Chaotic and Periodic Motions in a Vibro-Impacting System

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The present paper investigates the impact vibration in a single degree-of-freedom system having symmetric two-sided stops subjected to harmonic excitation. Periodic asymmetric two-impact/one-period motion, three-impact/one-period motion and symmetric four impact/one-period motion are determined analytically and numerically, and the corresponding stabilities are analyzed. Regions of stable periodic motion are given in the $\delta$–$\Omega$ plane, where $\delta$ denotes the clearance between the mass and the stop at the rest and $\Omega$ denotes the frequency of harmonic excitation. Bifurcation diagrams relating the impact-velocity and a system parameter are also presented. Period-doubling bifurcations, fold bifurcations, grazing bifurcations and chaotic motion are obtained. The grazing bifurcation is peculiar to the vibro-impacting system. In addition, the invariant curves for the system parameters for which chaotic motions arise are presented.

**Key Words**: Impact-Vibration, Chaos, Forced Vibration

1. Introduction

When an oscillator having amplitude constraints is subjected to harmonic excitation, impacts and noise may occur in the oscillator and the life of machine-components may be shortened due to these impacts. For example, a piping system of thermal fluid which has clearance at a support may break during an earthquake. Such a mechanical system having stops is referred to as a vibro-impacting system.

From the standpoint of the design of aseismatic structures, the study of the dynamics of vibro-impacting systems is very important. In particular, knowledge of the effect of the clearance and the frequency of the harmonic excitation on the impact velocity is important.

A single degree-of-freedom system having a one-sided stop has been investigated extensively by several authors\(^{(11-19)}\). The motions of a single degree-of-freedom system having two-sided stops have been analyzed by Shaw, Imanura and others\(^{(13-18)}\). In particular, symmetric two-impact motions during $n$ periods of excitation have been studied extensively.

The present paper investigates the motions of asymmetric two-impact/one-period motion and three-impact/one-period motion, as well as symmetric four-impact/one-period motion, both analytically and numerically. In addition, bifurcation diagrams, invariant curves, domains of attraction and regions of stable solutions are shown numerically.

2. Model of The Impact Oscillator

A simplified model of the vibro-impacting system is shown in Fig. 1. For small excitation amplitudes,

![Fig. 1 Model of a system](image-url)
impacts will not occur and the system will behave as a linear system having a single degree-of-freedom. As the amplitude increases, the mass starts to collide with the rigid stops and the system becomes nonlinear.

The nomenclature used in the present study is as follows:

\[ m: \text{mass} \]
\[ k: \text{spring constant} \]
\[ x: \text{displacement of the mass. The origin is taken to be the natural equilibrium of the spring} \]
\[ y = a \sin \omega t: \text{harmonic excitation} \]
\[ a: \text{amplitude} \]
\[ \omega: \text{frequency} \]
\[ t: \text{time} \]
\[ r: \text{clearance between the mass and symmetrically placed rigid stops} \]
\[ e: \text{coefficient of restitution} \]

The equation of motion is expressed as

\[ m \ddot{x} + kx = k \alpha \sin \omega t \quad (1) \]

where \( \alpha = \frac{d}{dt} \).

Using the relative displacement of the mass due to the excitation,

\[ z = x - y, \quad (2) \]

Equation (1) becomes

\[ m \ddot{z} + kz = ma_0^2 \sin \omega t, \quad (3) \]

and using the following relations

\[ \omega_0 = \sqrt{\frac{k}{m}}, \quad X = \frac{z}{a_0}, \quad \delta = \frac{\theta}{\phi}, \quad \Omega = \frac{\omega}{\omega_0}, \]

and \( \tau = \omega t \),

Equation (3) becomes

\[ X'' + X - \Omega^2 \sin (\Omega \tau + \phi) \quad (4) \]

where, \( \tau = \frac{d}{dt} \), \( \phi \) is the phase angle of excitation in an instance of impact \( X = -\delta \) or \( \delta \) when \( \tau \) is taken as zero.

The general solution of Eq. (4) is given by

\[ X = A \cos \tau + B \sin \tau + C \sin \Omega \tau \quad (5) \]

where \( A \) and \( B \) are arbitrary constants and \( C = \Omega^2 \frac{Q}{1 - \Omega^2} \).

3. Asymmetric Two-Impact/One-Period Periodic Motion

Let us denote a periodic motion of \( m \)-impacts in \( n \)-periods of excitation as \( m/n \)-motion. Since symmetric 2/1-motion has been analyzed previously, the present paper analyzes asymmetric 2/1-, 3/1- and symmetric 4/1-motions.

A schematic model of an asymmetric 2/1-motion is shown in Fig. 2. \( V_i (>0) \) and \( \phi \) denote the impact velocity and phase angle at \( X = -\delta \) and \( V_i (<0) \) denotes the impact velocity at the subsequent impact at \( \tau = \tau_1 \) and \( X = -\delta \). Based on the asymmetry assumption for the motion, \( V_1 = -V_2 \) and \( n = \pi/\Omega \) must hold.

![Fig. 2 Scheme of asymmetric 2/1-periodic motion](image)

From the definition of the coefficient of restitution \( e \), the initial velocity of the mass at \( X = -\delta \) is given by \(-eV_i\), and then

\[ A = -\delta - C \sin \phi, \quad B = -eV_i - C \Omega \cos \phi \]

are obtained. Thus, Eq. (5) becomes

\[ X = (\delta - C \sin \phi) \cos \tau - (eV_i + C \Omega \cos \phi) \sin \tau + C \sin (\Omega \tau + \phi) \quad (6) \]

Since we have assumed that the subsequent impact occurs at \( X = -\delta \) and \( \tau = \tau_1 \), the following equations are obtained.

\[ -\delta = (\delta - C \sin \phi) \cos \tau_1 - (eV_i + C \Omega \cos \phi) \sin \tau_1 + C \sin (\Omega \tau_1 + \phi) \quad (7) \]

and

\[ V_i = (\delta - C \sin \phi) \sin \tau_1 - (eV_i + C \Omega \cos \phi) \cos \tau_1 + C \Omega \cos (\Omega \tau_1 + \phi) \quad (8) \]

Taking \( \tau \) as zero again for the impact at \( X = -\delta \), in a manner similar to that mentioned above, we obtain

\[ X = - (\delta + C \sin (\Omega \tau_1 + \phi)) \cos \tau - (eV_i + C \Omega \cos (\Omega \tau_1 + \phi)) \sin \tau + C \cos (\Omega \tau + \Omega \tau_1 + \phi) \quad (9) \]

Since the subsequent impact occurs at \( X = \delta \) and \( \tau = \frac{2\pi}{\Omega} - \pi \), the following equations are obtained:

\[ \delta = - (\delta + C \sin (\Omega \tau_1 + \phi)) \cos \frac{2\pi}{\Omega} - \pi \]

\[ -(eV_i + C \Omega \cos (\Omega \tau_1 + \phi)) \sin \frac{2\pi}{\Omega} - \pi \]

\[ + C \sin \phi \quad (10) \]

\[ V_i = (\delta + C \sin (\Omega \tau_1 + \phi)) \sin \frac{2\pi}{\Omega} - \pi \]

\[ -(eV_i + C \Omega \cos (\Omega \tau_1 + \phi)) \cos \frac{2\pi}{\Omega} - \pi \]

\[ + C \Omega \cos \phi \quad (11) \]

By eliminating the unknown velocities \( V_i \) and \( \Omega \) from Eqs. (8) through (11), sin \( \phi \) and cos \( \phi \) are expressed as functions of the unknown \( \Omega \):

\[ \left[ -e \sin \Omega \cos \left( \frac{2\pi}{\Omega} - \pi \right) - \cos \Omega \right] \sin \theta \]

\[ - \sin \left( \frac{2\pi}{\Omega} - \pi \right) \cos \left( \cos \Omega \theta - \cos \Omega \right) \phi \]
\[ + \left[ e \sin \eta \left( \frac{2\pi}{Q} - \eta \right) + \sin \Omega \eta \right] \\
- \sin \left( \frac{2\pi}{Q} - \eta \right) \left( \sin \Omega \eta \cos \sin \eta \right) \cos \theta \\
= \frac{\delta}{C} \left[ \sin \left( \frac{2\pi}{Q} - \eta \right) (1 + \cos \eta) \right] \\
- e \sin \eta \left[ 1 + \cos \left( \frac{2\pi}{Q} - \eta \right) \right] \left( \sin \left( \frac{2\pi}{Q} - \eta \right) \right) \]

(12)

\[ + \left[ \sin \left( \frac{2\pi}{Q} - \eta \right) \right] \left( \sin \left( \frac{2\pi}{Q} - \eta \right) \cos \sin \eta \right) \sin \theta \\
= \frac{1}{C} \left[ \sin \left( \frac{2\pi}{Q} - \eta \right) \right] \cos \theta \]

(13)

Then, using the relationship \( \sin^2 \theta + \cos^2 \theta = 1 \), we can obtain an equation in terms of \( \eta \) only which can be solved numerically. If \( \eta \) is determined, we can easily determine the values of \( \sin \theta, \cos \theta, V_1, \) and \( V_2 \) and, therefore, the trajectory \( X \) of the Asym-2/1 periodic motion.

If \( V_1 > 0, V_2 < 0 \) and \( \delta < X < \delta \left( 0 < \eta \right) \) and \( 0 < \tau < \frac{2\pi}{Q} - \eta \) are satisfied, the motion may exist. However, in order for the motion to be realized, the motion must be stable.

4. Stability of Asym-2/1-Motion

Let us assume that small disturbances occur in \( \eta \) and \( V_1 \) at a certain impact at \( X = \delta \) and let those disturbances in the \( k \)-th period be \( \Delta \theta^{(k)} \) and \( \Delta V_1^{(k)} \). Then, after very complicated arrangement we obtain the following equation:

\[
\left( \frac{\Delta \theta^{(k+1)}}{\Delta V_1^{(k+1)}} \right) = A \left( \frac{\Delta \theta^{(k)}}{\Delta V_1^{(k)}} \right) \]

(14)

where \( A \) is a matrix of order two. A solution of Eq.(14) is assumed as

\[
\left( \frac{\Delta \theta^{(k)}}{\Delta V_1^{(k)}} \right) = \lambda^k \phi \]

(15)

where \( \lambda \) is an unknown parameter and \( \phi \) is an unknown column vector. Substituting Eq.(15) into Eq.(14), we obtain the quadratic characteristic equation

\[
\det (I - A) = 0 
\]

(16)

where \( I \) is a unit matrix of order two. This equation can be easily solved, i.e. its eigenvalues \( \lambda \) are easily determined. When both of the roots \( \lambda \) satisfy \( |\lambda| < 1 \), the above-mentioned periodic asymm-2/1-motion is stable.

Regions of the stable asymm-2/1-motion are easily drawn in the \((\delta, Q)\) plane.

Similarly, the stability of symm-4/1-motion, of which the trajectory is shown in Fig. 3, can be analyzed, and regions of stable motion can be easily determined. However, determining 3/1-motion, shown in Fig. 4, and the regions of stable motion in the \((\delta, Q)\) plane is very complicated. Therefore, description on 3/1-motion is omitted.

5. Symmetric 4-Impacts/1-Period Periodic Motion

A schematic model of a symmetric 4/1-motion is shown in Fig. 3. \( V_1(>0) \) and \( \theta \) denote the impact velocity and phase angle at \( X = \delta \) and \( V_2(>0) \) denotes the impact velocity at the subsequent impact at \( r = \eta \) and \( X = \delta \). Based on the symmetry assumption for the motion, the subsequent impact occurs at \( r = \pi/\Omega - \eta \) and \( X = \delta \).

From the definition of the coefficient of restitution \( e \), the initial velocity of the mass at \( X = \delta (r = 0) \) is given by \( -e V_1 \), and then

\[
A = \delta - C \sin \theta, B = -e V_1 - C \Omega \cos \theta \\
are obtained. Then Eq.(5) becomes

\[
X = (\delta - C \sin \theta) \cos r \\
- (e V_1 + C \Omega \cos \theta) \sin r + C \sin (\Omega r + \theta) 
\]

(17)

Since we have assumed that the subsequent impact occurs at \( X = \delta \) and \( r = \eta \), the following equations are obtained.

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\[ \delta = (\delta - C \sin \theta) \cos \eta - (eV_1 + C \Omega \cos \theta) \sin \eta + C \sin (\Omega \eta + \theta) \] (18)

and

\[ V_1 = -(\delta + C \sin \theta) \sin \eta - (eV_2 + C \Omega \cos \theta) \cos \eta + C \cos (\Omega \eta + \theta) \] (19)

Taking \( r \) as zero again for the impact at \( x = -\delta \), in a manner similar to that mentioned above, we obtain

\[ x = -[\delta - C \sin (\Omega \eta + \theta)] \cos r - (eV_1 + C \Omega \cos (\Omega \eta + \theta)) \sin r + C \sin (\Omega r + \Omega \eta + \theta). \] (20)

Since the subsequent impact occurs at \( x = -\delta \) and \( r = \frac{\pi}{\Omega} - \eta \), the following equations are obtained.

\[ \delta = -[\delta - C \sin (\Omega \eta + \theta)] \cos \left( \frac{\pi}{\Omega} - \eta \right) - (eV_1 + C \Omega \cos (\Omega \eta + \theta)) \sin \left( \frac{\pi}{\Omega} - \eta \right) - C \sin \theta \]

\[ V_1 = -[\delta - C \sin (\Omega \eta + \theta)] \sin \left( \frac{\pi}{\Omega} - \eta \right) - (eV_2 + C \Omega \cos (\Omega \eta + \theta)) \cos \left( \frac{\pi}{\Omega} - \eta \right) - C \cos \theta \] (21)

By eliminating the unknown velocities \( V_1 \) and \( V_2 \) from Eqs. (19) through (22), \( \sin \theta \) and \( \cos \theta \) are expressed as functions of the unknown \( \eta \):

\[ \begin{align*}
- \sin \left( \frac{\pi}{\Omega} - \eta \right) \sin \left[ \delta \left( \frac{\pi}{\Omega} - \eta \right) + \cos (\Omega \eta) \right] \\
- \sin \left( \frac{\pi}{\Omega} - \eta \right) \sin \left[ \delta \left( \frac{\pi}{\Omega} - \eta \right) - \cos (\Omega \eta) \right] \\
+ \left[ e \sin \eta \left( \Omega \sin \left( \frac{\pi}{\Omega} - \eta \right) - \sin (\Omega \eta) \right) \sin \theta \right] \\
- \sin \left( \frac{\pi}{\Omega} - \eta \right) \sin \left[ \delta \left( \frac{\pi}{\Omega} - \eta \right) - \cos (\Omega \eta) \right] \\
= \frac{\delta}{C} \sin \left[ \delta \left( \frac{\pi}{\Omega} - \eta \right) - \cos (\Omega \eta) \right] \\
+ e \sin \eta \left( \delta \left( \frac{\pi}{\Omega} - \eta \right) - \cos (\Omega \eta) \right) \sin \theta \\
+ \cos \left( \frac{\pi}{\Omega} - \eta \right) \sin (\Omega \eta) + C \Omega \cos (\Omega \eta) \sin \theta \\
+ \sin \left( \frac{\pi}{\Omega} - \eta \right) \sin (\Omega \eta) - e \sin \eta \cos (\Omega \eta) \cos \theta \\
= \frac{\delta}{C} \sin \left[ \delta \left( \frac{\pi}{\Omega} - \eta \right) - \cos (\Omega \eta) \right] + e \sin \eta \sin (\Omega \eta) \cos \theta \\
+ (1 - \cos \eta) \cos \left( \frac{\pi}{\Omega} - \eta \right) \] (22)

Then using the relationship \( \sin^2 \theta + \cos^2 \theta = 1 \), we can obtain an equation in terms of \( \eta \) only which can be solved numerically. If \( \eta \) is determined, we can easily determine the values of \( \sin \theta, \cos \theta, V_1 \) and \( V_2 \) and, therefore, the trajectory \( X \) of the Sym-4/1 periodic motion.

If \( V_i > 0, V_j > 0 \) and \( -\delta < X < \delta \left( 0 < r < \eta \right) \) are satisfied, the motion may exist. However, in order for the motion to be realized, the motion must be stable.

6. Numerical Examples

The results of numerical calculation for the case of the coefficient of restitution \( e = 0.6 \) are shown in this section. Figure 5 shows the regions of Asym-2/1-, Symm-2/1-, Symm-4/1-and the one-sided 1/1-periodic motions in the \( (\delta, \Omega) \) plane. Although these regions can be easily obtained, the regions of other types of motions can be obtained from a number of bifurcation diagrams relating the impact-velocity \( V \) with \( \delta \) or \( \Omega \).

Figure 6 shows a map of the stable motions and the sets of bifurcation in the \( (\delta, \Omega) \) plane. Bifurcation diagrams are shown in part (a) of Figs. 7 through 9, and curves of eigenvalues corresponding to these diagrams are shown in part (b) of Figs. 7 through 9. These figures reveal the existence of period-doubling

![Fig. 5 Regions of stable periodic motions](image1)

![Fig. 6 Regions of stable motions and sets of bifurcations](image2)
bifurcation when \( \lambda = -1 \) (boundaries with -1 in Fig. 6), fold bifurcation when \( \lambda = 1 \) (boundaries with 1) and grazing impacts (boundaries with gr). Bifurcation by grazing impacts is called grazing bifurcation.

Black zones in the bifurcation diagrams correspond to chaotic behavior. The tendency of the effect

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**Fig. 7** \( \Omega = 1.5 \)

**Fig. 8** \( \delta = 0.7 \)

**Fig. 9** \( \delta = 1 \)

**Fig. 10** Invariant curves and domains of attraction at point A (\( \Omega = 2, \delta = 0.95 \))
Fig. 11 Invariant curves and domains of attraction at point B (Ω=0.3, δ=0.3)

(a) Invariant curves  (b) Trajectories in phase plane

(c) Strange Attractor
Fig. 12 Point C (Ω=1.5, δ=0.8)

of clearance δ and frequency Ω of excitation on the impact-velocity V can not be expressed in a simple phrase. Generally speaking, the impact-velocity V increases as frequency Ω increases.

For the parameters corresponding to points A and B in Fig. 6, two kinds of stable periodic motions are found. Figures 10 and 11 show the invariant curves and the domains of attraction corresponding to points A and B, respectively, in the polar coordinate system (V, θ).

At points C and D in Fig. 6, different types of chaotic motion are found. Figures 12 and 13 show the invariant curves and the trajectories in the phase plane (X, X'). The chaotic motion is found to be asymmetric at point C and symmetric at point D. Although no homoclinic points appear in Fig. 12, these points are visible in Fig. 13.

7. Conclusions

(1) Asymmetric 2/1-, 3/1- and symmetric 4/1-periodic motions were determined analytically and numerically, and their stabilities were analyzed.

(2) Bifurcation diagrams relating V and δ or Ω are shown in Figs. 7 through 9.

(3) Period-doubling bifurcation, fold bifurcation, grazing impacts and chaotic motions were found in bifurcation diagrams.

(4) Generally, the impact-velocity, V, increases as frequency, Ω, increases.
References

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