General Form of Steady Response and Periodic Solution for an Impact Oscillator Having No Sticking*

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In this paper, we propose a method for obtaining the strict solution of a steady response and the periodic solution by which only the impact time of the impact oscillator system was assumed to be a parameter for which no sticking occurred. First, the response conversion which is the derivation operation of a steady response is defined. Two methods are shown to exist for the deriving process of a steady response, depending on the order of application of the response conversion and the "reference value expansion". By these methods, a general form of the periodic solution can be obtained from a steady response. In addition, the results obtained by these two methods show good agreement on the most fundamental periodic solution. Finally, the validity of the proposed method is confirmed using a numerical simulation example.

Key Words: Nonlinear Vibration, Forced Vibration, Piecewise Linear Oscillator, Global Representation, Self Reference Form, Steady Response, Response Steady Transform

1. Introduction

Generally, a steady response is a bounded solution that has the property whereby the solution is invariant over time. In the stable case, the solution agrees with the state of the system after an infinite amount of time has passed. For the problem concerning the vibration of a mechanical system, determining the steady response as the state that the system will eventually reach is important to determine the dynamic characteristics of the system for a practical design procedure. In particular, the periodic solution is the most basic form of steady response solution. In nonlinear oscillation theory, a great deal of research has been

devoted to finding this solution. However, rigorous determination of the steady response of a nonlinear oscillating system is generally impossible, and determining even the periodic solution requires the use of an approximation method. The situation is the same for a piecewise linear oscillating system, which has the special property that rigorous solutions can be found for each segment. In this paper, these solutions are referred as "local solutions".

We have shown that a steady response that is not accompanied by sticking, which is a solution that stops on the boundary between segments, can be rigorously determined throughout a segment by parameterizing the impact time of successive local solutions for the simplest piecewise linear system such as a prepressured spring mass system and a Coulomb friction oscillating system⁽¹⁾. In addition, the feedback superposition method has been proposed as a method in order to determine the general form of a periodic solution for these systems and for an impact oscillator system⁽²⁾. However, in the case of an impact oscillator system, the general form of a steady response or a periodic solution cannot be found by parameterizing only the impact time.

In the present paper, we first propose a method

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for rigorously determining the steady response that is not accompanied by sticking, by parameterizing only the impact time. For this purpose, we define a response steady transform as a means for deriving a steady response; for deriving the steady response, two methods are available, depending on the order of application of two processes; this conversion process and the "reference value expansion process" which was previously proposed⁽⁹⁾. Furthermore, we show that two methods exist by which to determine the general form of the steady response and periodic solution using only the impact time as a parameter. This is accomplished by using the reference value expansion and by solving the system of simultaneous linear equations found in relation to the periodic impact velocity. Both methods are shown to agree with respect to the most basic periodic solution and the appropriateness of these methods is confirmed using numerical calculations.

2. System Description and Global Representation of the General Solution

2.1 Equations of motion

The nondimensionalized equations of motion for the impact oscillator system considered in this paper are as follows.

$$\ddot{x} + 2\xi\dot{x} + \beta x = \eta \cos(\omega t); \quad x < \gamma \quad (1.a)$$

$$\dot{x}(t^+) = -r\dot{x}(t^-); \quad x = \gamma \quad (1.b)$$

Equation (1) indicates that the system is equivalent to a linear oscillator system between successive impacts and that at the instant of collision with a rigid wall at $x = \gamma$ (referred to below as the impact boundary) the mass bounces back with a velocity multiplied by the restitution coefficient $0 < r < 1$. Equation (1) can be combined into the equivalent feedback system expressed by Eq. (2)⁽⁹⁾.

$$\ddot{x} + 2\xi\dot{x} + \beta x = - \sum_{g_i \in Q} (1+r)\dot{x}(t_{g_i}^-)\delta(t-t_{g_i}) + \eta \cos \omega t; \quad x \leq \gamma \quad (2)$$

Here, Q is the set of impact times t_i ($i \in Z$) for which the solution becomes $x(t_i) = \gamma$, and g_i ($i \in Z$) is a sequence of impact times belonging to Q . Equation (2) shows that the nonlinearity of the system due to impact can be replaced by the impulsive force input $-(1+r)\dot{x}(t_i^-)\delta(t-t_i)$ that acts at the impact time t_i .

2.2 Global representation of the general solution in the case of no sticking

We have shown that in the case of a prepressured spring mass system, by shifting the initial value to a time that is in the infinite past, the steady response can be determined from the global representation of the general solution⁽¹⁾. In order to apply a similar procedure to an impact oscillator system, the initial time is shifted from t_0 to t_{-n} at n collisions ago.

Where we introduce the notation

$$t_{i-1} < t_i \leq \bar{t}_i \leq \bar{t}_{i+1} < t_{i+1} \quad (3)$$

for the impact times. Then the general solution, in the case of no sticking, for the initial values $x(t_{-n})$ and $\dot{x}(t_{-n})$ during the impact interval $t_{-n} \leq t \leq t_m$ (we denote this interval I_{t_{-n}, t_m}) can be determined globally by the following equation⁽⁴⁾.

$$x_{t_{-n}, t_m}(t) = u(t - t_{-n})\mathcal{S}_{t_{-n}}(t) - \mathcal{N}_{t_{-n}, t_m}(t) \quad (4.a)$$

$$\mathcal{N}_{t_{-n}, t_m}(t) = (1+r) \sum_{i=1-n}^{m-1} u(t-t_i)\mathcal{S}(t-t_i)\dot{x}_{t_{-n}, t_i}(t_i^-) \quad (4.b)$$

The term $\mathcal{S}_{t_{-n}}(t)$ in Eq. (4) is the response of the base linear system for the initial values $x(t_{-n})$ and $\dot{x}(t_{-n})$, and can be expressed as follows:

$$\mathcal{S}_{t_{-n}}(t) = \mathcal{S}(t - t_{-n})\{x(t_{-n}) - \mathcal{S}(t_{-n})\} + \mathcal{S}(t - t_{-n})\{\dot{x}(t_{-n}) - \dot{\mathcal{S}}(t_{-n}) + \mathcal{S}(t)\} \quad (5)$$

In this Eq. (5), the state transition functions $\mathcal{S}(t)$, $\dot{\mathcal{S}}(t)$, and the input response function $\mathcal{S}(t)$ are defined by:

$$\mathcal{S}(t) = e^{-\xi t} \left(\cos \Omega t + \frac{\xi}{\Omega} \sin \Omega t \right) \quad (6)$$

$$\dot{\mathcal{S}}(t) = e^{-\xi t} \frac{1}{\Omega} \sin \Omega t \quad (7)$$

$$\mathcal{S}(t) = \frac{\eta}{q} \cos(\omega t + \kappa) \quad (8)$$

The symbols Ω and q used here are defined as:

$$\Omega = \sqrt{\beta - \xi^2}, \quad \kappa = \tan^{-1} \left(\frac{-2\xi\omega}{\beta - \omega^2} \right), \\ q = \sqrt{(\beta - \omega^2)^2 + (2\xi\omega)^2}$$

The term $-\mathcal{N}_{t_{-n}, t_m}(t)$ in Eq. (4) is the sum of impulse responses to the impulsive force inputs $-(1+r)\dot{x}(t_i^-)\delta(t-t_i)$ that occur in sequence at the impact times $t_{1-n} < \dots < t_{m-1}$. This function expresses the total effect of nonlinearity due to the impacts. $u(x)$ is the unit step function defined by the following equation.

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (9)$$

Equation (4) is recursive, with the impact velocity referring to itself at the instance of impact, so that this general solution is referred to as the recursive general solution⁽³⁾. The term $\dot{x}_{t_{-n}, t_i}(t_i^-)$ in Eq. (4) is calculated by substitution up to $i=1$ using the relationship:

$$\dot{x}_{t_{-n}, t_i}(t_i^-) = \dot{\mathcal{S}}_{t_{-n}}(t_i) - \dot{\mathcal{N}}_{t_{-n}, t_i}(t_i^-) \quad (10)$$

and then expanding and rearranging using the fact that $t_{1-n} < \dots < t_{m-1}$, giving the following expression for impact velocity, which is not explicitly recursive:

$$x_{t_{-n}, t_m}(t) = u(t - t_{-n})\mathcal{S}_{t_{-n}}(t) - (1+r) \sum_{k_1=1-n}^{m-1} u(t-t_{k_1})\mathcal{S}(t-t_{k_1}) \\ \times \left[\sum_{i=1}^{k_1+n} \left\{ \prod_{j=1}^{i-1} \sum_{\substack{k_{j+1}=i-j-n \\ k_j > k_{j+1}}}^{k_j-1} -(1+r)\mathcal{S}(t_{k_j}-t_{k_{j+1}}) \right\} \right. \\ \left. \times \dot{\mathcal{S}}_{t_{-n}}(t_{k_i}) \right] \quad (11)$$

The series of operations through which Eq. (4) is

transformed into Eq.(11) is called a reference value expansion⁽³⁾ and is expressed by $\mathcal{O}_d$. Equation (11) is called the general solution of hierarchical parameter form. By transforming the subscript parameters according to $k=k_1$, $p=i$, $q_j=k_{p-(j-1)}$ ($1 \leq j \leq p$), Eq.(11) can be transformed as follows:

$$\begin{aligned} x_{\overline{t-n}, \overline{t_m}}(t) &= u(t - \overline{t-n}) \mathcal{S}_{\overline{t-n}}(t) - (1+r) \\ &\times \sum_{k=1-n}^{m-1} u(t - t_k) \mathcal{S}(t - t_k) \\ &\times \left[\sum_{p=1}^{k+n} \sum_{j=1}^{p-1} \sum_{\substack{q_j=j-n \\ q_{j+1}>q_j}}^{p-j} \left\{ \prod_{j=1}^{p-1} - (1+r) \mathcal{S}(t_{q_{j+1}} - t_{q_j}) \right\} \right. \\ &\times \left. \mathcal{S}_{\overline{t-n}}(t_{q_1}) \right] \\ &= u(t - \overline{t-n}) \mathcal{S}_{\overline{t-n}}(t) - (1+r) \\ &\times \sum_{k=1-n}^{m-1} u(t - t_k) \mathcal{S}(t - t_k) \\ &\times \left[\sum_{p=1}^{k+n} \sum_{1-n \leq q_1 < \dots < q_{p-1} \leq k-1 < k=q_p} \left\{ \prod_{j=1}^{p-1} - (1+r) \mathcal{S}(t_{q_{j+1}} - t_{q_j}) \right\} \mathcal{S}_{\overline{t-n}}(t_{q_1}) \right] \end{aligned} \quad (12)$$

The sum between the brackets on the right-hand side of Eq.(12) is taken over $p-1$ impact times out of the total of $k+n-1$ impact times from $1-n$ to $k-1$, and Eq.(12) is called the general solution of normal parameter form. In this study, resolved solutions given in Eqs.(11) and (12) are generically referred to as "general solution of reference value expansion form". Hereafter, the symbol ($\hat{}$) will be used to indicate an expression that has been derived through a reference value expansion.

3. General Form of the Steady Response

Hereafter, the steady response is expressed as $\mathcal{S}_\infty(t)$. The method previously described for deriving the steady response of a prepressured spring mass system has the initial value and final value of the general solution converted to a global representation shifted to the time in the infinite past and the time in the infinite future, respectively⁽¹⁾. This is equivalent to taking the limits of the general solution $x_{\overline{t-n}, \overline{t_m}}$, given in Eq.(13).

$$m, n \rightarrow +\infty \quad (13.a)$$

$$\mathcal{S}_{\overline{t-n}}(t) \rightarrow \mathcal{S}(t) \quad (13.b)$$

Using Eq.(13a), the response of a linear system becomes:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathcal{S}_{\overline{t-n}}(t) &= \mathcal{S}_\infty(t) \\ &= \mathcal{S}(t - t_\infty) \{x(t_\infty) - \mathcal{S}(t_\infty)\} \\ &\quad + \mathcal{S}(t - t_\infty) \{\dot{x}(t_\infty) - \dot{\mathcal{S}}(t_\infty)\} + \mathcal{S}(t) \\ &= \mathcal{S}(t) \end{aligned} \quad (14)$$

so that the indeterminate terms occurring in $\mathcal{S}_\infty(t)$, including the initial value $x(t_\infty)$, $\dot{x}(t_\infty)$ and the initial input response $\mathcal{S}(t_\infty)$, $\dot{\mathcal{S}}(t_\infty)$, are annihilated by the damping effect, with the result that the derivation of

Eq.(13b) is justified. If this limiting operation is applied to the general solution of reference value expansion form, an explicit expression for the steady response with only the impact time as a parameter can be obtained. Here, we assumed that $t_\infty = -\infty$, $t_\infty = \infty$, and that the sequence of impact times does not converge to a fixed value. This condition is equivalent to the condition that the solution does not converge to a stuck state after chattering.

However, in the double sum in brackets on the right-hand side of Eq.(12), $1-n \leq q_1$, so that when the limit $t_n \rightarrow -\infty$ ($n \rightarrow +\infty$) is taken, q_1 becomes asymptotically $-\infty$ as n approaches $+\infty$. This means that the possibility exists that $\mathcal{S}_{\overline{t-n}}(t_{q_1})$ cannot decrease sufficiently for t_{q_1} , for which $t_n \rightarrow -\infty$, $t_{q_1} \rightarrow -\infty$ ($-n < q_1$, $n \rightarrow +\infty$). Consequently, although the response term of the base linear system of the first term on the right-hand side of the general solution (12) becomes:

$$u(t - \overline{t-n}) \mathcal{S}_{\overline{t-n}}(t) \rightarrow \mathcal{S}(t) \quad (n \rightarrow +\infty) \quad (15)$$

when the limit (13) is taken, whether

$$\mathcal{S}_{\overline{t-n}}(t_{q_1}) \rightarrow \mathcal{S}(t_{q_1}) \quad (n \rightarrow +\infty) \quad (16)$$

will hold in the pseudo feedback response term is unclear.

This point can be interpreted as follows. From the pseudo feedback point of view, the global solution is expressed as the sum of the response $\mathcal{S}_{\overline{t-n}}(t)$ and the impulse response sequence $\mathcal{N}_{\overline{t-n}, \overline{t_m}}(t)$ formed by feedback velocity to the base linear system that originated in the impact. Therefore, when the response $\mathcal{S}_{\overline{t-n}}(t)$ converges to $\mathcal{S}(t)$, all pseudo-feedback response terms related to $\mathcal{S}_{\overline{t-n}}(t_{q_1})$ that appear in $\mathcal{N}_{\overline{t-n}, \overline{t_m}}(t)$ must be replaced by the corresponding value of $\mathcal{S}(t)$. Thus, in the process of derivation of the steady response, the substitution of (16) in the pseudo-feedback response term of Eq.(12) is confirmed. In addition, for the case in which Eq.(13) is applied to Eq.(11), a similar process to that mentioned above can be applied. In this procedure, the limit operation of Eq.(13) for the general solution is carried out, and the transformation used to determine the steady response is defined as the steady response conversion $\hat{\mathcal{O}}_s$.

Based on the above argument, when the steady response conversion of Eq.(13) is applied to Eq.(11), the following equation is obtained as the general form of the steady response.

$$\begin{aligned} \mathcal{S}_\infty(t) &= \mathcal{S}(t) - (1+r) \sum_{k_1=-\infty}^{+\infty} u(t - t_{k_1}) \mathcal{S}(t - t_{k_1}) \\ &\times \left[\sum_{i=1}^{+\infty} \left\{ \prod_{j=1}^{i-1} \sum_{\substack{k_{j+1}=-\infty \\ k_j > k_{j+1}}}^{k_j-1} - (1+r) \mathcal{S}(t_{k_j} - t_{k_{j+1}}) \right\} \mathcal{S}(t_{k_i}) \right] \end{aligned} \quad (17)$$

In addition, when a parameter conversion similar to that used to obtain Eq.(12) from Eq.(11) is applied to Eq.(17), the following expression is obtained for the

steady response.

$$\begin{aligned} \mathcal{X}_\infty(t) = & \mathcal{S}(t) - (1+r) \sum_{k=-\infty}^{+\infty} u(t-t_k) \mathcal{S}(t-t_k) \\ & \times \left[\sum_{p=1}^{+\infty} -\infty \leq q_1 < \dots < q_{p-1} \leq k-1 < k = q_p \right. \\ & \left. \times \left[\prod_{j=1}^{p-1} -(1+r) \mathcal{S}(t_{q_{j+1}} - t_{q_j}) \right] \mathcal{S}(t_{q_1}) \right] \quad (18) \end{aligned}$$

When the steady response conversion $\mathcal{O}_s$ is applied directly to the recursive type general solution (4), the following equation is obtained.

$$\mathcal{X}_\infty(t) = \mathcal{S}(t) - (1+r) \sum_{i=-\infty}^{+\infty} u(t-t_i) \mathcal{S}(t-t_i) \mathcal{X}_\infty(t_i^-) \quad (19)$$

When the reference value expansion $\mathcal{O}'_d$ is applied to Eq. (19), we can obtain:

$$\begin{aligned} \mathcal{X}_\infty(t) = & \mathcal{S}(t) - (1+r) \sum_{i=-\infty}^{+\infty} u(t-t_i) \mathcal{S}(t-t_i) \mathcal{X}_\infty(t_i^-) \\ = & \mathcal{S}(t) - (1+r) \sum_{k_1=-\infty}^{+\infty} u(t-t_{k_1}) \mathcal{S}(t-t_{k_1}) \\ & \times [\mathcal{S}(t_{k_1}) - (1+r) \sum_{k_2=-\infty}^{k_1-1} \mathcal{S}(t_{k_1} - t_{k_2}) \\ & \times [\mathcal{S}(t_{k_2}) - (1+r) \sum_{k_3=-\infty}^{k_2-1} \mathcal{S}(t_{k_2} - t_{k_3}) \\ & \vdots \\ & \times [\mathcal{S}(t_{k_i}) - (1+r) \sum_{k_{i+1}=-\infty}^{k_i-1} \mathcal{S}(t_{k_i} - t_{k_{i+1}}) \\ & \vdots \quad (20) \end{aligned}$$

Equation (20) is rearranged to obtain Eq. (17), which reconfirms the appropriateness of the replacement of $\mathcal{X}_\infty(t)$ by $\mathcal{S}(t)$ in the process of the derivation of the steady response. Thus, the steady response conversion and the reference value expansion are transformable. The process of deriving the steady response (17) obtained from the reference value expansion of the recursive general solution (4) is described using a commutative diagram (21).

$$\begin{array}{ccc} \mathcal{X}_{t-n, t_m}(t) & \xrightarrow{\mathcal{O}_s} & \mathcal{X}_\infty(t) \\ \mathcal{O}_d \downarrow & & \downarrow \mathcal{O}'_d \\ \hat{\mathcal{X}}_{t-n, t_m}(t) & \xrightarrow{\hat{\mathcal{O}}_p} & \hat{\mathcal{X}}_\infty(t) \end{array} \quad (21)$$

In order for Eqs. (17), (18) and (19) to be realized as actual steady responses, the following condition must be satisfied:

$$\mathcal{X}_\infty(t) < \gamma \quad (t_i < t < t_{i+1} \quad (i \in \mathbb{Z})) \quad (22)$$

Since the inequality (22) cannot be analytically evaluated, this inequality must be verified numerically. Equations (17) and (18) completely govern all steady responses when no sticking occurs in the system under the condition of inequality (22). Consequently, the responses that are expressed by these equations can be recognized to contain not only periodic solutions but also chaotic solutions. For the method of expressing the general solution based on the connecting method, a traditional representation method of a general solution for piecewise linear systems, explicit expression of the steady response is

impossible. Therefore, herein, Eqs. (17) and (18) are referred to for the canonical steady response.

4. General Form of the Periodic Solution

The periodic solution when an external force exerted through n periods ($T = 2\pi n/\omega$) causes m collisions, with the impact boundary at $x = \gamma$, is expressed as $\mathcal{X}_m^n(t)$. Here, it should be noted that m and n (which are included in N) indicate quantities other than those expressed by the symbols in Eq. (4). When the condition for the formation of a periodic solution, specifically the condition in which the impact time is periodic, is applied to the general form of the steady response determined in section 3, the periodic solution is derived using the feedback superposition method⁽²⁾. This operation is called a periodic feedback superposition transformation. In this section, we will discuss the rigorous method for deriving the periodic solution based on the above-described method.

4.1 Derivation of a periodic solution of reference value expansion form

When the feedback superposition method is directly applied to the reference value expansion type steady response, evaluation of an infinite sum and an infinite product becomes necessary. Therefore, determination of the general form of the periodic solution via this method is unlikely. However, just as in the process of deriving the steady response, the steady response conversion and the reference value expansion are transformable. Therefore, in the process of deriving the periodic solution, the periodic feedback superposition transformation and the reference value expansion can be shown as below:

$$\begin{array}{ccc} \mathcal{X}_\infty(t) & \xrightarrow{\mathcal{O}_p} & \mathcal{X}_m^n(t) \\ \mathcal{O}'_d \downarrow & & \downarrow \mathcal{O}''_d \\ \hat{\mathcal{X}}_\infty(t) & \xrightarrow{\hat{\mathcal{O}}_p} & \hat{\mathcal{X}}_m^n(t) \end{array} \quad (23)$$

Here $\mathcal{O}'_d$ and $\mathcal{O}''_d$ express reference value expansions, and $\mathcal{O}_p$ and $\hat{\mathcal{O}}_p$ express the periodic feedback superposition transformations. By successively applying the periodic feedback superposition transformation $\mathcal{O}_p$ and the reference value expansion $\mathcal{O}''_d$ to Eq. (19), the periodic solution $\mathcal{X}_m^n(t)$ is determined as $\mathcal{O}''_d \circ \mathcal{O}_p(\mathcal{X}_\infty)$.

If we assume that the periodic solution $\mathcal{X}_m^n(t)$ exists, then the following equation holds with regard to the impact time:

$$t_{im+j} = t_j + iT \quad (0 \leq j \leq m-1, i \in \mathbb{Z}) \quad (24)$$

Thus, Eq. (19) can be rewritten as follows:

$$\begin{aligned} \mathcal{X}_m^n(t) = & \mathcal{S}(t) - (1+r) \sum_{k=-\infty}^{+\infty} \sum_{i=0}^{m-1} u(t-t_i-kT) \\ & \times \mathcal{S}(t-t_i-kT) \mathcal{X}_m^n(t_i+kt) \\ = & \mathcal{S}(t) - (1+r) \sum_{k=-\infty}^{+\infty} \sum_{i=0}^{m-1} u(t-t_i-kT) \\ & \times \mathcal{S}(t-t_i-kT) \mathcal{X}_m^n(t_i^-) \end{aligned} \quad (25)$$

Here, we use

$$\dot{\mathcal{X}}_m^n(t_i^- + kT) = \dot{\mathcal{X}}_m^n(t_i^-) \quad (26)$$

because $\mathcal{X}_m^n(t)$ is assumed to be the periodic solution having period T . When the feedback superposition method⁽²⁾ is applied to Eq.(25) and, with i in the double sum term fixed, we first evaluate the sum related to k , which can be given by the following equations:

$$\begin{aligned} & \sum_{k=-\infty}^{+\infty} u(t-t_i-kT) \mathcal{F}(t-t_i-kT)|_{t \in [t_0, t_0+T]} \\ &= \chi_{[t_0, t_0+T]}(t) \sum_{k=-\infty}^0 u(t-t_i-kT) \mathcal{F}(t-t_i-kT) \\ &= \chi_{[t_0, t_0+T]}(t) \sum_{k=0}^{+\infty} u(t-t_i+kT) \mathcal{F}(t-t_i+kT) \\ &= \chi_{[t_0, t_0+T]}(t) \sum_{k=1}^{+\infty} \mathcal{F}(t-t_i+kT) \\ &+ \chi_{[t_0, t_0+T]}(t) \sum_{k=0}^{+\infty} \mathcal{F}(t-t_i+kT) \\ &= \chi_{[t_0, t_0+T]}(t) \sum_{k=0}^{+\infty} \mathcal{F}(t-t_i+T+kT) \\ &+ \chi_{[t_0, t_0+T]}(t) \sum_{k=0}^{+\infty} \mathcal{F}(t-t_i+kT) \quad (27) \\ &= \chi_{[t_0, t_0+T]}(t) Q(t-t_i+T) + \chi_{[t_0, t_0+T]}(t) Q(t-t_i) \quad (28) \end{aligned}$$

In the process of deriving Eq.(28) from Eq.(27), the following relationship was used.

$$\begin{aligned} \sum_{k=0}^{+\infty} \mathcal{F}(\tau+kT) &= \frac{1}{\Omega} e^{-\zeta\tau} \sum_{k=0}^{+\infty} (e^{-\zeta T})^k \sin(\Omega\tau + k\Omega T) \\ &= \frac{1}{\Omega} e^{-\zeta\tau} \frac{\sin \Omega\tau - e^{-\zeta T} \sin \Omega(\tau-T)}{1-2e^{-\zeta T} \cos \Omega T + e^{-2\zeta T}} \\ &= p^{-1} \{ \mathcal{F}(\tau) - e^{-2\zeta T} \mathcal{F}(\tau-T) \} \quad (29) \end{aligned}$$

Also, in the process of deriving the sum in Eq.(29), the formula⁽⁵⁾

$$\sum_{i=0}^{+\infty} a^i \sin(x+i\theta) = \frac{\sin x - a \sin(x-\theta)}{1-2a \cos \theta + a^2} \quad (|a| < 1)$$

was used. It is noted that, in the expression of Eqs. (28) and (29), the following simplifications are used:

$$p \equiv 1 - 2e^{-\zeta T} \cos \Omega T + e^{-2\zeta T} \quad (30)$$

$$Q(t) = p^{-1} \{ \mathcal{F}(t) - e^{-2\zeta T} \mathcal{F}(t-T) \} \quad (31)$$

By applying the floor function $\lfloor x \rfloor$, which gives the largest integer that does not exceed x , Eq.(25) can be transformed into a recursive type periodic solution such as

$$\begin{aligned} \mathcal{X}_m^n(t) &= \mathcal{F}(t) - (1+r) \sum_{i=0}^{m-1} Q\left(t-t_i - T \left\lfloor \frac{t-t_i}{T} \right\rfloor\right) \\ &\times \dot{\mathcal{X}}_m^n(t_i^-) \quad (32) \end{aligned}$$

The m element functions that make up the sum over i in Eq.(32) are defined as the basic function. Then, applying a reference value expansion to Eq.(32) yields:

$$\begin{aligned} \mathcal{X}_m^n(t) &= \mathcal{F}(t) - (1+r) \sum_{k_1=0}^{m-1} Q\left(t-t_{k_1} - T \left\lfloor \frac{t-t_{k_1}}{T} \right\rfloor\right) \\ &\times \left[\dot{\mathcal{F}}(t_{k_1}) - (1+r) \sum_{k_2=0}^{m-1} Q\left(t_{k_1}-t_{k_2} - T \left\lfloor \frac{t_{k_1}-t_{k_2}}{T} \right\rfloor\right) \right. \\ &\times \left[\dot{\mathcal{F}}(t_{k_2}) - (1+r) \sum_{k_3=0}^{m-1} Q\left(t_{k_2}-t_{k_3} - T \left\lfloor \frac{t_{k_2}-t_{k_3}}{T} \right\rfloor\right) \right. \\ &\vdots \end{aligned}$$

$$\begin{aligned} &\times \left[\dot{\mathcal{F}}(t_{k_1}) - (1+r) \sum_{k_{i+1}=0}^{m-1} Q\left(t_{k_i}-t_{k_{i+1}} - T \left\lfloor \frac{t_{k_i}-t_{k_{i+1}}}{T} \right\rfloor\right) \right] \\ &\vdots \quad (33) \end{aligned}$$

Rearranging the terms in Eq.(33) yields

$$\begin{aligned} \mathcal{X}_m^n(t) &= \mathcal{F}(t) - (1+r) \sum_{k_1=0}^{m-1} Q\left(t-t_{k_1} - T \left\lfloor \frac{t-t_{k_1}}{T} \right\rfloor\right) \\ &\times \left[\sum_{i=1}^{+\infty} \left(\prod_{j=1}^{i-1} \sum_{k_{j+1}=0}^{m-1} - (1+r) Q(t_{k_j}-t_{k_{j+1}} \right. \right. \\ &\left. \left. + T u(-(t_{k_j}-t_{k_{j+1}}))) \right) \right] \dot{\mathcal{F}}(t_{k_1}) \quad (34) \\ &= \mathcal{F}(t) - (1+r) \sum_{k=0}^{m-1} Q\left(t-t_k - T \left\lfloor \frac{t-t_k}{T} \right\rfloor\right) \\ &\times \left[\sum_{i=1}^{+\infty} \sum_{0 \leq q_1, \dots, q_{i-1} \leq m-1} \left(\prod_{j=1}^{i-1} - (1+r) Q(t_{q_j}-t_{q_{j+1}} \right. \right. \\ &\left. \left. + T u(-(t_{q_j}-t_{q_{j+1}}))) \right) \right] \dot{\mathcal{F}}(t_{q_1}) \quad (35) \end{aligned}$$

Here, we have used the fact that the following relationship holds:

$$\begin{aligned} &Q\left(t_{k_j}-t_{k_{j+1}} - T \left\lfloor \frac{t_{k_j}-t_{k_{j+1}}}{T} \right\rfloor\right) \\ &= \begin{cases} Q(t_{k_j}-t_{k_{j+1}}+T) & ; t_{k_j}+kT \leq t_{k_{j+1}}+kT \\ Q(t_{k_j}-t_{k_{j+1}}) & ; t_{k_j}+kT > t_{k_{j+1}}+kT \end{cases} \\ &= Q(t_{k_j}-t_{k_{j+1}}+T(-(t_{k_j}-t_{k_{j+1}}))) \quad (36) \end{aligned}$$

The sum enclosed in brackets in Eq.(35) is the sum over all combinations extracted from $p-1$ impact times, with duplication permitted, out of the m impact times $t_0 < \dots < t_{m-1}$.

Equations (34) and (35) completely govern all periodic solutions of the impact oscillator system in the case of no sticking with the sum of the basic functions having the steady response of the basic linear system $\mathcal{F}(t)$ and the m impact times that appear as parameters in the sum. In the previously described method of determining a periodic solution by the connecting method, only the impact times and the impact velocity that become the solution are given. However, a direct expression for the periodic solution in the form of a function of time cannot be obtained. These solutions are referred to as canonical periodic solutions. Since, $\mathcal{X}_m^n(t)$ exists only in the parameter ranges in which the infinite sum and infinite product inside the brackets in Eq.(35) converge, these expressions can be used to provide formulas for evaluating the conditions of existence of the periodic solution.

4.2 Derivation of the periodic solution using the periodic impact velocity matrix

Equations (34) and (35) require an infinite number of sums and products in order to express the periodic solution. Consequently, although they are well suited to theoretical analysis, using such expressions is not necessarily a useful strategy for determining actual periodic solutions. Therefore, let us now discuss a method of expressing a periodic solution

using only a finite number of procedures.

The periodic solution reaches the impact boundary at times $t = t_i$ ($0 \leq i \leq m-1$). Applying this condition to the recursive periodic solution (32) yields the following equations:

$$\begin{aligned} & \sum_{i=0}^{j-1} \dot{Q}(t_j - t_i) \dot{x}_m^n(t_i^-) + \dot{Q}(T) \dot{x}_m^n(t_j^-) \\ & + \sum_{i=j+1}^{m-1} \dot{Q}(t_j - t_i + T) \dot{x}_m^n(t_i^-) \\ & = \frac{\mathcal{S}(t_j) - \gamma}{1+r} \quad (0 \leq j \leq m-1) \end{aligned} \quad (37)$$

Here, if we take:

$$\mathcal{A}_m^n = \begin{pmatrix} \dot{Q}(T) & \cdots & \dot{Q}(t_0 - t_{m-1} + T) \\ \vdots & \ddots & \vdots \\ \dot{Q}(t_{m-1} - t_0) & \cdots & \dot{Q}(T) \end{pmatrix} \quad (38)$$

the system of simultaneous linear equations (37) for the periodic impact velocity $\dot{x}_m^n(t_0^-), \dots, \dot{x}_m^n(t_{m-1}^-)$ can be rewritten as the following matrix equation:

$$\begin{pmatrix} \dot{Q}(T) & \cdots & \dot{Q}(t_0 - t_{m-1} + T) \\ \vdots & \ddots & \vdots \\ \dot{Q}(t_{m-1} - t_0) & \cdots & \dot{Q}(T) \end{pmatrix} \cdot \begin{pmatrix} \dot{x}_m^n(t_0^-) \\ \vdots \\ \dot{x}_m^n(t_{m-1}^-) \end{pmatrix} = \frac{1}{1+r} \begin{pmatrix} \mathcal{S}(t_0) - \gamma \\ \vdots \\ \mathcal{S}(t_{m-1}) - \gamma \end{pmatrix} \quad (39)$$

The periodic impact velocity is given, as a solution to the above equation, by the following equation:

$$\dot{x}_m^n(t_i^-) = \frac{1}{1+r} \frac{\det(\mathcal{A}_m^n(j))}{\det(\mathcal{A}_m^n)} \quad (40)$$

Here, $\det(\cdot)$ indicates a determinant, and $\mathcal{A}_m^n(j)$ is an adjoint matrix in which the j th column of $\mathcal{A}_m^n$ is replaced by the right-hand side of Eq. (39). Consequently, the following equation is obtained as a new expression for the general form of the periodic solution.

$$\begin{aligned} \dot{x}_m^n(t) &= \mathcal{S}(t) - \sum_{i=0}^{m-1} \dot{Q}\left(t - t_i - T \left\lfloor \frac{t - t_i}{T} \right\rfloor\right) \\ &\times \frac{\det(\mathcal{A}_m^n(j))}{\det(\mathcal{A}_m^n)} \end{aligned} \quad (41)$$

If the periodic extension of the period T of the derivative $\dot{Q}(t)$ is expressed as $\dot{Q}(t - T \lfloor t/T \rfloor)$, the derivative of the periodic solution (32) is given by

$$\dot{x}_m^n(t) = \mathcal{S}(t) - \sum_{i=0}^{m-1} \dot{Q}\left(t - t_i - T \left\lfloor \frac{t - t_i}{T} \right\rfloor\right) \dot{x}_m^n(t_i^-) \quad (42)$$

Since the values at $t = t_i$ ($0 \leq i \leq m-1$) in the above equation agree with the periodic impact velocity, the following equations are obtained:

$$\sum_{i=0}^{j-1} \dot{Q}(t_j - t_i) \dot{x}_m^n(t_i^-) + \left(\frac{1}{1+r} + \dot{Q}(T)\right) \dot{x}_m^n(t_j^-)$$

$$\begin{aligned} & + \sum_{i=j+1}^{m-1} \dot{Q}(t_j - t_i + T) \dot{x}_m^n(t_i^-) \\ & = \frac{\mathcal{S}(t_j)}{1+r} \quad (0 \leq j \leq m-1) \end{aligned} \quad (43)$$

Here, if we let:

$$\mathcal{B}_m^n = \begin{pmatrix} \frac{1}{1+r} + \dot{Q}(T) & \cdots & \dot{Q}(t_0 - t_{m-1} + T) \\ \vdots & \ddots & \vdots \\ \dot{Q}(t_{m-1} - t_0) & \cdots & \frac{1}{1+r} + \dot{Q}(T) \end{pmatrix} \quad (44)$$

then, from Eq. (43), the following system of simultaneous linear equations is obtained:

$$\begin{pmatrix} \frac{1}{1+r} + \dot{Q}(T) & \cdots & \dot{Q}(t_0 - t_{m-1} + T) \\ \vdots & \ddots & \vdots \\ \dot{Q}(t_{m-1} - t_0) & \cdots & \frac{1}{1+r} + \dot{Q}(T) \end{pmatrix} \cdot \begin{pmatrix} \dot{x}_m^n(t_0^-) \\ \vdots \\ \dot{x}_m^n(t_{m-1}^-) \end{pmatrix} = \frac{1}{1+r} \begin{pmatrix} \mathcal{S}(t_0) \\ \vdots \\ \mathcal{S}(t_{m-1}) \end{pmatrix} \quad (45)$$

Therefore, the periodic impact velocity can also be expressed as

$$\dot{x}_m^n(t_i^-) = \frac{1}{1+r} \frac{\det(\mathcal{B}_m^n(j))}{\det(\mathcal{B}_m^n)} \quad (46)$$

Here, $\mathcal{B}_m^n(j)$ is an adjoint matrix formed by replacing the j th column of $\mathcal{B}_m^n$ by the right-hand side of Eq. (45). Consequently, the following expression, which is of the same form as the general form of the periodic solution Eq. (41), is obtained.

$$\begin{aligned} \dot{x}_m^n(t) &= \mathcal{S}(t) - \sum_{i=0}^{m-1} \dot{Q}\left(t - t_i - T \left\lfloor \frac{t - t_i}{T} \right\rfloor\right) \\ &\times \frac{\det(\mathcal{B}_m^n(j))}{\det(\mathcal{B}_m^n)} \end{aligned} \quad (47)$$

From Eqs. (40) and (46), the periodic impact velocity exists in the parameter region in which the matrices $\mathcal{A}_m^n$ and $\mathcal{B}_m^n$ are regular. These matrices are thus referred to as the periodic impact velocity matrices. These matrices can be conveniently utilized to provide an evaluation formula under the condition of existence of a periodic solution.

5. General Form of the Periodic Solution for the Case $m=1$, and Example of Numerical Calculation

The case $m=1$ is considered to be the most basic periodic solution $\dot{x}_1^n(t)$ that occurs after one impact due to an external force acting over n periods. In this case only, the impact time t_0 can be expressed explicitly, as in the following equation.

$$t_0(\pm) = \frac{1}{\omega} \tan^{-1} \left(\frac{\gamma U \{(\beta - \omega^2)V + (2\xi\omega)U\} \pm \{-(\beta - \omega^2)U + (2\xi\omega)V\}\sqrt{D}}{\gamma U \{(\beta - \omega^2)U - (2\xi\omega)V\} \pm \{(\beta - \omega^2)V + (2\xi\omega)U\}\sqrt{D}} \right) \quad (48)$$

Here D , U , and V express

$$D \equiv \frac{\eta^2}{q^2} (U^2 + V^2) - \gamma^2 U^2, \quad V \equiv \omega(1+r)\mathcal{S}(T), \quad U \equiv 1 - \mathcal{S}(T) - r[p - \{1 - \mathcal{S}(T)\}] \quad (49)$$

Substituting $m=1$ into the canonical periodic solution (35), yields

$$\begin{aligned} \mathcal{X}_1^n(t) &= \mathcal{S}(t) - (1+r) \mathcal{Q} \left(t - t_0 - T \left\lfloor \frac{t-t_0}{T} \right\rfloor \right) \\ &\times \left[\sum_{i=1}^{+\infty} \sum_{q_1, \dots, q_i=0} \left\{ \prod_{j=1}^{i-1} - (1+r) \mathcal{Q}(t_{q_{j+1}} - t_{q_j}) \right. \right. \\ &\quad \left. \left. + Tu(-(t_{q_{j+1}} - t_{q_j})) \right\} \dot{\mathcal{S}}(t_{q_1}) \right] \\ &= \mathcal{S}(t) - (1+r) \mathcal{Q} \left(t - t_0 - T \left\lfloor \frac{t-t_0}{T} \right\rfloor \right) \\ &\times \left[\sum_{i=1}^{+\infty} \left\{ \prod_{j=1}^{i-1} - (1+r) \mathcal{Q}(T) \right\} \dot{\mathcal{S}}(t_0) \right] \\ &= \mathcal{S}(t) - (1+r) \mathcal{Q} \left(t - t_0 - T \left\lfloor \frac{t-t_0}{T} \right\rfloor \right) \\ &\times [1 + \{-(1+r)\} \mathcal{Q}(T) + \{-(1+r)\}^2 \mathcal{Q}^2(T) \\ &\quad + \dots] \dot{\mathcal{S}}(t_0) \\ &= \mathcal{S}(t) - (1+r) \mathcal{Q} \left(t - t_0 - T \left\lfloor \frac{t-t_0}{T} \right\rfloor \right) \\ &\times \frac{\dot{\mathcal{S}}(t_0)}{1 + (1+r) \mathcal{Q}(T)} \quad (50) \end{aligned}$$

From Eq.(50), it is recognized that $\mathcal{X}_1^n(t)$ can be given only in the region under the convergence condition $|(1+r)\mathcal{Q}(T)| < 1$. When m is set equal to in the recursive type general solution (32), $\mathcal{X}_1^n(t)$ is given by

$$\mathcal{X}_1^n(t) = \mathcal{S}(t) - (1+r) \mathcal{Q} \left(t - t_0 - T \left\lfloor \frac{t-t_0}{T} \right\rfloor \right) \mathcal{X}_1^n(t_0^-) \quad (51)$$

From Eq.(46), the periodic impact velocity $\dot{\mathcal{X}}_1^n(t_0^-)$ is given by

$$\dot{\mathcal{X}}_1^n(t_0^-) = \frac{\dot{\mathcal{S}}(t_0)}{1 + (1+r) \mathcal{Q}(T)} \quad (52)$$

By substituting Eq.(52) into Eq.(51), the general form of the periodic solution $\mathcal{X}_1^n(t)$, which can be expressed by the determinant form of the periodic impact velocity determinant, agrees with the result of Eq.(50). From Eq.(52), $\mathcal{X}_1^n(t)$ can be calculated among the parameter region whereby $(1+r)\mathcal{Q}(T) \neq -1$ is satisfied. From the above discussion, it is confirmed that the canonical periodic solution for the case $m=1$ agrees with the periodic solution expressed by the periodic impact velocity matrix.

Figure 1 shows an example of calculation of the periodic solution $\mathcal{X}_1^2(t)$ for the impact oscillator system having parameters of $\gamma=1$, $\omega=3$, $\eta=10$, $\xi=0.1$, $r=0.7$ and $n=2$. From Eqs.(48) and (52), the time and the velocity at the time of impact and that immediately before impact for $m=1$ can be determined, respectively, as:

$$\begin{aligned} t_0(+) &\simeq 0.155475 \dots, \\ \dot{\mathcal{X}}_1^2(t_0(+)) &\simeq 4.549013 \dots, \\ t_0(-) &\simeq 1.042533 \dots, \\ \dot{\mathcal{X}}_1^2(t_0(-)) &\simeq -0.537070 \dots \end{aligned}$$

However, $\dot{\mathcal{X}}_1^2(t_0(-)) < 0$ violates the physical condition that the velocity immediately before impact must be nonnegative. Therefore, we can discuss only $t_0(+)$ and $\dot{\mathcal{X}}_1^2(t_0(+))$. Substituting the above impact time

Fig. 1 Periodic solution $\mathcal{X}_1^2(t)$

$t_0(+)$ into the general form of the periodic solution (50) and superimposing the steady response of the base linear system $\mathcal{S}(t)$ ((b) Fig. 1) and the basic functions (c) yields (a). This solution agrees with that obtained using the connecting method with initial values of $\mathcal{X}_1^2(t_0(+)) = \gamma$ and $\dot{\mathcal{X}}_1^2(t_0(+)) = -r \dot{\mathcal{X}}_1^2(t_0(-))$. From the above discussion, it is concluded that the result obtained for the canonical periodic solution can be appropriately calculated for the periodic solution $\mathcal{X}_1^2(t)$ for the case of $m=1$.

6. Conclusions

We have proposed theoretical methods by which the steady response and the periodic solution of an impact oscillator system can be rigorously calculated. Major results obtained in this study are summarized as below.

1. From the recursive type general solution, we obtained the general form of the steady response using the reference value expansion and the steady response conversion, which can be expressed taking only the impact time as a parameter. This solution completely governs all steady responses in the case of no sticking that occur in an impact oscillator system.

2. By applying, in sequence, a periodic feedback superposition transformation and a reference value expansion to the recursive type periodic solution, we achieved the general form of the periodic solution.

3. By expressing the periodic impact velocity as the solution of a system of simultaneous linear equations

and substituting this solution into the recursive type periodic solution, we obtained an expression for the periodic solution that requires only a finite number of operations.

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