}(\mathbf{FCG})^{-1}\mathbf{FC})\mathbf{G} = \mathbf{0} \quad (7)$$

where $\mathbf{J} \in \mathcal{C}^{(n-p) \times (n-p)}$ is the Jordan form for the eigenvalues $\{\lambda_1, \dots, \lambda_{n-p}\}$ and $\mathbf{W} \in \mathcal{C}^{n \times (n-p)}$, being the corresponding eigenvector matrix, is of full rank. Note that the matrix $[\mathbf{W} \quad \mathbf{G}]$ is invertible.

Lemma 3⁽¹²⁾: The matrix $\mathbf{W}$ can be decomposed into:

$$\mathbf{W} = \mathbf{U}\mathbf{F} \quad (8)$$

where $\mathbf{U} \in \mathcal{R}^{n \times (n-p)}$ is full rank and $\mathbf{F} \in \mathcal{C}^{(n-p) \times (n-p)}$ is invertible.

Since $[\mathbf{W} \quad \mathbf{G}]$ is invertible, it follows from lemma 3 that $[\mathbf{U} \quad \mathbf{G}]$ is a nonsingular matrix. Let

$$\begin{bmatrix} \mathbf{U}^g \\ \mathbf{G}^g \end{bmatrix} = [\mathbf{U} \quad \mathbf{G}]^{-1} \quad (9)$$

where $\mathbf{G}^g \in \mathcal{R}^{p \times n}$ and $\mathbf{U}^g \in \mathcal{R}^{(n-p) \times n}$ can be taken as the generalized inverse of $\mathbf{G}$ and $\mathbf{U}$. Clearly,

$$\begin{aligned} G^g G &= I_p, & U^g U &= I_{n-p} \\ U^g G &= 0, & G^g U &= 0 \end{aligned} \quad (10)$$

and

$$UU^g + GG^g = I_n \quad (11)$$

From Eqs. (7), (8), and (10), it is easy to verify that

$$U^g(A - AG(FCG)^{-1}FC)U = \Gamma J \Gamma^{-1} \quad (12)$$

Post-multiplying Eq. (12) by U^g and using Eq. (11), we obtain

$$U^g(A - AG(FCG)^{-1}FC) = \Gamma J \Gamma^{-1} U^g = R U^g \quad (13)$$

where $R = \Gamma J \Gamma^{-1} \in \mathcal{H}^{(n-p) \times (n-p)}$ is a stable matrix with eigenvalues $\{\lambda_1, \dots, \lambda_{n-p}\}$. Having established the above facts, we are ready to introduce a state estimator. From Eqs. (10) and (11), it is easy to check

$$\begin{aligned} & \begin{bmatrix} (FCG)^{-1}FC \\ U^g \end{bmatrix} [G(I - G(FCG)^{-1}FC)U] \\ &= \begin{bmatrix} I_p & 0 \\ 0 & I_{n-p} \end{bmatrix} = I_n \end{aligned} \quad (14)$$

Now, define the system transformation as

$$\begin{aligned} z &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} (FCG)^{-1}FC \\ U^g \end{bmatrix} x \\ &= \begin{bmatrix} (FCG)^{-1}Fy \\ U^g x \end{bmatrix} \end{aligned} \quad (15)$$

and

$$\begin{aligned} x &= Gz_1 + (I - G(FCG)^{-1}FC)Uz_2 \\ &= G(FCG)^{-1}Fy + Mz_2 \end{aligned} \quad (16)$$

where $z_1 = (FCG)^{-1}FCx \in \mathcal{H}^p$, $z_2 = U^g x \in \mathcal{H}^{n-p}$, and $M = (I - G(FCG)^{-1}FC)U \in \mathcal{H}^{n \times (n-p)}$. Evidently, in order to estimate x , it is required to set up a reduced-order observer for z_2 from Eq. (16) only. Pre-multiplying Eq. (1) by U^g and using Eq. (13), we have

$$\begin{aligned} \dot{z}_2 &= U^g Ax + U^g Bu + U^g Gd \\ &= U^g(A - AG(FCG)^{-1}FC)x \\ &\quad + U^g AG(FCG)^{-1}Fy + U^g Bu \\ &= Rz_2 + U^g AG(FCG)^{-1}Fy + U^g Bu \end{aligned} \quad (17)$$

Now, the reduced-order observer can be constructed as

$$\dot{\hat{z}}_2 = R\hat{z}_2 + U^g AG(FCG)^{-1}Fy + U^g Bu \quad (18)$$

where $\hat{z}_2$ is the estimate of z_2 . Define the estimation error as $e_z = z_2 - \hat{z}_2$ and, from Eqs. (17) and (18), it leads to

$$\dot{e}_z = Re_z \quad (19)$$

Since R is a stable matrix, we can conclude that $e_z(t) = z_2(t) - \hat{z}_2(t) \rightarrow 0$ as $t \rightarrow \infty$. This means that the estimated states $\hat{z}_2$ successfully approach the true states z_2 . Finally, define the estimate of x as

$$\hat{x} = G(FCG)^{-1}Fy + M\hat{z}_2 \quad (20)$$

where $\hat{x} \in \mathcal{H}^n$. From Eqs. (16) and (20), we have

$$x - \hat{x} = Me_z \quad (21)$$

Since $e_z(t) \rightarrow 0$ as $t \rightarrow \infty$ is guaranteed, it follows from Eq. (21) that $\hat{x}(t) \rightarrow x(t)$ as $t \rightarrow \infty$.

4. Sliding-Mode Controller Design

First, from Eqs. (1) and (21), the system

dynamics for $\hat{x}$ can be obtained as

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + Gd + (AM - MR)e_z \\ &= A\hat{x} + Bu + G(d + Ee_z) \end{aligned} \quad (22)$$

where $E = -(FCG)^{-1}FCUR \in \mathcal{H}^{p \times (n-p)}$. Define the sliding surface as follows:

$$s = B^+ \hat{x} - \int_0^t (B^+ A \hat{x} + v) dt \quad (23)$$

where $B^+ = (B^T B)^{-1} B^T \in \mathcal{H}^{m \times n}$ is the pseudo-inverse for B and $v \in \mathcal{H}^m$ is a term to be designed. Differentiating Eq. (23) with respect to time and using Eq. (22), it can follow

$$\dot{s} = u - v + B^+ Gd + B^+ GEe_z \quad (24)$$

When the system is in the sliding motion, $s = \dot{s} = 0$, the so-called equivalent control⁽¹⁾ $u_{eq}|_{s=\dot{s}=0}$ can be obtained as

$$u_{eq} = -B^+ Gd + v - B^+ GEe_z \quad (25)$$

Substituting it into Eq. (22) obtains the system dynamics for $\hat{x}$ in the sliding motion as

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bv + (I - BB^+)Gd + (I - BB^+)GEe_z \\ &= A\hat{x} + Bv + G_1 d + G_2 e_z \end{aligned} \quad (26.a)$$

$$y = [C \quad CM] \begin{bmatrix} \hat{x} \\ e_z \end{bmatrix} \quad (26.b)$$

where $G_1 = (I - BB^+)G$ and $G_2 = (I - BB^+)GE$. Our objective is to design v such that the robust disturbance attenuation can be achieved. Hence, for the system (26), the robust disturbance attenuation is modified as

$$\int_0^t (y^T y + v^T W_v v) dt \leq \gamma^2 \int_0^t (d^T W_d d) dt \quad (27)$$

where $\gamma \geq 0$, $W_v > 0$, and $W_d > 0$. The main approach employed here is the standard Hamilton-Jacobi-Isaac method⁽¹⁴⁾. First, we define a quadratic energy function

$$E(\hat{x}, e_z) = \hat{x}^T P_x \hat{x} + e_z^T P_z e_z \quad (28)$$

where $P_x > 0$ and $P_z > 0$ are to be determined. In addition, define the Hamiltonian function

$$H(v, d) = y^T y - \gamma^2 d^T W_d d + v^T W_v v + \frac{dE}{dt} \quad (29)$$

It is known that a sufficient condition for achieving robust disturbance attenuation Eq. (27) is that⁽¹⁴⁾

$$H(v, d) < 0 \quad \text{for all } d \in L_2 \quad (30)$$

We will use the standard method of Riccati inequalities, which have been used in linear H_∞ control, to establish conditions under which

$$\inf_v \sup_{d \in L_2} H(v, d) < 0 \quad (31)$$

Theorem: Consider the following systems

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bv + G_1 d + G_2 e_z \\ \dot{e}_z &= Re_z \\ y &= [C \quad CM] \begin{bmatrix} \hat{x} \\ e_z \end{bmatrix} \end{aligned} \quad (32)$$

If there exist two positive definite solutions $P_x > 0$ and $P_z > 0$ to the following Riccati equations

$$A^T P_x + P_x A + 2C^T C + \frac{1}{\gamma^2} P_x G_1 W_d^{-1} G_1^T P_x$$

$$+P_x G_2 G_2^T P_x - P_x B W_v^{-1} B^T P_x < 0 \quad (33)$$

$$R^T P_z + P_z R + 2M^T C^T C M + I < 0 \quad (34)$$

and the control inputs v are chosen as

$$v = -W_v^{-1} B^T P_x \hat{x} \quad (35)$$

then the robust disturbance attenuation Eq.(27) can be guaranteed.

Proof: From the expression $E(\hat{x}, e_z) = \hat{x}^T P_x \hat{x} + e_z^T P_z e_z$ and $y = C\hat{x} + C M e_z$, we can rewritten $H(v, d)$ as

$$\begin{aligned} H(v, d) = & \hat{x}^T C^T C \hat{x} + e_z^T M^T C^T C M e_z + 2\hat{x}^T C^T C M e_z \\ & - \gamma^2 d^T W_d d + v^T W_v v + 2\hat{x}^T P_x B v \\ & + \hat{x}^T (A^T P_x + P_x A) \hat{x} + 2\hat{x}^T P_x G_1 d + 2\hat{x}^T P_x G_2 e_z \\ & + e_z^T (R^T P_z + P_z R) e_z \end{aligned}$$

From the H_∞ control theory⁽¹⁶⁾, it is shown that the worst case $\sup_{d \in L_2} H(v, d)$ occurs when

$$d = \frac{1}{\gamma^2} W_d^{-1} G_1^T P_x \hat{x}$$

It follows that

$$\begin{aligned} H_1(v) = & \sup_{d \in L_2} H(v, d) = F(\hat{x}) + e_z^T M^T C^T C M e_z \\ & + 2\hat{x}^T C^T C M e_z + v^T W_v v + 2\hat{x}^T P_x B v \\ & + 2\hat{x}^T P_x G_2 e_z + e_z^T (R^T P_z + P_z R) e_z \end{aligned}$$

where

$$\begin{aligned} F(\hat{x}) = & \hat{x}^T \left(C^T C + A^T P_x + P_x A \right. \\ & \left. + \frac{1}{\gamma^2} P_x G_1 W_d^{-1} G_1^T P_x \right) \hat{x} \end{aligned}$$

In addition, it is trivial to show⁽¹³⁾

$$2\hat{x}^T P_x G_2 e_z \leq \hat{x}^T P_x G_2 G_2^T P_x \hat{x} + e_z^T e_z$$

and

$$2\hat{x}^T C^T C M e_z \leq \hat{x}^T C^T C \hat{x} + e_z^T M^T C^T C M e_z$$

Then $H_1(v)$ becomes

$$\begin{aligned} H_1(v) \leq & F(\hat{x}) + 2\hat{x}^T C^T C \hat{x} + v^T W_v v \\ & + 2\hat{x}^T P_x B v + \hat{x}^T P_x G_2 G_2^T P_x \hat{x} \\ & + e_z^T (R^T P_z + P_z R + 2M^T C^T C M + I) e_z \end{aligned}$$

Consequently, if there exists a positive definite solution $P_z > 0$ to the following inequality

$$R^T P_z + P_z R + 2M^T C^T C M + I < 0$$

then the term $H_1(v)$ can be rewritten as

$$\begin{aligned} H_1(v) \leq & F(\hat{x}) + \hat{x}^T C^T C \hat{x} + v^T W_v v \\ & + \hat{x}^T P_x G_2 G_2^T P_x \hat{x} + 2\hat{x}^T P_x B v \end{aligned}$$

The optimal control law, which minimizes the right-hand side of the above equation, is given by

$$v = -W_v^{-1} B^T P_x \hat{x}$$

and

$$\begin{aligned} \inf_v H_1(v) \leq & F(\hat{x}) + \hat{x}^T C^T C \hat{x} - \hat{x}^T P_x B W_v^{-1} B^T P_x \hat{x} \\ & + \hat{x}^T P_x G_2 G_2^T P_x \hat{x} \end{aligned}$$

If there exists a positive definite solution $P_x > 0$ the following Riccati equation

$$\begin{aligned} A^T P_x + P_x A + 2C^T C + \frac{1}{\gamma^2} P_x G_1 W_d^{-1} G_1^T P_x \\ + P_x G_2 G_2^T P_x - P_x B W_v^{-1} B^T P_x < 0 \end{aligned}$$

then we have $H(v, d) < 0$ for all $d \in L_2$. $\square$

Similar to work of the traditional H_∞ control⁽¹⁴⁾, it is necessary for solving two Riccati equations. From Eq.(33) and (34), it is clear that the determination of the matrix F will affect whether the solution of P_z exists or not. If F can be found to stabilize $A - G(FCG)^{-1}FCA$, then it is not guaranteed the existence of $P_z > 0$ in Eq.(34). How to find a suitable matrix F such that $P_z > 0$ is solvable is very important. The research of this part could be found in Edwards's paper⁽¹⁸⁾.

Having the sliding surface design, we attempt to find the control law such that the reaching and sliding can be guaranteed⁽¹⁾. From Eq.(19) and the linear system theory, there exists a real known numbers $\alpha > 0$ and $\kappa > 0$ such that

$$\|e_z(t)\| \leq \kappa e^{-\alpha t} e_z(0) \quad \text{for } t > 0 \quad (36)$$

Let the control algorithm be

$$\begin{aligned} u = & -W_v^{-1} B^T P_x \hat{x} - (\eta + \beta \|B^+ G\| \\ & + \kappa e^{-\alpha t} \|e_z(0)\| \|B^+ G E\|) \text{sat}(s, \epsilon) \\ \text{sat}(s, \epsilon) = & \begin{cases} \frac{s}{\|s\|} & \text{if } \|s\| > \epsilon \\ \frac{s}{\epsilon} & \text{if } \|s\| \leq \epsilon \end{cases} \end{aligned} \quad (37)$$

where $\eta > 0$ is a constant. When $\|s\| > \epsilon$, we substitute the control inputs Eq.(37) into Eq.(24) and multiply (24) by s^T to obtain

$$\begin{aligned} s^T \dot{s} = & s^T (B^+ G d + B^+ G E e_z \\ & - (\eta + \kappa \|B^+ G\| + \beta e^{-\alpha t} \|e_z(0)\| \|B^+ G E\|) \text{sat}(s, \epsilon)) \\ \leq & \|s\| (\beta \|B^+ G\| + \|B^+ G E\| \|e_z\|) \\ & - (\eta + \kappa \|B^+ G\| + \beta e^{-\alpha t} \|e_z(0)\| \|B^+ G E\|) \|s\| \\ \leq & -\eta \|s\| \end{aligned} \quad (38)$$

From Eq.(38), it is easy to verify that the reaching and sliding condition is satisfied when $\|s\| > \epsilon$. Hence, the state will be forced to the sliding layer $\|s\| \leq \epsilon$ in finite time.

5. Simulation Results

Consider the following plant:

$$\begin{aligned} \dot{x}(t) = & A x(t) + B u(t) + G d(t) \\ y(t) = & C x(t) \end{aligned} \quad (39)$$

where

$$A = \begin{bmatrix} -0.277 & 1 & -0.0002 \\ -1.71 & -0.178 & -12.2 \\ 0 & 0 & -6.67 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 4 \\ 6.67 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$d(t) = 0.5 \times (\sin(\pi t) + \cos(\pi t/4))$$

The disturbance term is bounded by $\|d\|_2 \leq 1 = \beta$. It is easy to check that the plant (39) satisfies three assumptions, proposed in Section 2. First, we design the state estimator. Choose $F = [-1 \ 3]$ and verify that the eigenvalues for $A - AG(FCG)^{-1}FC$ are $\{0, -1.6545 \pm 0.8169i\}$. Following to the proposed proce-

dures in section 3, the corresponding matrices for the estimator design are found to be

$$U^g = \begin{bmatrix} -1.1406 & 0 & 0 \\ -0.2743 & 0.7906 & -0.7906 \end{bmatrix},$$

$$R = \begin{bmatrix} 0.2434 & -2.1640 \\ 1.9729 & -3.5524 \end{bmatrix}, M = \begin{bmatrix} -0.8767 & 0 \\ -0.4563 & 1.8974 \\ -0.1521 & 0.6325 \end{bmatrix} \quad (40)$$

It is easy to check the eigenvalues for R are $-1.6545 \pm 0.8169i$. The reduced-order observer can be constructed as

$$\dot{\hat{z}}_2 = R\hat{z}_2 + U^g A G (FCG)^{-1} Fy + U^g B u \quad (41)$$

Fig. 1 Outputs y_1 and y_2

Fig. 2 Control input u

Fig. 3 The trajectory of s

and the estimated state $\hat{x}$ is

$$\hat{x} = G(FCG)^{-1}Fy + M\hat{z}_2 \quad (42)$$

Now the sliding surface is chosen as

$$s = B^+ \hat{x} - \int_0^t (B^+ A \hat{x} + v) dt \quad (43)$$

where the term v is designed such that the following equation of robust disturbance attenuation can be satisfied:

$$\int_0^t (y^T y + v^T v) dt \leq 0.01 \int_0^t (d^T d) dt \quad (44)$$

Comparing Eqs.(27) and (44), we know that $\gamma=0.1$ and $W_v=W_d=I$. Then the solutions to two Riccati

Fig. 4(a) State x_1 and estimated state $\hat{x}_1$

Fig. 4(b) State x_2 and estimated state $\hat{x}_2$

Fig. 4(c) State x_3 and estimated state $\hat{x}_3$

Eqs.(33) and (34) can be calculated to obtain

$$\mathbf{P}_x = \begin{bmatrix} 0.2515 & -0.1544 & 0.1972 \\ -0.1544 & 0.6642 & -0.6554 \\ 0.1972 & -0.6554 & 0.8338 \end{bmatrix}$$

$$\text{and } \mathbf{P}_z = \begin{bmatrix} 4.1241 & 0.5797 \\ 0.5797 & 2.1931 \end{bmatrix} \quad (45)$$

and

$$\mathbf{v} = -\mathbf{B}^T \mathbf{P}_x \hat{\mathbf{x}} = [-0.6981 \ 1.7147 \ -2.9399] \hat{\mathbf{x}} \quad (46)$$

Moreover, the control law is designed as

$$\mathbf{u} = -\mathbf{B}^T \mathbf{P}_x \hat{\mathbf{x}} - (\eta + \beta \|\mathbf{B}^+ \mathbf{G}\| + \kappa e^{-\alpha t} \|\mathbf{e}_z(0)\| \|\mathbf{B}^+ \mathbf{G}\mathbf{E}\|) \text{sat}(\mathbf{s}, \varepsilon) \quad (47)$$

where $\alpha=1$, $\kappa=1$, $\eta=0.1$, and $\varepsilon=0.05$. Figures 1-4 show the simulation results using the initial state $\mathbf{x}(0)=[0 \ 2 \ 0]^T$ and $\hat{\mathbf{z}}_2(0)=[0 \ 0]^T$. The time response of $\mathbf{y}$ is shown in Fig. 1 and the control input $\mathbf{u}$ is shown in Fig. 2. Figure 3 shows the trajectory of $\mathbf{s}$. The trajectories of true states and estimated states are given in Figs. 4, which clearly shows that the estimated states $\hat{\mathbf{x}}(t)$ successfully approach the true states $\mathbf{x}(t)$ as $t \rightarrow \infty$. From Figs. 1 and 4, although the system (39) is in the presence of disturbance, the performance of system output $\mathbf{y}$ is very well and the state observer can effectively estimate the true states.

6. Conclusions

In this paper, we have presented a sliding mode control with using output feedback only for linear systems with mismatched disturbance. Applying the H_∞ theory and solving two algebraic Riccati equations, we found that the desired sliding surface can guarantee some level of robust stability and disturbance attenuation for the system once it is in sliding mode. Moreover, in this paper, a state observer is developed to effectively estimate the system states. Hence, the overall sliding mode control design can be divided into two separate steps. First, the observer is used to estimate the states, and the control law is then designed with respect to the estimated states.

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