Variable Structure Model Following Control of Robot Manipulators with High-Gain Observer∗

Chih-Jer LIN∗∗

The conventional sliding mode control motion has two phases, i.e., the reaching phase and the sliding phase. However, the invariance of the sliding mode control only applies to the sliding phase on the sliding surface. In the reaching phase, the switching control is applied to provide the robustness but there usually exists chattering. The conventional continuous approximation method can alleviate chattering but spoil the robustness. In this paper, a sliding mode control with global invariance is studied to solve the above problem. On the other hand, velocity sensors are often omitted to reduce the cost and reject measuring noise. In this paper, a Variable-Structure Model Following Control with output-feedback is proposed to solve tracking problems with modelling uncertainties and external disturbances.

Key Words: Sliding Mode Control, Chattering, Robot Manipulator, Invariance, Trajectory Tracking Problem, High-Gain Observer

1. Introduction

Tracking control of robot manipulators has attracted much attention in the last two decades. To achieve good tracking in the presence of uncertainties and disturbance, many control approaches have been developed, such as adaptive control(1)–(5), nonlinear state feedback linearization(6),(7), variable structure control method(8),(9), learning control(10) and feed-forward method. Sliding mode control of dynamical systems has a long history of theoretical and practical development since the 1970’s(8). One of the advantages of sliding mode control is its invariance against parametric uncertainties and external disturbances. However, it also introduces actuator chattering phenomenon that should be avoided in many physical systems, such as servo control systems, structure vibration control systems and robotic systems.

The sliding mode control algorithms consist of enforcing the states in some manifolds in system spaces. Traditionally, the manifolds are constructed by the intersection of some hyper surfaces, which is called switching plane. The order of the motion equation in the conventional sliding motion is equal to \( n-m \) with \( n \) being the dimension of the state space and \( m \) the dimension of the control input. Therefore, it results that the sliding mode control motion has two phases, i.e., the reaching phase and the sliding phase. The invariance of the sliding mode control only applies to the sliding phase on the switching plane. During the reaching phase, the system is sensitive to parameter uncertainty and disturbance. The robustness during the reaching phase is improved by high-gain feedback control, however, it also introduces chattering phenomenon and arises stability problems.

On the other hand, the concept of a dynamic sliding mode control with global invariance is studied by many researchers(11)–(14). In 1992, Chan and Yao(7) proposed a method of global linearization of the nonlinear mechanical system. This approach can establish a globally linearized model by introducing a linear dynamic compensator. In 1993, Hebert(12) proposed a dynamical sliding mode control using Fließ’s generalized controller canonical form(11) of the nonlinear system. This approach can be applied to a nonlinear dynamical system control provided that the internal dynamics of the system are stable. However, Sira-Ramirez’s approach may result higher order dynamics for control input, which is not necessary in general. An approach(13) to reduce the effect of reaching phase dynamics on control performance is to make the sliding mode occur while the control is applied. However, for some applications, the above approach may fail to eliminate chattering with an auxiliary control input associated with the plant uncertainty. In 1996, Utkin and Shi(14) proposed a concept of Integral Sliding Mode, and the order of the motion equation is equal to the dimension...
of the state space. Therefore, the robustness of the system can be guaranteed throughout an entire response from the initial time instance. However, the velocity measurements are needed in this approach, and the velocity signals obtained by tachometers are often contaminated by noise. It may reduce the dynamic performance of the systems.

In robotic control, velocity sensors are often omitted to reduce the cost and weight. The problem of velocity measurements can be solved by using observers to estimate joint velocities from position measurements. Nonlinear observers such as sliding observers and high-gain observers have been used to estimate joint velocities for robots. The sliding observer has been studied and modified for the propose approach. An idea to overcome peaking is suggested by Esfandiari and Khalil (1992), it applies the saturated control inputs to prevent peaking from affecting the states. This approach has been applied to adaptive output feedback control of nonlinear systems (Khalil 1997).

In this paper, a dynamical sliding mode control with output feedback is studied to the trajectory control of robot manipulators with uncertainties and disturbances. First, a variable structure model-following control strategy using a dynamic sliding mode is proposed, and the order of the sliding motion equation is equal to the dimension of the state space. Based on the proposed method, the resulting dynamic sliding surface across the initial state is applied to robotic tracking control problems. Without reaching phase, a global invariance is achieved by the proposed algorithm. Second, the concept of a high-gain observer is studied and modified for the propose approach.

2. Robot Model and Problem Statement

The dynamic equations of an $n$-link robot manipulator can be written as,

\[ M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + G(q) + F \dot{q} = \tau, \]

where $q$, $\dot{q}$, $\ddot{q}$ are the joint position, velocity and acceleration vectors; $M(q) \in \mathbb{R}^{n \times n}$ is a positive-definite inertia matrix; $C(q, \dot{q}) \in \mathbb{R}^{n \times 2n}$ is a matrix containing the Coriolis and centrifugal terms; $G(q) \in \mathbb{R}^{n}$ is a vector of gravitational terms; $F \in \mathbb{R}^{n \times 2}$ is a positive definite diagonal matrix of frictional terms (Coulomb friction), $\tau \in \mathbb{R}$ is the input control vector.

If the desired system states are available, the desired angle, angular velocity, and angular acceleration vectors can, respectively, be denoted by $\dot{q}_d$, $\ddot{q}_d$, $\dddot{q}_d$. The error dynamic equations for the robot manipulator of (1) can be represented in the state variable form as follows:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \ddot{q} - \dddot{q}_d \\
&= -\ddot{q}_d - M^{-1}(x_1, \dot{q}_d)[(C(q, \dot{q}_d, \ddot{q}_d) + F)(x_2 + \dot{q}_d) \\
&\quad + G(q, \dot{q}_d)] + M^{-1}(x_1, \dot{q}_d)\tau,
\end{align*}
\]

where $x_1 = e = q - \dot{q}_d$, $x_2 = \dot{e} = \ddot{q} - \dddot{q}_d$; and $x = [x_1, x_2]^T$.

Using the Computed Torque Method based on the model without perturbation, the required joint torque for the tracking control will be

\[
\tau = M(e + \dot{q}_d)(\ddot{q}_d - K_e e - K_p \dot{e})
\]

\[+ C(e + q_d, \dot{e} + \dot{q}_d)(\dot{e} + \dot{q}_d) + G(e + \dot{q}_d) + F(\dot{e} + \ddot{q}_d) \]

where $K_p$ and $K_e$ are constant diagonal $n \times n$ gain matrices with $k_{p}$ and $k_{e}$ on the diagonals. Without considering external disturbances and modelling uncertainties, the closed-loop equation can be obtained from (1) and (3) as

\[
\dot{e} + K_e \dot{e} + K_p e = 0.
\]

Remark 1:

The tracking problem of robot manipulator can be considered as a model-following problem. The resulting closed-loop equation of (5) is the reference model of the error dynamic of the robot manipulator. Without external disturbances and modelling uncertainties, the error state will follow the model of Eq. (5) if the control of Eq. (3) is applied.

Remark 2:

For the variable structure control (VSC) method, the switching plane is a specified invariant manifolds in the system space. The sliding mode control algorithms drive the error states into this invariant manifolds, and then the error states are enforced to follow the sliding mode dynamics. Traditionally, the switching plane can be represented as:

\[
S(e, \dot{e}) = \dot{e} + \Lambda e = 0,
\]

where $\Lambda$ can be chosen according to the desired system specifications. However, the switching plane of Eq. (6) is a reduced-order subspace of the error space. By choosing the constant matrix $\Lambda$, it cannot guarantee that the initial error state $(\epsilon(0), \dot{e}(0))$ is on the switching surface of Eq. (6), or the system is unstable. Therefore, there exists a reaching phase motion.

To make the sliding mode dynamics has the same order as the original system and a sliding-mode control with global invariance is described in the following section.
3. Variable Structure Model Following Control

Consider the tracking problem of robot manipulator as a model-following problem. Let the following model, i.e. the switching plane, be introduced as follows.

\[ \dot{X} = \begin{bmatrix} 0 & I \\ -K_p & -K_v \end{bmatrix} X, \]  

(7)

where \( X = [e, \dot{e}]^T \) and \( X(0) = [e(0), \dot{e}(0)]^T \).

The model of Eq. (7) is determined by the initial error state \( (e(0), \dot{e}(0)) \) and the specified parameters \( K_p, K_v \); the states determined by the Eq. (7) are the reference states of the error dynamic for the robotic system. Without external disturbances and modeling uncertainties, the error states will follow the model of Eq. (7) if the equivalent control law of the sliding-mode controller is chosen as follows.

\[ \tau_{eq} = M(q)(\dot{q}_d - K_v \dot{e} - K_p e) + C(q, \dot{q})\dot{q} + G(q) + F_0 \dot{q}. \]

(8)

With \( M(q) \neq 0 \), the closed-loop dynamics can be obtained as

\[ \ddot{e} + K_v \dot{e} + K_p e = 0 \]

(10)

using the control law (8).

It means that the equivalent control (8) could make the error dynamics of the system, which is described in Eq. (2), follow the desired model of the switching plane (7).

After integration of the Eq. (7), the switching surface could be chosen as:

\[ s(e, \dot{e}) = (\dot{e}(t) - \dot{e}(0)) + K_v(e(t) - e(0)) + K_p \int_0^t e(t) \, dt \]

\[ = (\dot{e}(t) + K_v e(t)) + K_p \int_0^t e(t) \, dt - (\dot{e}(0) + K_v e(0)) \]

(11)

From Eq. (11), this switching surface will be across the initial error state \( (e(0), \dot{e}(0)) \) and the origin of the phase plane; it indicates no reaching phase motion in this algorithm.

However, the perfect model of the robots can hardly be obtained and usually only the nominal model of the robot can be estimated; on the other hand, there usually exist external disturbances. Suppose that the nominal model matrices of the robotic system are denoted by \( M_0(q), C_0(q, \dot{q}), G_0(q) \) and \( F_0 \), respectively; \( d \in \mathbb{R}^n \) is the vector of disturbances referred to the actuator input. Therefore, the equivalent control law according to the nominal model can be described by

\[ \tau_{eq} = M_0(q)(\dot{q}_d - K_v \dot{e} - K_p e) + C_0(q, \dot{q})\dot{q} + G_0(q) + F_0 \dot{q}. \]

(12)

Combining (1) and (12), the resulted error dynamics is obtained that

\[ \ddot{e} + K_v \dot{e} + K_p e = M_0^{-1}(\Delta M(q)\dot{q} + \Delta C(q, \dot{q})\dot{q} + \Delta G(q) + \Delta F_0 \dot{q} + d) \]

(13)

As one can see from (13), no matter how the constant matrices \( K_v \) and \( K_p \) are designed, the tracking error will not tend to zero or even not be stable. Since the external disturbances and the model uncertainties always exist, a switching control law \( \tau_{sw} \) is needed to compensate these uncertainties and to keep the error states staying on the sliding surface. The additional switching control law is

\[ \tau_{sw} = -M_0(q)W \text{sgn}(S). \]

(14)

where \( W \in \mathbb{R}^{n^{sa}} \) is a positive definite diagonal matrix

\[
\begin{bmatrix}
0 & \cdots & 0 \\
0 & \ddots & \vdots \\
\vdots & \ddots & 0 \\
0 & \cdots & 0
\end{bmatrix}
\]

and \( \text{sgn}(S) = [\text{sgn}(s_1), \ldots, \text{sgn}(s_n)]^T \).

Then, the Variable Structure Model Following Control (VSMFC) law is defined as:

\[ \tau = \tau_{eq} + \tau_{sw} \]

\[ = M_0(q)(\dot{q}_d - K_v \dot{e} - K_p e) + C_0(q, \dot{q})\dot{q} + G_0(q) + F_0 \dot{q} - M_0(q)W \text{sgn}(S). \]

(15)

By Eqs. (15) and (1), it results that

\[ M(q)\dddot{q} + C(q, \dot{q})\dot{q} + G(q) + F_0 \dot{q} \]

\[ = M_0(q)(\dot{q}_d - K_v \dot{e} - K_p e) + C_0(q, \dot{q})\dot{q} + G_0(q) + F_0 \dot{q} + d - M_0(q)W \text{sgn}(S). \]

(16)

The modelling uncertainties be denoted as \( \Delta M = M_0 - M, \Delta C = C_0 - C, \Delta G = G_0 - G \), and \( \Delta F = F_0 - F \).

Then, the resulted error dynamics is

\[ \dddot{e} + K_v \ddot{e} + K_p \dot{e} = M_0^{-1}(\Delta M(q)\ddot{q} + \Delta C(q, \dot{q})\dot{q} + \Delta G(q) + \Delta F_0 \dot{q} + d) \]

(17)

From (17), if the switching control is designed to make the error state \( (e, \dot{e}) \) stay on the sliding surface \( S(e, \dot{e}) = 0 \), as Eq. (11), and then the reaching condition can be derived by defining the Lyapunov function as

\[ V = \frac{1}{2} S^T S. \]

(18)

Differentiating \( V(t) \) with respect to time yields

\[ \dot{V} = S^T \dot{S} \]

\[ = S^T (\dddot{e} + K_v \ddot{e} + K_p \dot{e}) \]

\[ = S^T [M_0^{-1}(\Delta M(q)\ddot{q} + \Delta C(q, \dot{q})\dot{q} + \Delta G(q) + \Delta F_0 \dot{q} + d) - W \text{sgn}(S)]. \]
An immediate choice for $W$ to make $\dot{V}$ a negative semi-definite function of $S$ would be

$$
\nu_i \geq \left[ M_0^{-1} (\Delta M \dot{q} + \Delta C \dot{q} + \Delta G + \Delta F \dot{q} + d) \right] + \eta_i; \quad (20)
$$

where the constants $\eta_i$ are strictly positive. Then the reaching condition should be satisfied as follows

$$
\dot{V} \leq -\sum_{i=1}^{n} \eta_i |s_i|.
$$

Now, if $\dot{V}$ is negative definite with respect to $S$ (i.e., $\dot{V} < 0$ for $S \neq 0$ and $\dot{V} = 0$ for $S = 0$) with large enough weighting factors $\nu_i$, then the sliding mode should be satisfied. In practice, if $\left[ M_0^{-1} (\Delta M \dot{q} + \Delta C \dot{q} + \Delta G + \Delta F \dot{q} + d) \right]$ can be estimated exactly, the lowest $\nu_i$ can be obtained to the switching control law. However, the condition can reasonably be achieved with large enough $\nu_i$. If the disturbance can be estimated more accurately, then the resulting control input to keep the state trajectory on sliding surface will be more appropriate. This situation is similar to the conventional sliding mode control law.

From Eq. (14), the switching control would usually introduce the high frequency chattering in practical implementation. To alleviate this phenomenon, the continuous approximation of the discontinuous sign function is the most direct approach (21), where the sign function can be simply replaced by a saturation function as follows.

$$
\text{sat}(s_i) = \begin{cases} 
\text{sign}(s_i) & \text{if } |s_i| > \delta \\
 \frac{2s_i}{|s_i|+\delta} & \text{otherwise}
\end{cases}
$$

However, this continuous approximation method cannot be guaranteed in the neighbourhood of the sliding surface ($|s_i| < \delta$), so this is often referred to as pseudo-sliding. It indicates that the approach eliminates chattering at the expense of robustness and accuracy. To overcome this drawback of traditional methods, some methods have been proposed (15)–(20). In this paper, a method is studied to find the average value of the switching control by a first order linear filter with the real discontinuous control signals. Therefore, the Eq. (15) is modified to the following:

$$
\tau = \tau_{eq} + \tau_{av}
$$

where $\tau_{eq}$ is the original discontinuous switching control, and $\tau_{av}$ is the average value measured by a first order linear filter.

$$
\mu \dot{\tau}_{av} + \tau_{av} = \tau_{sw},
$$

where $\tau_{sw} = [\tau_{sw1}, \tau_{sw2}, \ldots, \tau_{swn}]^T$, $\tau_{av} = [\tau_{av1}, \tau_{av2}, \ldots, \tau_{avn}]^T$. That is,

$$
\tau_{av1} = \frac{1}{\mu} \left( e^{-\frac{\mu}{\tau_{sw1}}} + \tau_{sw1}(0) \cdot e^{-\frac{\mu}{\tau_{sw1}} \cdot \tau} \right)
$$

where the time constant $\mu$ should be made so small that the linear filter could not distort the slow component of the switching control law. However, if $\mu$ is too small, it could not filter the chattering. With appropriate selection of $\mu$, this approach does not spoil the robustness and accuracy of the system at the same time to alleviate the chattering.

4. Output Feedback Control with the High-Gain Observer

The above design method provides a dynamic sliding-mode control with global invariant if the error states $(e, \dot{e})$ are available. However, the velocity sensors are often omitted to save cost and to prevent noise from the tachometers. The problem of velocity measurements can be solved by using the high-gain observer to estimate joint velocities form position measurements. The high-gain observer is usually used to estimate joint velocities; however, it exhibits a peaking phenomenon in its transient behaviour. This peaking phenomenon may cause the system to become unstable, and an idea to overcome peaking is studied by some researchers (15)–(20). Esfandiari and Khalil suggested designing the state feedback control to be globally bounded by the saturated control inputs. Another approach to avoid peaking, called a semi-high-gain observer, is proposed by Lu and Spurgeon.

Consider the Eq. (1) and represent it to the state-space form:

$$
\begin{align*}
\dot{\xi}_1 &= \dot{\xi}_2 \\
\dot{\xi}_2 &= -M_0^{-1}(\xi_1)[C_0(\xi_1, \xi_2)\xi_2 + G_0(\xi_1) + F_0\xi_2] + \tau + d
\end{align*}
$$

where $\xi_1 = q$.

If the joint velocity $\dot{\xi}_2$ is not measurable and the dynamics of the robot are unknown, high-gain observer may be used to estimate $\dot{\xi}_2^{(19)}$.

$$
\begin{align*}
\dot{\hat{\xi}}_1 &= \hat{\dot{\xi}}_2 + \frac{1}{\epsilon}K_1(\dot{\xi}_1 - \hat{\xi}_1) \\
\dot{\hat{\xi}}_2 &= \frac{1}{\epsilon^2}K_2(\dot{\xi}_1 - \hat{\xi}_1)
\end{align*}
$$

However, if the nominal model of the manipulator can be obtained, a modified high-gain observer can be designed as the following:

$$
\begin{align*}
\dot{\hat{\xi}}_1 &= \dot{\hat{\xi}}_2 + \frac{1}{\epsilon}K_1(\dot{\xi}_1 - \hat{\xi}_1) \\
\dot{\hat{\xi}}_2 &= \phi(\hat{\xi}, \hat{\dot{\xi}}) + \frac{1}{\epsilon^2}K_2(\dot{\xi}_1 - \hat{\xi}_1)
\end{align*}
$$

where $\hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2]^T$ denote the estimated states of $\xi = [\xi_1, \xi_2]^T$; $\epsilon$ is chosen as a small positive parameter; $K_1, K_2$ are positive definite matrices chosen such that the matrix $\begin{bmatrix} -K_1 & I \\ -K_2 & 0 \end{bmatrix}$ is Hurwitz, and $\phi(\hat{\xi}, \hat{\dot{\xi}}) = -M_0^{-1}(\xi_1)[C_0(\xi_1, \hat{\xi}_2)\hat{\xi}_2 + G_0(\xi_1) + F_0\xi_2] + \dot{\tau}$; $\dot{\tau}$ is the control input obtained with estimated velocities. The observer error is defined as:

$$
\begin{align*}
\hat{\xi}_1 &= \dot{\hat{\xi}}_1 - \xi_1 \\
\hat{\xi}_2 &= \dot{\hat{\xi}}_2 - \xi_2
\end{align*}
$$
and $z_1 = \hat{\xi}_1$, $z_2 = \hat{\xi}_2$.

From the Eqs. (25) and (27) the observer error equation can be described as:

$$\varepsilon \ddot{z}_1 = z_2 - K_1 z_1$$
$$\varepsilon \ddot{z}_2 = -K_2 z_1 + \varepsilon^2 (A \varphi + d)$$

where $A \varphi = \varphi(\hat{\xi}, \dot{\varphi}) - \varphi(\xi, \varphi)$.

Rewrite the Eq. (28) in the matrix form:

$$\varepsilon \ddot{z} = A \varepsilon + \varepsilon^2 B (A \varphi + d)$$

where $A = \begin{bmatrix} -K_1 & 1 \\ -K_2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The observer error $\varepsilon$ of the high gain observer (27) will converge to the residual set:

$$\Omega_\varepsilon = \left\{ |\varepsilon| \leq \tilde{K}(\varepsilon) \right\}$$

where $\tilde{K}(\varepsilon) = 2 \varepsilon^2 \sup_{r \in [0, T]} ||B(A \varepsilon + d)|| \cdot ||P||$, $P$ is the solution of Lyapunov equation:

$$A^T P + PA = -I$$

**Remark 3:**

If $\sup_{r \in [0, T]} ||B(A \varepsilon + d)|| \cdot ||P||$ is bounded, we can select $\varepsilon$ arbitrary small to make $\tilde{K}(\varepsilon)$ small enough. So the observer error can be arbitrary small.

Therefore, the proposed VSMF controller is modified as:

$$\ddot{\varphi}_e = \ddot{\varphi}_r + \dot{\varphi}_r w$$

$$= M_\varphi(\xi_1) \ddot{q}_d - K_\varphi(\tilde{\xi}_2 - \dot{\varphi}_d) - K_p(\xi_1 - q_d) + C_\varphi(\xi_1, \tilde{\xi}_2) + G_\varphi(\xi_1) + F_\varphi(\xi_1, \tilde{\xi}_2) - M_\varphi(\xi_1) W \text{sgn}(\tilde{S})$$

where $\tilde{S} = S(\varepsilon, \dot{\varphi}) = (\tilde{\xi}_2 - \dot{\varphi}_d + K_{\varphi}(\xi_1 - \dot{\varphi}_d) + K_p \int (\xi_1 - q_d) dt - \dot{\varphi}(0) + K_{\varphi}(\xi_1 - \dot{\varphi}_d)(0))$

The structure of the proposed VSMF controller with output feedback is shown in Fig. 1. The proposed controller is similar to the normal PD control with velocity estimation, but the difference is the switching controller term $-M_\varphi(\xi_1) W \text{sgn}(\tilde{S})$, which compensates the modelling uncertainties and disturbances.

Combining the Eqs. (2) and (33) with external disturbances, the system dynamics can be obtained as follows:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\dot{q}_d - \dot{M}^{-1}(x_1, q_d)(C(x_2, q_d, \ddot{q}_d) + \dot{F}(x_2 + \ddot{q}_d) + G(x, \ddot{q}_d)) + M^{-1}(x_1, q_d)$$

$$+ G_\varphi(\xi_1) + F_\varphi(\xi_1, \tilde{\xi}_2) - M_\varphi(\xi_1) W \text{sgn}(\tilde{S}) + d$$

From (34), if $\tilde{\xi}_2 = \xi_2 + \Delta \xi_2$, and then we have

$$M \ddot{e} = -M_\varphi \ddot{q}_d - [(C + F) \xi_2 + G] + M_\varphi(\ddot{q}_d - K_\varphi(\dot{\varphi}_d + \tilde{\xi}_2 - K_p e)$$

$$+ C_\varphi(\tilde{\xi}_2 + \tilde{\xi}_2) + G_\varphi(\xi_1, \tilde{\xi}_2) - M_\varphi W \text{sgn}(\tilde{S}) + d$$

and

$$M_\varphi - \Delta M \ddot{e} + M_\varphi K_\varphi \ddot{\varphi}_d + K_p e$$

$$= \Delta M \ddot{q}_d - [(C + F) \xi_2 + G] + M_\varphi(\ddot{q}_d - K_\varphi(\dot{\varphi}_d + \tilde{\xi}_2 - K_p e) + C_\varphi(\tilde{\xi}_2 + \tilde{\xi}_2) + G_\varphi(\xi_1, \tilde{\xi}_2) - M_\varphi W \text{sgn}(\tilde{S}) + d$$

Then, $\ddot{e} + K_{\varphi} \ddot{\varphi}_d + K_p e = M_\varphi(\Delta M \ddot{q} + \Delta C + \Delta F) \xi_2$

$$+ \Delta G + \Psi_\varphi(\tilde{\xi}_2 + d) + W \text{sgn}(\tilde{S})$$

where $\Psi_\varphi = C_\varphi + F_\varphi - M_\varphi K_\varphi$

Let the Lyapunov function be defined as

$$V = \frac{1}{2} \tilde{S}^T \tilde{S}$$

Then,

$$\dot{V} = \tilde{S}^T \tilde{S}$$

$$= \tilde{S}^T \left( \ddot{\varphi}_e + K_{\varphi} \ddot{\varphi}_d + K_p e \right)$$

$$= \tilde{S}^T \left( M_\varphi^{-1}(\Delta M \ddot{q} + (\Delta C + \Delta F) \xi_2 + \Delta G + \Psi_\varphi(\tilde{\xi}_2 + d)$$

$$- W \text{sgn}(\tilde{S}) + \frac{d}{dt}(\tilde{\xi}_2) + K_{\varphi} \xi_2 \right)$$

$$= \tilde{S}^T \left( M_\varphi^{-1}(\Delta M \ddot{q} + (\Delta C + \Delta F) \xi_2 + \Delta G + \Psi_\varphi(\tilde{\xi}_2 + d$$

$$+ \frac{d}{dt}(\tilde{\xi}_2) + K_{\varphi} \xi_2 \right) - \sum_{i=1}^{n} w_i |\tilde{S}_i|$$

If $V$ is negative definite with respect to $\tilde{S}$ (i.e., $\dot{V} < 0$ for $\tilde{S} \neq 0$, and $\dot{V} = 0$ for $\tilde{S} = 0$) with a large enough $W$, then the existence of the sliding mode will be satisfied. An immediate choice for $W$ to make $\dot{V}$ a negative semi-definite function of $S$ would be

$$w_i \geq \left| M_\varphi^{-1}(\Delta M \ddot{q} + (\Delta C + \Delta F) \xi_2 + \Delta G + \Psi_\varphi(\tilde{\xi}_2 + d$$

$$+ \frac{d}{dt}(\tilde{\xi}_2) + K_{\varphi} \xi_2) \right| + \eta_i$$

where the constants $\eta_i$ are strictly positive. Then the reaching condition should be satisfied as follows

$$\dot{V} \leq - \sum_{i=1}^{n} \eta_i |\tilde{S}_i|$$

In practice, if the uncertainties and disturbances can be estimated exactly and the observer can estimate the
states more accurate, the lower $w_i$ can be obtained to the switching control law. However, the condition can reasonably be achieved with a large enough $w_i$. On the other hand, the details about estimating the uncertainties and disturbances are not addressed here and this will be considered in future work.

5. Case Study

To develop the simulations, a two-link manipulator, as shown in Fig. 2, is considered. The manipulator is in vertical position, with gravity and friction. The robot parameters are listed in Table 1. The dynamic model of the manipulator can be described as follows.

$$
M = \begin{bmatrix}
m_1l_1^2 + m_2(l_1^2 + l_2^2 + 2l_1l_2\cos(q_2)) + I_1 + I_2 \\
m_2(l_1^2 + l_1l_2\cos(q_2)) + I_2
\end{bmatrix}
$$

$$
C = \begin{bmatrix}
-2m_2l_1l_2\sin(q_2)\dot{q}_2 \\
-m_2l_1l_2\sin(q_2)\dot{q}_2 \\
m_2lg_2\cos(q_1 + q_2) + (m_1 + m_2)gl_1\cos(q_1) \\
m_2lg_2\cos(q_1 + q_2)
\end{bmatrix}
$$

$$
G = \begin{bmatrix}
m_2lg_2\cos(q_1 + q_2) + (m_1 + m_2)g l_1\cos(q_1) \\
m_2lg_2\cos(q_1 + q_2)
\end{bmatrix}
$$

$$
F = \begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}
$$

The simulation studies are conducted to compare the tracking performances and the control inputs by using the proposed method with different type of switching control law. In the simulation studies, the desired joint trajectory is:

$$
\begin{bmatrix}
q_{d1} \\
q_{d2}
\end{bmatrix} = \begin{bmatrix}
4\sin t \\
2\sin t
\end{bmatrix}.
$$

(41)

In the simulation studies, the unmatched model uncertainties and the matched disturbance are considered; the model uncertainties are listed in Table 1 and the external disturbances to the manipulator joints are supposed to be

$$
d(t) = \begin{bmatrix}
d_1 \\
d_2
\end{bmatrix} = \begin{bmatrix}
0.5\sin(\pi t) \\
0.5\sin(\pi t)
\end{bmatrix}
$$

(42)

Using the proposed VSMF controller with output feedback, two design parameters $K_p$ and $K_v$ are assigned according to the system requirements. For the control system design, the gains $K_p$ and $K_v$ can be chosen as $K_p = \begin{bmatrix}
31 & 0 \\
0 & 45
\end{bmatrix}$, $K_v = \begin{bmatrix}
60 & 0 \\
0 & 80
\end{bmatrix}$. The weighting matrix of the switching control in Eq. (14) are designed as $W = \begin{bmatrix}
5 & 0 \\
0 & 5
\end{bmatrix}$.

The parameters of the high-gain observer are chosen as $\varepsilon = 0.08$, $K_1 = 10$, $K_2 = 10$. Figure 3 shows the tracking results of both links by applying the proposed method with the conventional switching control, and it show that the robot can follow the desired trajectories accurately under the external disturbance. However, from Fig. 3, there is the chattering phenomenon in resulting torque outputs, and it

![Fig. 2 The schematic diagram of a planar robot](image)

![Fig. 3 The tracking results under the external disturbances and the modelling uncertainties](image)
should be avoided in practical implementation. To alleviate this phenomenon, the continuous approximation of the discontinuous sign function is the most direct approach, where the sign function can be simply replaced by a saturation function as Eq. (21). Figure 4 shows the resulting torque output by applying the proposed method with the saturation function instead of the sign function, and the parameter $\delta$ in Eq. (21) is chosen as 0.1, where the chattering phenomenon has been alleviated perfectly. However, the approximating method cannot be guaranteed in the neighbourhood of the sliding surface ($|s_i| < \delta$); it indicates that the approximating approach eliminates chattering at the expense of robustness and accuracy. To overcome this drawback of pseudo sliding, a method is studied to find the average value of the switching control by Eq. (23), and the time constant is chosen as $\mu = 0.02$, and Fig. 6 shows the resulting torque output by applying the proposed method with the first order linear filtered switching control. From the simulating results in Fig. 6, there exists some low-frequency switching and they make the sliding plane with invariant property. This approach is not as the same with the approximating approach which eliminates chattering at the expense of robustness and accuracy; the differences between these three methods can be detailed in Figs. 7 and 8. From the simulation results of the tracking errors in Fig. 7, the approximating method has larger tracking errors, the traditional switching method and the filtered switching method has better tracking performance. From the simulation results of the sliding variables in Fig. 8, the traditional switching method has the chattering phenomena; the approximating approach can alleviate the chattering, however, the robustness is spoiled; the filtered switching method can alleviate the chattering and maintain the robustness. From the above simulations, the conclusion is the proposed VSMF controller with the filtered switching control law can maintain the invariance of the sliding mode control and avoid the chattering.

In the following simulations in Figs. 9 – 11, the influences of the parameters $\mu$, $W$, $\varepsilon$ in the proposed method are discussed. At the beginning, the other parameters are chosen as the same of the above simulations: $K_p = \begin{bmatrix} 31 & 0 \\ 0 & 45 \end{bmatrix}$, $K_v = \begin{bmatrix} 60 & 0 \\ 0 & 80 \end{bmatrix}$, $K_1 = 10$, $K_2 = 10$. To simplify the designing procedure, the weighting matrix of the switching control is defined as the following: $W = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}$. Figure 9 shows the resulting control torques of the proposed method with different $\mu$. From the Eq. (24), $\mu$ is the time constant of the linear low-pass filter and it could not distort the slow component of the switching control law. However, if $\mu$ is too small, it could not filter the chattering and the resulting control torques are the same as the conventional switching method in Fig. 4. With appropriate

**Fig. 4** The torque outputs of the proposed method with conventional switching control

**Fig. 5** The torque outputs of the proposed method with approximating approach

**Fig. 6** The torque outputs of the proposed method with the first order linear filter of $\mu = 0.02$
Fig. 7 The comparison between the three methods for the tracking errors

Fig. 8 The comparison between the three methods for the sliding variable

selection of $\mu$, this approach does not spoil the robustness and accuracy of the system at the same time to alleviate the chattering. From the Eq. (40), the parameter $w$ dominates the robustness of the sliding mode control; it can be detailed from the simulation results in Fig. 10. It represents the larger $w$ with the larger robustness and the less tracking errors. From Eqs. 30 and 31, if the parameter $\varepsilon$ of the high-gain observer is lower, the observer error $\hat{\xi}$ will converge to the smaller residual set. The problem is that, with the perturbation method, one can only expect that small enough $\varepsilon$ will guarantee local stability. Most researchers discuss the upper bound of $\varepsilon$, $(0 < \varepsilon < \bar{\varepsilon})$, to guaranteed the stability of the system\textsuperscript{(4)},(9),(18)\textsuperscript{–}(20). However, if $\varepsilon$ is too small, then the gain of the observer is too high and a peaking phenomenon will occur. The question is “how small $\varepsilon$ is small enough?”. This is important
because the peaking phenomenon due to high-gain may damage global stability of the proposed method. The following simulation results are demonstrated the choice of $\varepsilon$ will affect the observer error and too small $\varepsilon$ will degrade the performance of the observer. Figure 11 shows the influence of the parameter $\varepsilon$ in the observing errors. The simulating results show the smaller $\varepsilon$ makes the observer have smaller errors. However, when $\varepsilon$ of the 1st Joint is lower than 0.04 and $\varepsilon$ of the 2nd Joint is lower than 0.03, the smaller $\varepsilon$ will generate the larger observer errors of joint velocities, and this is cause by the peaking phenomenon. From the simulating results, therefore, it is suggested that the parameter $\varepsilon$ of the high-gain observer should be consider the lower bound and upper bound ($\varepsilon < \bar{\varepsilon}$) when the high-gain observer is implemented.

Fig. 9 The comparison of control torque with $\mu = [0.0001, 0.005, 0.02]$, $w = 5$, $\varepsilon = 0.08$

Fig. 10 The tracking error with $w = [1, 5, 10]$, $\mu = 0.02$, $\varepsilon = 0.08$
6. Conclusions

In the conventional sliding mode control, a discontinuous switching control signal is applied to make the system invariant, and it also causes chattering. Modified approaches are introduced to eliminate chattering, but it loses the invariance property in the neighbourhood of the sliding surface. In this paper, an alternative approach to the design of sliding mode control with output-feedback is proposed. A systematic design procedure based on the concept of a dynamic sliding surface is presented. The proposed scheme results in a continuous control signal such that chattering output is avoided. The sliding surface synthesized using the proposed approach goes across the initial state and the origin, and this guarantees that the invariance property exists as the control is applied; the concept of a high-gain observer is studied and modified invariance property exists as the control is applied; the initial state and the origin, and this guarantees that the face synthesized using the proposed approach goes across such that chattering output is avoided. The sliding surface is proposed. A systematic design procedure based on the concept of a dynamic sliding surface is presented. The proposed scheme results in a continuous control signal such that chattering output is avoided. The sliding surface synthesized using the proposed approach goes across the initial state and the origin, and this guarantees that the invariance property exists as the control is applied; the concept of a high-gain observer is studied and modified for the propose approach to estimate joint velocities using position measurements. The simulation results show that the proposed approach achieves good performance with global invariance and chattering free.

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References

