Robust Stability of Linear Continuous/Discrete-Time Output Feedback Systems with Both Time-Varying Structured and Unstructured Uncertainties*

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This paper investigates the stability robustness of linear continuous/discrete-time systems with output feedback controllers as well as both time-varying structured and unstructured parameter uncertainties by directly considering the mixed quadratically-coupled uncertainties in the problem formulation. Based on the Lyapunov approach and some essential properties of matrix measures, some new sufficient conditions are proposed for ensuring that the linear output feedback systems with both time-varying structured and unstructured parameter uncertainties are asymptotically stable. Three numerical examples for continuous-time and discrete-time systems are given to illustrate the application of the presented sufficient conditions, and for the case of only considering time-varying structured parameter uncertainties, the proposed sufficient conditions are shown to be less conservative than the existing ones reported in the literature.

Key Words: Stability Robustness, Output Feedback Systems, Time-Varying Uncertainties, Structured and Unstructured Parameter Uncertainties

1. Introduction

In general, a mathematical description is only an approximation of the actual physical system and deals with fixed nominal parameters. Usually, these parameters are not known exactly due to imperfect identification or measurement, aging of components and/or changes in environmental conditions. Thus, it is almost impossible to get an exact model for the system due to the existence of various parameter uncertainties. Here, we consider linear state-space systems with time-varying uncertain parameters in the system matrix, input matrix, and output matrix. Because the output feedback controller design is usually based on the nominal values of these system matrices, it is interesting to know whether the closed-loop system remains asymptotically stable in the presence of time-varying uncertain parameters. Applying those previous robust stability analysis results[1,2,3,4,5,6,7,8,9,10,11,12,13,14] to solve this problem is not easy, in that after output feedback, there will be coupled terms of parameters in the closed-loop system matrix because of the uncertain parameters in both input and output matrices[11]. Although we may regard these coupled terms as new independent parameters if we insist on using those previous robust stability analysis results, Su and Fong[11] and Tseng et al.[12] have showed that a conservative analysis conclusion may be reached. Therefore, Su and Fong[11] and Tseng et al.[12] investigated the robust stability problem of linear systems with constant output feedback in the presence of time-varying uncertain parameters by directly considering the coupled terms in the problem formulation. Su and Fong[11] used the Lyapunov method to analyze the robust stability of linear continuous/discrete-time systems with quadratically-coupled structured uncertainties. Tseng et al.[12] applied the structured singular value technique to solve the robust stability analysis problem of linear continuous-time systems with quadratically-coupled structured uncertainties. Here it should be noticed that, to the
authors' best knowledge, only the articles of Su and Fong and Tseng et al. studied the robust stability of linear systems with output feedback controllers and time-varying uncertain parameters by directly considering the coupled terms in the problem formulation. That is, the research on the stability robustness of linear systems with output feedback controllers and time-varying uncertain parameters by directly considering the coupled terms in the problem formulation is considerably rare and almost embryonic.

On the other hand, it is well known that an approximate system model is always used in practice and sometimes the approximation error should be covered by introducing both structured (elemental) and unstructured (norm-bounded) parameter uncertainties in control system analysis and design. That is, it is not unusual that at times we have to deal with a system simultaneously consisting of two parts: one part has only the structured parameter uncertainties, and the other part has the unstructured parameter uncertainties. But, to the authors' best knowledge, none of the research works published in the literature proposes any robust stability criterion to study the problem of stability robustness for linear systems with output feedback controllers as well as both time-varying structured and unstructured parameter uncertainties by directly considering the mixed quadratically-coupled uncertainties (i.e., quadratically-coupled structured, quadratically-coupled unstructured, and coupled structured-unstructured uncertainties appearing together) in the problem formulation. Therefore, the purpose of this paper is to investigate the robust stability problem of linear continuous/discrete-time systems with output feedback controllers as well as both time-varying structured and unstructured parameter uncertainties by directly considering the mixed quadratically-coupled uncertainties in the problem formulation. The main results are presented in Section 2. In Section 3, some numerical examples for continuous-time and discrete-time systems are given to illustrate the application of the proposed sufficient conditions. For the case of linear systems only subject to structured parameter uncertainties, a conservatism comparison between the criteria, which are, respectively, proposed in the paper and by Su and Fong and Tseng et al., is also given in this section. Finally, Section 4 offers some conclusions.

2. Robust Stability Analysis

Consider the linear uncertain systems with the state-space model

\[
\dot{x}(t) = Ax(t) + Bu(t),
\]
(1)

\[
y(t) = Cx(t),
\]
(2)

where \( \delta \) denotes the differentiation operator for continuous-time systems, or the shift operator for discrete-time systems, \( x \in \mathbb{R}^n \) is the state vector, \( y \in \mathbb{R}^q \) is the output vector, \( u \in \mathbb{R} \) is the input vector,

\[
A = A_0 + \sum_{i=1}^{n} k_i(t)A_i + \bar{A}(t),
\]

\[
B = B_0 + \sum_{i=1}^{n} k_i(t)B_i + \bar{B}(t),
\]

\[
C = C_0 + \sum_{i=1}^{n} k_i(t)C_i + \bar{C}(t)
\]

are the system matrices, \( k_i(t) \) is the \( i \)th time-varying uncertain parameter, and \( n \) is the number of independent time-varying uncertain parameters. The time-varying unstructured uncertain matrices \( \bar{A}(t), \bar{B}(t) \) and \( \bar{C}(t) \) are assumed to be bounded, i.e.,

\[
\|\bar{A}(t)\| \leq \beta_a, \quad \|\bar{B}(t)\| \leq \beta_b, \quad \|\bar{C}(t)\| \leq \beta_c
\]

(4)

where \( \beta_a, \beta_b, \) and \( \beta_c \) are non-negative real constant numbers, and \( \| \cdot \| \) denotes any matrix norm.

Any dynamic controllers of order \( p \) for the system (1) and (2) can be viewed as an output feedback gain of an augmented system with dimension \( n + p \) (Su and Fong), so, in this paper, we only discuss the static output feedback gain controllers. Let the output feedback gain matrix be \( K \), then the closed-loop system equations of the linear uncertain systems can be expressed as

\[
\dot{x}(t) = \left[ A_{0} + B_{0}KC_{0} + \sum_{i=1}^{n} k_i(t)(A_{i} + B_{i}KC_{i} + B_{i}KC_{0}) \right] x(t)
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} k_i(t)k_j(t)(B_{i}KC_{j} + B_{i}KC_{0} + B_{i}KC_{i} + B_{i}KC_{j}) x(t)
\]

\[
+ \sum_{i=1}^{n} k_i(t)(B_{i}KC_{0} + B_{i}KC_{i} + B_{i}KC_{j}) x(t)
\]

\[
= \left[ \tilde{A}_{0} + \sum_{i=1}^{n} k_i(t)E_{i} + \sum_{i=1}^{n} \sum_{j=1}^{n} k_i(t)k_j(t)E_{ij} \right] x(t)
\]

\[
+ F(t) x(t),
\]

(5)

where

\[
\tilde{A}_{0} = A_{0} + B_{0}KC_{0},
\]

\[
E_{i} = A_{i} + B_{i}KC_{i} + B_{i}KC_{0},
\]

\[
E_{ij} = \frac{1}{2} (B_{i}KC_{j} + B_{i}KC_{j}),
\]

and

\[
F(t) = \bar{A}(t) + B_{0}K \bar{C}(t) + \bar{B}(t)KC_{0} + \bar{B}(t)KC_{i} + \sum_{i=1}^{n} k_i(t)(B_{i}KC_{0} + B_{i}KC_{i} + B_{i}KC_{j})
\]

(6)

The problem considered in this paper is similar to those considered by Su and Fong and Tseng et al. Su and Fong and Tseng et al. only considered the case of structured uncertainties, whereas here we consider both cases of structured and unstructured uncertainties. For continuous-time systems, the robust stability analysis problem of (5) reduces to that of Refs. (1), (2), (4), (8), (10), (13) and (15) or that of Chou and Chen if \( F(t) = 0 \) and \( E_{ij} = 0 \) for
all $i, j \in \{1, 2, \ldots, m\}$. This may happen if either of the following conditions is satisfied:

1) there are no unstructured uncertainties (i.e., $\tilde{A}(t) = 0$, $\tilde{B}(t) = 0$, and $\tilde{C}(t) = 0$) as well as there are no structured uncertain parameters in the input matrix $B$ and/or in the output matrix $C$ (i.e., $\beta_i = 0$ and/or $\gamma_i = 0$ for all $i \in \{1, 2, \ldots, m\}$);

2) there are no unstructured uncertainties as well as the perturbation structure matrices $\beta_i$ and $\gamma_i$ happen to make $P_0 = 0$ and $E_0 = 0$ for all $i, j \in \{1, 2, \ldots, m\}$. Thus, it can readily be seen from Eq. (5) the deficiencies of previous problem formulations. Under the assumption that $K(t)$ is $\geq 0$ and $E_0 \neq 0$ for some $i, j \in \{1, 2, \ldots, m\}$, our problem can be formulated as: given a stabilizing feedback gain matrix $K$, find the robust stability properties of the closed-loop system matrix $\tilde{A}_0$ subject to the time-varying uncertain parameters $k_i(t)$ with the structural matrices $E_i$ and $E_j$ as well as the unstructured uncertain matrix $F(t)$ in Eq. (5).

Before we analyze the robust stability problem of the closed-loop systems in Eq. (5), the following lemmas need to be introduced first.

Lemma 1\textsuperscript{(3)}

The matrix measures of the matrices $W$ and $V$, $\mu(W)$ and $\mu(V)$, are well defined for any norm and have the following properties:

(i) $\mu(\pm I) = \pm 1$, for the identity matrix $I$;

(ii) $-\|W\| \leq -\mu(-W) \leq \text{Re}(\lambda(W)) \leq \mu(W) \leq \|W\|$, for any norm $\|\cdot\|$ and any matrix $W \in C^{**}$;

(iii) $\mu(W + V) \leq \mu(W) + \mu(V)$, for any two matrices $W, V \in C^{**}$;

(iv) $\mu(\gamma W) = \gamma \mu(W)$, for any matrix $W \times C^{**}$ and any non-negative real number $\gamma$; where $\lambda(W)$ denotes any eigenvalue of $W$, and $\text{Re}(\lambda(W))$ denotes the real part of $\lambda(W)$.

Lemma 2\textsuperscript{(3)}

For any $\gamma < 0$ and any matrix $W \in C^{**}$, $\mu(\gamma W) = -\gamma \mu(-W)$.

In what follows, we assume an output feedback gain matrix $K$ has been previously designed to make $\tilde{A}_0$ a stable matrix, and present two new sufficient criteria for ensuring that the closed-loop uncertain systems (5) remain asymptotically stable.

Theorem 1:

For continuous-time systems, the closed-loop system (5) remains asymptotically stable, if the following condition is satisfied

\begin{equation}
\sum_{i=1}^{m} \mu(k_i(t)) \mu(aP_0 + P_0) + \sum_{i=1}^{m} \sum_{j=1}^{m} \beta_i \beta_j \|k_i(t)\|_2 \|k_j(t)\|_2 + 2\beta \|P\| < \frac{2a-1}{\alpha},
\end{equation}

where

$$a > 0.5,$$

$$P\tilde{A}_0 + \tilde{A}_0 P = -2P,$$

$$P = \frac{1}{2}(PE + EP),$$

$$P_0 = PE_0 + E_0 P,$$

$$k_i(t)k_i(t) \geq 0;$$

$$\beta = \beta_1 + \beta_2 \|K\|_2 + \beta_3 \|B_0 \|_2 + \beta_4 \|K\|_2 + \sum_{i=1}^{m} \|k_i(t)\|_2 \|K\|_2;$$

Proof:

Let the Lyapunov function candidate be $V(x(t)) = x(t)^T P x(t)$, where $P$ is the unique solution of Eq. (8), then, differentiating the Lyapunov function and from Eqs. (4), (9), (10) and (12), we have

$$\dot{V}(x(t)) = x(t)^T \left[ P\tilde{A}_0 + \tilde{A}_0 P + \sum_{i=1}^{m} \beta_i \|k_i(t)\|_2 (PE + EP) + \sum_{i=1}^{m} \sum_{j=1}^{m} \beta_i \beta_j \|k_i(t)\|_2 \|k_j(t)\|_2 + 2\beta \|P\| \right] x(t)$$

$$\leq x(t)^T \left[ -2P + \beta \|k_i(t)\|_2 \left( PE + EP + \sum_{i=1}^{m} \sum_{j=1}^{m} \beta_i \beta_j \|k_i(t)\|_2 \|k_j(t)\|_2 + 2\beta \|P\| \right) \right] x(t)$$

$$\leq 2P \|x(t)\|^2 \leq 2\|x(t)\|^2.$$
Thus,
\[
\dot{V}(x(t)) \leq x(t)^T \left[ -2I + \frac{1}{\alpha} \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) P_j P_k + \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) P_j U_k + 2\beta \|P\|I \right] x(t)
\]
\[
= x(t)^T \left[ \frac{1-2\alpha}{\alpha} I + \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) (aP_j P_k + P_j U_k) + 2\beta \|P\|I \right] x(t).
\]
Since \( \frac{1-2\alpha}{\alpha} I + \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) (aP_j P_k + P_j U_k) + 2\beta \|P\|I \) is symmetric, its eigenvalues are all real. Thus, using the properties in Lemmas 1 and 2, and from Eq. (7) and the assumption \( \alpha > 0.5 \), we can get
\[
\dot{V}(x(t)) \leq \mu \left( \frac{1-2\alpha}{\alpha} I + \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) (aP_j P_k + P_j U_k) + 2\beta \|P\|I \right)
\]
\[
+ 2\beta \|P\|I \leq \mu \left( \frac{1-2\alpha}{\alpha} + \sum_{j=1}^{n} \sum_{k=1}^{n} (k_j(t) aP_j + P_j U_k) + 2\sum_{j=1}^{n} \sum_{k=1}^{n} (k_j(t) k_k(t) (P_j + P_k + aP_j P_k + P_j U_k) + 2\beta \|P\|I \right)
\]
\[
+ 2\beta \|P\|I \mu ( \frac{1-2\alpha}{\alpha} + \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) aP_j + P_j U_k + 2\sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) (P_j + P_k + aP_j P_k + P_j U_k) + 2\beta \|P\|I \right)
\]
\[
+ 2\beta \|P\|I \mu ( \frac{1-2\alpha}{\alpha} + \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) aP_j + P_j U_k + 2\sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) (P_j + P_k + aP_j P_k + P_j U_k) + 2\beta \|P\|I \right)
\]
This implies that \( \dot{V}(x(t)) \leq 0 \). So, we have proved that the uncertain continuous-time system (5) is asymptotically stable if the inequality (7) is satisfied. Thus, the proof is completed. Q.E.D.

Theorem 2:

For discrete-time systems, the closed-loop system (5) remains asymptotically stable, if the following condition is satisfied
\[
\sum_{j=1}^{n} k_j(t) (aP_j + P_j U_k) + \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) \theta_{aj} + \sum_{j=1}^{n} k_j(t) \phi_{aj} + \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) \phi_{aj}
\]
\[
+ \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} k_j(t) k_k(t) k_l(t) \phi_{ajl} + \sum_{j=1}^{n} k_j(t) P_{aj} + \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) \phi_{aj} + \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) \phi_{ajl} + \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} k_j(t) k_k(t) k_l(t) \phi_{ajll}
\]
\[
\leq \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} k_j(t) k_k(t) k_l(t) \phi_{ajll} + \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{m=1}^{n} k_j(t) k_k(t) k_l(t) k_m(t) \phi_{ajml} + \beta_d < \frac{2\alpha - 1}{2\alpha},
\]
where
\[
a > 0.5,
\]
\[
A^T P_a A = P_a = -I,
\]
\[
P_a = A^T P_a A + E^T P_a A
\]
\[
P_{au} = A^T P_a E_a + E^T P_a A
\]
\[
P_{au} = E^T P_a E_a + E^T P_a E_a
\]
\[
P_{au} = E^T P_a E_a + E^T P_a E_a
\]
\[
P_{au} = E^T P_a E_a + E^T P_a E_a
\]
\[
P_d(t) = A^T P_d F(t) + \sum_{j=1}^{n} k_j(t) E^T P_d F(t) + \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) E^T P_d F(t) + F^T(t) P_d A_0
\]
\[
+ \sum_{j=1}^{n} k_j(t) k_j(t) F^T(t) P_d E_a + \sum_{j=1}^{n} \sum_{k=1}^{n} k_j(t) k_k(t) F^T(t) P_d E_a + F^T(t) P_d F(t)
\]
\[
\theta_{aj} = \begin{cases} 
\mu (P_{au} + P_{au} + \frac{\alpha}{2} (P_{au} P_{au} + P_{au} P_{au})), & \text{for } k(t) k(t) \geq 0; \\
-\mu (P_{au} - P_{au} - \frac{\alpha}{2} (P_{au} P_{au} + P_{au} P_{au})), & \text{for } k(t) k(t) < 0;
\end{cases}
\]
\[
\phi_{aj} = \begin{cases} 
\mu (P_{au} + P_{au} + P_{au}), & \text{for } k(t) k(t) \geq 0; \\
-\mu (P_{au} - P_{au} - P_{au}), & \text{for } k(t) k(t) < 0;
\end{cases}
\]
\[
\phi_{ajl} = \begin{cases} 
\mu (P_{au} + P_{au} + P_{au} + P_{au} + P_{au} + P_{au}), & \text{for } k(t) k(t) k(t) \geq 0; \\
-\mu (P_{au} - P_{au} - P_{au} - P_{au} - P_{au} - P_{au}), & \text{for } k(t) k(t) k(t) < 0;
\end{cases}
\]
\[
\phi_{ajml} = \begin{cases} 
\mu (P_{au} + P_{au} + P_{au} + P_{au} + P_{au} + P_{au}), & \text{for } k(t) k(t) k(t) \geq 0; \\
-\mu (P_{au} - P_{au} - P_{au} - P_{au} - P_{au} - P_{au}), & \text{for } k(t) k(t) k(t) < 0;
\end{cases}
\]
\[
\phi_{ajll} = \begin{cases} 
\mu (M_{ajll}), & \text{for } k(t) k(t) \geq 0; \\
-\mu (M_{ajll}), & \text{for } k(t) k(t) < 0;
\end{cases}
\]
By using the method of Su and Fong, we can obtain
\[ k(t) + k^2(t) \leq 0.2069 \leq \max \left( \frac{2a-1}{\lambda_{\max}(AP^2 + Q)} \right) \]
= 0.1542.

Thus, no conclusion can be made. That is, the robust stability condition of Su and Fong cannot be applied in this example.

By adopting the method of Tseng et al. and the software of Matlab Toolbox for structured singular value, we can obtain
\[ |k(t)| < 0.2324, \]
and
\[ |k_0(t)| < 0.2324. \]

Thus, we cannot reach any conclusion for guaranteeing the robust stability. That is, the robust stability condition of Tseng can also not be applied in this example.

Now, applying the sufficient condition (7) with the 2-norm-based matrix measure and \( \varepsilon = 1 \), we have
\[ \sum_{i=1}^{3} k(t) \mu(aP^2 + P_0) + \sum_{i=1}^{3} k(t) k(t) 0 \leq 0.9988 \]
< 1, for \( k(t) k(t) \geq 0 \);
\[ \sum_{i=1}^{3} k(t) \mu(aP^2 + P_0) + \sum_{i=1}^{3} k(t) k(t) 0 \leq 0.7138 \]
< 1, for \( k(t) k(t) < 0 \).

So, we can conclude that the continuous-time system with closed-loop system matrix (33) is asymptotically stable. From above results, it can be shown that our proposed sufficient condition (7) for the case of only considering time-varying structured parameter uncertainties is less conservative than those of Su and Fong and Tseng et al.

**Example 2:**

Consider a discrete-time system \( \{A, B, C\} \) with time-varying uncertain parameters \( k(t) (i=1, 2) \) and controlled by the output feedback gain \( K = 1 \), where
\[
K = \begin{bmatrix}
-1 & -1 \\
0 & 2
\end{bmatrix} + k(t) \begin{bmatrix}
5 & -2 \\
-8 & 3
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
1 \\
-3
\end{bmatrix} + k(t) \begin{bmatrix}
1 \\
0
\end{bmatrix},
\]
\[
C = \begin{bmatrix}
0 & 1 + k(t) [-3 & 1] + k(t) [1 & 0]
\end{bmatrix},
\]
\[
k(t) \in [-0.38, 0.38],
\]
and
\[
k(t) \in [-0.25, 0.25].
\]

It is seen that the closed-loop system matrix becomes
\[
\bar{A} + k(t) E_1 + k(t) E_2 + k(t) k(t) E_{12} + k(t) k(t) E_{13} = \begin{bmatrix}
-1 & 0 & 0 & k(t) \ [2 & 0 & 1 & 0] \\
0 & -1 & -1 & k(t) \ [0 & 1 & 1 & 0] \\
-3 & 0 & 0 & k(t) k(t) \ [0 & 0 & 1 & 0]
\end{bmatrix},
\]
\[
+ k^2(t) \begin{bmatrix}
3 & 1 \\
0 & 0 \\
1 & 0 \\
0 & 0
\end{bmatrix} + k(t) k(t) \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

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By using the method of Su and Fong\cite{11}, we can obtain

\[ k(t) + \tilde{k}(t) \leq 1.1909 \times \max(\gamma_2(a, k), \gamma_2(a)) = 0.3442. \]

Then, no conclusion can be made. That is, the robust stability condition of Su and Fong\cite{11} cannot be applied in this example.

Now, applying the sufficient condition (15) with the 2-norm-based matrix measure and \( a = 1.3 \), we have

\[
\begin{align*}
(\text{i}) & \quad 1.0189 k(t) + 0.0654 \tilde{k}(t) + 0.5339 k(t) \tilde{k}(t) + 0.0004 k(t) \tilde{k}(t) + 0.0007 \tilde{k}(t) \leq 0.5990 \\
& \quad < 0.6154, \text{ for } k(t) \in [0, 0.5] \text{ and } \tilde{k}(t) \in [0, 0.9]; \\
(\text{ii}) & \quad 1.0189 k(t) + 0.0654 \tilde{k}(t) - 0.1930 k(t) \tilde{k}(t) + 0.0004 k(t) \tilde{k}(t) + 0.0007 \tilde{k}(t) \leq 0.4418 \\
& \quad < 0.6154, \text{ for } k(t) \in [-0.5, 0] \text{ and } \tilde{k}(t) \in [0, 0.9]; \\
(\text{iii}) & \quad 1.0189 k(t) + 0.0654 \tilde{k}(t) + 0.5339 k(t) \tilde{k}(t) - 0.0004 k(t) \tilde{k}(t) + 0.0007 \tilde{k}(t) \leq 0.6135 \\
& \quad < 0.6154, \text{ for } k(t) \in [-0.5, 0] \text{ and } \tilde{k}(t) \in [-0.9, 0]; \\
(\text{iv}) & \quad 1.0189 k(t) + 0.0654 \tilde{k}(t) - 0.1930 k(t) \tilde{k}(t) - 0.0004 k(t) \tilde{k}(t) + 0.0007 \tilde{k}(t) \leq 0.4341 \\
& \quad < 0.6154, \text{ for } k(t) \in [0, 0.5] \text{ and } \tilde{k}(t) \in [-0.9, 0].
\end{align*}
\]

So, we can conclude that the discrete-time system with closed-loop system matrix (34) is asymptotically stable. This illustrates that the proposed sufficient condition (15) can overcome the conservatism of the sufficient condition given by Su and Fong\cite{11}.

**Remark 2:** From the above examples, we can see that the proposed sufficient conditions (7) for continuous-time systems and (15) for discrete-time systems may obtain less conservative results than those of Su and Fong\cite{11} and Tseng et al.\cite{12, 13}. The reason the proposed sufficient conditions (7) and (15) are less conservative is that it takes the directional information into consideration. This can be explained by the fact that as a parameter varies in different directions, it affects the system's properties differently. That is, the effect of a single parameter \( k \) on the system's properties can be completely different for the same \( |k| \) and opposite sign. This idea has been also used by the authors\cite{14} to analyze the stability robustness of linear uncertain time-variant systems. Therefore, any sufficient conditions, that ignore the signs, may obtain more conservative results.

**Example 3:**

Consider the robust stability problem of a continuous-time system \( \{ A, B, C \} \) having both time-varying structured and unstructured parameter uncertainties, and controlled by the output feedback gain \( K = 1 \), where

\[
\begin{align*}
A &= A_0 + k(t)A_1 + \tilde{A}(t), \\
B &= B_0 + k(t)B_1 + \tilde{B}(t), \\
C &= C_0 + k(t)C_1 + \tilde{C}(t),
\end{align*}
\]

in which \( |\tilde{A}(t)| \leq \beta, |\tilde{B}(t)| \leq \beta \) and \( |\tilde{C}(t)| \leq \beta \), \( \beta > 0 \), and the matrices \( A_0, A_1, B_0, B_1, C_0, C_1, \) and \( C_2 \) are the same as those given in Example 1. Then, we have the closed-loop system as

\[
\dot{x}(t) = A_{Cl}(t),
\]

where

\[
\begin{align*}
A_{Cl} &= A_0 + k(t)E_1 + k(t)E_2 + \tilde{k}(t)E_3 \\
&+ (k(t) - k_0)A_1 + \tilde{A}(t) + k(t)C_1 + B(t)C(t) \\
&+ \tilde{B}(t)(C(t) + k(t)(B(t)C(t) + B_1C(t)) \\
&+ k(t)B_0C(t) \\
&= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + k(t) \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} + k(t) \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \\
&+ k(t) \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} + k(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \tilde{A}(t) \\
&+ \tilde{B}(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \tilde{C}(t) \\
&+ k(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \tilde{B}(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tilde{C}(t) \\
&+ k(t)\tilde{B}(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\end{align*}
\]

Using the sufficient condition (7) with the 2-norm-based matrix measure, the spectral norm and \( a = 1 \), we can get that the closed-loop system (36) is asymptotically stable, if \( k(t), k_0(t) \) and \( \tilde{k} \) satisfy the following conditions

\[
\begin{align*}
1.4861 k(t) + 4.3311 k(t) \tilde{k}(t) + 5.4051 k(t) \tilde{k}(t) + 8.2346 k(t) \tilde{k}(t) + 2 k(t) < 1, \\
&\text{for } k(t), k_0(t) > 0; \quad (36.\text{a}) \\
1.4861 k(t) + 4.3311 k(t) \tilde{k}(t) - 2.4051 k(t) \tilde{k}(t) + 8.2346 k(t) \tilde{k}(t) + 2 k(t) < 1, \\
&\text{for } k(t), k_0(t) < 0. \quad (36.\text{b})
\end{align*}
\]

The sufficient conditions in Eq. (36) give the explicit relationship of the bounds on \( k(t), k_0(t) \) and \( \tilde{k} \) for ensuring that the closed-loop system (36) is asymptotically stable.

4. **Conclusions**

In this paper, the stability robustness of linear continuous/discrete-time systems with output feedback controllers as well as both time-varying structured and unstructured uncertainties is investigated by directly considering the mixed quadratically-coupled uncertainties in the problem formulation. Based on the Lyapunov approach and some essential
properties of matrix measures, two new sufficient conditions is proposed for ensuring that the linear output feedback systems with both time-varying structured and unstructured parameter uncertainties is asymptotically stable. Although the analysis results are derived for linear systems using constant output feedback gain matrix, they apply equally well to linear systems using dynamic output feedback controllers. Three numerical examples for continuous-time and discrete-time systems are given to illustrate the application of the presented sufficient conditions, and for the case of only considering time-varying structured parameter uncertainties, the proposed sufficient conditions are shown to be less conservative than those of Su and Yong and and Tseng et al.

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Appendix : Proof of Theorem 2

Let the Lyapunov function candidate be \( V(x(t)) = x(t)^T P_a x(t) \), where \( P_a \) is the positive definite solution of Eq. (16). From Eqs. (13), (17)–(21), the time difference of the Lyapunov function \( V(x(t)) \) along the state trajectory of system (5) can be obtained as

\[
\Delta V(x(t)) = x(t)^T \left[ A_d^T P_a A_d + P_d + \sum_{j=1}^{m} k_i(t) \sum_{j=1}^{m} k_j(t) \right] x(t) \\
+ E_i^T P_a E_i + \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) \left( A_d^T P_a E_i + E_i^T P_a A_d \right) \\
+ E_i^T P_a E_r + \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) \left( A_d^T P_a E_r + E_r^T P_a A_d \right) \\
+ E_r^T P_a E_d + \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) \left( A_d^T P_a E_d + E_d^T P_a A_d \right) \\
+ \sum_{j=1}^{m} k_i(t) F^T(t) P_d E_i + \sum_{j=1}^{m} k_i(t) k_j(t) F^T(t) P_d E_i \\
+ F^T(t) P_d F(t) \] 

where

\[
A_d = \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
+ \sum_{j=1}^{m} \sum_{j=1}^{m} k_i(t) k_j(t) k_i(t) k_j(t) P_{air} \\
\[
\begin{align*}
&+ \sum_{j=1}^{n} \sum_{j=1}^{n} \mu(k_j(t)k_j'(t)(P_{d_{ij}} + P_{d_{ji}} + P_{d_{ij}} + P_{d_{ji}})) \\
&+ \sum_{j=1}^{n} \sum_{j=1}^{n} \mu(k_j(t)k_j'(t)(P_{d_{ij}} + P_{d_{ji}} + P_{d_{ij}} + P_{d_{ji}})) \\
&+ \sum_{j=1}^{n} \mu(k_j(t)k_j'(t)(M_{d_{ij}} + \mu(\beta_1))) \\
&+ \mu(\beta_1) \left( \sum_{j=1}^{n} k_j(t)k_j'(t) + \sum_{j=1}^{n} k_j'(t) \phi_{d_{ij}} \right) \\
&+ \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} k_j(t)k_j'(t) + \sum_{j=1}^{n} k_j'(t) \phi_{d_{ij}} \\
&+ \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} k_j(t)k_j'(t) + \sum_{j=1}^{n} k_j'(t) \phi_{d_{ij}} \\
&+ \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} k_j(t)k_j'(t) + \sum_{j=1}^{n} k_j'(t) \phi_{d_{ij}} \\
&+ \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} k_j(t)k_j'(t) \phi_{d_{ij}} + \mu(\beta_1) \\
&+ \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} k_j(t)k_j'(t) \phi_{d_{ij}} + \sum_{j=1}^{n} k_j'(t) \phi_{d_{ij}} \\
&+ \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} k_j(t)k_j'(t) \phi_{d_{ij}} + \sum_{j=1}^{n} k_j'(t) \phi_{d_{ij}} \\
&+ \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} k_j(t)k_j'(t) \phi_{d_{ij}} + \sum_{j=1}^{n} k_j'(t) \phi_{d_{ij}} \\
&+ \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} k_j(t)k_j'(t) + \sum_{j=1}^{n} k_j'(t) \phi_{d_{ij}} + \beta_1 < 0.
\end{align*}
\]

This implies that \( \Delta V(x(t)) < 0 \). So, we have proved that the uncertain discrete-time system (5) is asymptotically stable if the inequality (15) is satisfied. Thus, the proof is completed. \( \text{Q.E.D.} \)

References


