Stability Analysis of High Speed Railway Vehicles*

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The lateral mathematical model of a high speed vehicle system with 17 degrees of freedom is set up and the nonlinearities arising due to suspension parameters and wheel/rail interactions are considered. The coupler traction force in tension and compression cases which is the function of vehicle speed and position in the trainset is also taken into account in the model. The critical speed at the Hopf bifurcation point on straight and circular curved tracks are studied by utilizing efficient numerical methods. The influence of track curve radius and superelevation on the critical speed is investigated.

Key Words: Railway Vehicle, Stability, Critical Speed, Coupler Force

1. Introduction

The stability is an important dynamic problem for railway vehicle system that determines the maximum operating speeds of vehicles on tracks. Instability must be avoided for any vehicle at its normal operating speed range otherwise severe hunting oscillation will occur and that will worsen the dynamic performance of the vehicle system or even cause derailment. Therefore, the nonlinear stability problems of railway vehicles have been paid high attention by researchers(1)–(8) and also the stability problems of some unconventional bogies are carefully studied(9)–(11).

In the past, researchers carried out a lot of work on the stability studies of railway vehicles running on straight track, but paid little attention to the stability problems of vehicles on curved tracks(12). In the vehicle system model, normally the traction force of coupler is not taken into account. The purpose of this paper is to apply efficient numerical methods to study the nonlinear stability problems of high speed vehicles running on straight and circular curved tracks by consideration of the coupler traction force. The critical speeds of the vehicle in different position in the trainset with coupler traction force in tension or compression are studied and compared with the results of single vehicle system without consideration of coupler traction force. The influences of the curve radius and the superelevation of the track on the stability of the vehicle system are also investigated.

2. Mathematical Model

2.1 Nonlinear wheel/rail contact geometry

The wheel/rail contact geometry parameters shown as Fig. 1 are considered as nonlinear functions of wheelset lateral displacement $y_w$, which include the wheel rolling radius ($r_1, r_2$), transverse radius of wheel profile ($R_1, R_2$), contact angle ($\delta_1, \delta_2$), wheelset roll angle ($\phi_w$) and transverse radius of rail profile ($R_{T1}, R_{T2}$). Because the profiles of wheel and rail can have any arbitrary shapes, the geometry parameters are difficult to be expressed as explicit functions of the lateral displacement $y_w$. Thus, the nonlinear functions are described as a tabular form in terms of $y_w$. The intermediate values of the contact parameters are calculated using linear interpolation.

2.2 Nonlinear wheel/rail interactive forces

Each wheel experiences lateral, longitudinal and spin creepages that are defined as relative motions between wheel and rail. The nonlinear expressions of the longitudinal, lateral and spin creepages for the left wheel ($j = 1$) and right wheel ($j = 2$) can be written as

![Fig. 1 Wheel/rail contact geometry parameters](image-url)
\[
\begin{align*}
\xi_{ij} &= 1 - r_j/r_0 + (-1)^j \omega \psi_0/v - (-1)^j a/R \\
\xi_{ij} &= (\gamma_0 - v \psi_0) \cos(\delta_j - (-1)^j \phi_0) \\
\xi_{ij} &= (-1)^j \sin(\delta_j - (-1)^j \phi_0)/r_0 + (\psi_0/v - 1)/R
\end{align*}
\]

where \(v\) is the vehicle forward speed, \(R\) denotes the curve radius of the track and \(r_0\) is the wheel nominal rolling radius.

Firstly according to Kalker’s linear creep theory, the longitudinal creep force \(F_{c x j}\), lateral creep force \(F_{c y j}\) and spin creep moment \(M_{c z j}\), which are defined in the wheel/rail contact point axes \(0_x'x' y'z'_{ij}\) shown in Fig. 2, can be taken as the linear functions of creepages. The relationships between creep forces and creepages are given as

\[
\begin{align*}
F_{c x j} &= -f_{11} \xi_{sj} \\
F_{c y j} &= -f_{22} \xi_{sj} - f_{23} \xi_{sj} \\
M_{c z j} &= f_{23} \xi_{sj} - f_{33} \xi_{sj}
\end{align*}
\]

where \(f_{11}, f_{22}, f_{23}\) and \(f_{33}\) are the creep coefficients which depend on the wheel/rail contact geometry parameters, instantaneous normal contact force \(F_{c n j}\), and materials of wheel and rail. In order to calculate the wheel/rail creep forces more accurately, modifications for the creep forces in Eq. (2) can be made according to Shen-Hedrick-Elkins’ nonlinear creep force model(13). Then the lateral forces (in \(y\)), vertical forces (in \(z\)) and yaw moment (about \(z\)) acting on the wheelset by the rails in the wheelset axes \(0x'y'z'_{ij}\) (Fig. 2) can be derived according to the creep forces, creep moments and wheel/rail normal contact forces through coordinate transformation between axes \(0_x'x' y'z'_{ij}\) and \(0x'y'z_{ij}\).

2.3 Centrifugal force on curved track

When the vehicle travels on a curved track at a constant forward speed \(v\), each body in the vehicle system also moves at speed \(v\) along the track and there is centrifugal force acting on each body. Because the outer rail of the track is superelevated relative to the inner rail with superelevation \(h\) or angle \(\alpha\), the centrifugal force can be partly balanced by the gravitational force of the body. The forces acting on each body are shown as Fig. 3. The unbalanced centrifugal force in the lateral direction (in \(y\)) can be obtained as

\[F_{gi} = m_i \frac{v^2}{R} \cos \alpha - m_i g \sin \alpha\]

where \(m_i\) \((i = 1 - 7)\) indicates the mass of the \(i\)th body, \(a\) is the half of lateral distance between wheel/rail contact points and \(h\) or \(\alpha\) is the superelevation value or cant angle of the outer rail.

2.4 Coupler traction force

The traction force of coupler of each vehicle in the trainset is related to the vehicle mass, forward speed, position of the vehicle, mechanical resistance and air resistance. It is assumed that all vehicles in the trainset have the same mass, then the traction force of the \(i\)th coupler which is between the vehicle no. \(i\) and vehicle no. \(i+1\) can be described by the following expression(14)

\[F_i = (n - i)a_1 m + [(n - i)a_2 m + a_3] v + [(n - i - 1)b_1 + b_2] v^2\]

where \(m\): vehicle mass
\(n\): number of vehicles
\(v\): vehicle forward speed
\(a_j\): mechanical resistance coefficient, \(j = 1, 2, 3\)
\(b_j\): air resistance coefficient of middle vehicles

Thus, the lateral components of the coupler traction force caused by the carbody lateral, roll and yaw motions and the track curve radius can be derived, and which can be added in the equations of motions of vehicle system.

2.5 Equations of motion of vehicle system

A high speed railway vehicle is taken as an example to set up its nonlinear mathematical model. The schematic diagram of the vehicle system is shown as Fig. 4. The vehicle is considered as a multiple rigid body system com-
posed of four wheelsets, two bogie frames, one carbody, and primary and secondary suspensions. The nonlinear wheel/rail interactive forces in the vehicle system are considered in the model. The coordinate system of each body in the vehicle system is chosen to have its origin at the center of gravity of the body and moves along the track center line at a constant forward speed \(v\) in the \(x\) direction. The \(y\) axis is parallel to the track plane and directed upward. It is assumed that the lateral and vertical vibrations of the system are uncoupled, thus only the lateral vibration of the system needs to be taken into consideration for the stability analysis. The total number of degrees of freedom of the vehicle system is 17, they are lateral (in \(y\)) and yaw (about \(z\)) motions of each wheelset; lateral, roll (about \(x\)) and yaw motions of each bogie frame and carbody. The only nonlinearity considered in the model are the nonlinear wheel/rail contact geometry and nonlinear wheel/rail interactive forces that are described in section 2.1 and 2.2. Then the equations of motion of the vehicle system can be derived and expressed as the following vector form.

\[
M \ddot{y} + C \dot{y} + K y = F_e(y, y) + F(y) + F_g
\]

where \(y \in \mathbb{R}^{17}\) is coordinate vector and \(M, C, K\) are \(17 \times 17\) diagonal mass, damping matrix and stiffness matrix respectively. \(F_e\) denotes the nonlinear force vector consisting of the wheel/rail interactive forces, \(F\) the vector of component of the coupler traction force on carbody and \(F_g\) the unbalanced centrifugal force vector.

Through coordinate transformation, \(x = (\dot{y}, y)^T\), the second order differential equation (5) can be rewritten as the following state space form

\[
\frac{dx}{dt} = f(x)
\]

The main parameter values of the high speed vehicle system are listed in the Appendix.

3. Hopf Bifurcation and Limit Cycle

3.1 Calculation of equilibrium position

The stability problem of the railway vehicle on circular curved tracks is more complicated than that on straight track because of the unknown equilibrium position of the system, while the equilibrium position of the vehicle on straight track is known to be \(y = 0\). Hence, for the vehicle running on curved tracks, the equilibrium point should be firstly determined, and then the eigenvalues of the system in the equilibrium position can be calculated in order to determine the stability of the vehicle system.

By setting the velocity and acceleration vectors \(\dot{y} = \ddot{y} = 0\) in Eq. (5), the vehicle steady state curving equations can be obtained as below

\[
H(y) = Ky - F_e(y) + F(y) - F_g = 0
\]

By solving the above nonlinear algebraic equations using Newton-Raphson iteration, the equilibrium position of the vehicle system can be obtained as \(y = y'\).

3.2 Determination of Hopf bifurcation point

According to the stability theory, the stability of nonlinear system (6) in the neighbourhood of its equilibrium point can be determined by its linearized equation

\[
\frac{dx}{dt} = Ax
\]

where \(A\) is the Jacobian matrix of the system, \(A = \left[\frac{\partial f_i}{\partial x_j}\right]_{x=0,y=y'}\) \((i, j = 1, 2, \ldots, 34)\). The stability of the equilibrium position of nonlinear system (6) can be determined by the eigenvalues of the Jacobian matrix \(A\). If all the eigenvalues have negative real parts, the system is asymptotically stable. While if there is at least one eigenvalue having positive real part, then the system becomes unstable. When the maximum real part turns to be zero, then we say the vehicle system comes to a critical state. Usually for railway vehicles, there always appears a pair of complex conjugate eigenvalues passing through the imaginary axis when gradually increasing the speed, then we say Hopf bifurcation occurs and the vehicle speed at the Hopf bifurcation point is defined as the linear critical speed \(v_l\) of the system.

The occurrence of Hopf bifurcation satisfies the following condition\(^{(15)}\)

\[
\begin{cases}
\lambda_{1,2} = \alpha(v_l, R, h) \pm i \omega(v_l, R, h) \\
\alpha(v_l, R, h) = 0 \quad \omega(v_l, R, h) > 0 \\
d\alpha(v_l, R, h)/dt \neq 0
\end{cases}
\]

where \(\lambda_{1,2}\) is the pair of conjugate eigenvalues.

3.3 Calculation of limit cycles

The linear critical speed can only be used to describe the local stability behaviour around the neighbourhood of the equilibrium position of the vehicle system. In order to better understand the stability characteristics of the vehicle system in a large area, the limit cycle oscillations emerged via the Hopf bifurcation should be investigated. The variable-step fourth-order Runge-Kutta method is employed to solve the nonlinear differential equation (6) to obtain the trajectories of the system, and the Poincaré map is used to find the limit cycles.

To construct a Poincaré map in space \(R^34\), the trajectory of system (6) is intercepted with hyperplane defined by

\[
\sum: = \{x|x_1 = 0, x_{18} > 0\}
\]

where \(x_1\) and \(x_{18}\) indicate the lateral velocity and displacement of the leading wheelset respectively. Then the Poincaré map consists of those points in the hyperplane for which the trajectory penetrates with the same sense. A limit cycle of the vehicle system which means a fixed point in the hyperplane can be found if it satisfies the condition

\[
||X^{(k)} - X^{(k+1)}||_2 < \varepsilon
\]

where \(X^{(k)}\) is for the \(k\)-th penetration.
with the hyperplane and $\varepsilon$ is a small positive value. The period $T$ of the limit cycle can be determined by the time spent from $X^{(k)}$ to $X^{(k+1)}$. The first limit cycle can be found by choosing $v = v_0$. The relation of limit cycles with respect to speed $v$ is determined through numerical integrations by always using the final condition for the integration of the former limit cycle as the initial condition for solving the next limit cycle. Then the turning point can be found at the speed where the limit cycle dies out to the equilibrium position. Normally the turning point speed that is defined as nonlinear critical speed $v_n$ is lower than the Hopf bifurcation point speed $v_h^{(1)-(4)}$, therefore it should be taken as the speed limitation for vehicles operating on tracks.

4. Numerical Simulations

A high speed trainset with one locomotive and ten trailer cars travelling on straight track and circular curved tracks is taken as an application example. The locomotive is located at the front of the trainset with pull or push conditions. The critical speed of each trailer car, numbered as No.1 to No.10 from front (starting from the locomotive) to rear in the trainset, is investigated through numerical calculations in consideration of the effect of the coupler traction force.

The limit cycle oscillations of the leading wheelset versus speed $v$ for the single vehicle on straight track and curved track with radius of curvature $R = 4000$ m and outer rail superelevation $h = 120$ mm are shown as Figs. 5 and 6. The solid lines in the figures indicate the amplitudes of stable limit cycles and stable equilibrium positions, while the dashed lines indicate the unstable equilibrium positions. Point A is the Hopf bifurcation point, while point B where the first stable limit cycle appears is the turning point. It is obvious that there must exist unstable limit cycles between point A and point B which can not be obtained directly through numerical integrations. Therefore the Hopf bifurcation is subcritical. The amplitude of the bifurcated limit cycle increases as the raising of vehicle speed. The nonlinear critical speed on straight track is $v_n = 428$ km/h while the linear critical speed is $v_l = 477$ km/h. The nonlinear critical speed on curved tracks with $R = 4000$ m and $h = 120$ mm is 320 km/h, whereas the corresponding linear critical speed is 334 km/h. It is known from the bifurcation diagrams of Figs. 5 and 6 that the vibrations of the nonlinear vehicle system possess hysteresis effects, which means that the vehicle system suddenly undergoes a jump from stable equilibrium to limit cycle orbit when vehicle speed $v$ is increased past the speed value $v_l$ of Hopf bifurcation point A, and then jumps back from stable limit cycle orbit to stable equilibrium when vehicle speed $v$ is decreased past the speed value $v_n$ of turning point B. For the speed values in the interval $v_n \leq v \leq v_l$ we have a bistable situation which means the jump to higher amplitude can occur before $v_l$ is reached if the vehicle system is subjected to sufficiently large track perturbations. Because the nonlinear critical speed $v_n$ is lower than the linear critical speed $v_l$, so it should be taken as the speed limitation for the vehicle running on track.

Figures 7 and 8 shows the linear and nonlinear critical speeds of each vehicle in the trainset running on straight track with coupler traction force in tension or compression, i.e., the trainset is in pull or push condition. No.0 in the figures indicates the single vehicle without considering the coupler traction force. It can been seen from the figures that the critical speeds between single vehicle (car No.0), middle vehicles (car No.4 and No.7), front and rear vehicles (car No.1 and No.10) on straight track have little difference. The critical speeds of the vehicles of the trainset in push condition are a little higher than the vehicles in pull condition. That is to say that the coupler force does not very much affect the stability of the vehicle system on straight track because the lateral components of coupler force caused by carbody vibrations are small. It is also known that the difference between linear and nonlinear critical speeds of each car on straight track is large to be around 50 km/h.

For the trainset running on the circular curved track with curve radius $R = 4000$ m and outer rail superelevation $h = 120$ mm, the calculating results of the linear and nonlinear critical speeds are illustrated in Figs. 9 and 10.
It is known that the critical speeds of the vehicles in the trainset in pull condition are higher than the vehicles in push condition on the circular curved track. The middle vehicles have higher critical speeds than the front and rear vehicles for pull condition, but have lower critical speeds for push condition. It can be seen from the figures that the coupler force has a strong influence on the stability of the vehicle system on circular curved track because the coupler force on carbody has large lateral component that is mainly caused by the curve radius.

The influences of curve radius $R$ with $h = 120$ mm and superelevation $h$ with $R = 4000$ m on the linear and nonlinear critical speeds of the vehicle system are shown as Figs. 11 – 14. The trainset is in the pull condition. It can be seen that the linear and nonlinear critical speeds of the vehicle system increase as the curve radius $R$ and outer rail superelevation $h$ increase. The vehicles in the middle of the trainset (car No.4 and No.7) usually have higher critical speeds on curved track, while the front and rear vehicles have lower critical speeds. The reason maybe is that...
the middle cars have coupler constrains by both ends. The single vehicle without considering coupler traction force (car No.0) has lower critical speed for curve radius below 6000 m compared with other vehicles of the trainset. It is clearly indicated from the figures that the curve radius and outer rail superelevation have tremendous effects on the stability of the vehicle systems. This is because the stability of the vehicle system is related to the creep forces between wheels and rails, and the creep forces on straight track and different curved tracks are quite different. When the vehicle is running at a certain speed, the smaller the curve radius and outer rail superelevation are, and the larger the lateral displacement of the wheelset equilibrium positions. Then large lateral wheelset displacement will cause large effective conicity on wheel and rail contact point that will reduce the vehicle system stability.

5. Conclusion

The loss of stability of the equilibrium position of the vehicle system can happen not only on straight track, but also on large radius curved tracks. Limit cycle oscillations appear both on straight track and on curved tracks due to Hopf bifurcation. The traction force of coupler has little influence on the critical speed of the vehicle in the

trainset on straight track, but has a certain effect for the vehicle on curved track. The vehicles in the trainset have different critical speeds for pull or push condition. The critical speed of the vehicle in the trainset increases as the track curve radius and outer rail superelevation increase. The nonlinear critical speeds of the vehicle on straight track and curved tracks are smaller than the linear critical speeds, therefore the nonlinear critical speed should be taken as the limit speed when the vehicle runs on tracks.

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6. Appendix: Main Parameter Values

<table>
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<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>$M_c$</td>
<td>Cabody mass: 34 000 kg</td>
</tr>
<tr>
<td>$M_b$</td>
<td>Bogie frame mass: 3 000 kg</td>
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<tr>
<td>$M_w$</td>
<td>Wheelset mass: 1 400 kg</td>
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<tr>
<td>$I_{c_3}$</td>
<td>Cabody roll inertia: 75 060 kgm$^2$</td>
</tr>
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<td>$I_{c_2}$</td>
<td>Cabody yaw inertia: 2 086 000 kgm$^2$</td>
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<tr>
<td>$I_{b_3}$</td>
<td>Bogie frame roll inertia: 2 260 kgm$^2$</td>
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<td>$I_{b_2}$</td>
<td>Bogie frame yaw inertia: 3 160 kgm$^2$</td>
</tr>
<tr>
<td>$I_w$</td>
<td>Wheelset yaw inertia: 915 kgm$^2$</td>
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<td>$k_{px}$</td>
<td>Primary longitudinal stiffness per axle: 10 000 kN/m</td>
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<td>$k_{py}$</td>
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<td>$k_{pc}$</td>
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<td>Height of wheelset C.G. to primary supers.: 0.15 m</td>
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<td>$A$</td>
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<td>$B$</td>
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<td>$D_s$</td>
<td>Secondary suspension lateral semi-spacing: 1.21 m</td>
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<td>$R_0$</td>
<td>Wheel rolling radius: 0.4575 m</td>
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References


