Hexahedral Mesh Improvement Algorithm*

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In this paper, we present a novel method, based on an implementation of quasi-statistical modeling, for improving hexahedral solid mesh. A method for improving meshes by producing elements with a Gaussian (normal) distribution of the mesh quality parameter values is discussed. The main intention is to attain a fairly smooth change from one mesh element to another without creating a significant difference between the shapes of neighboring elements. As regards the initial distribution of the mesh quality parameter values, we assume that after improvement the distribution varies from a rather random distribution to a smoother one, such as a normal distribution. The preliminary choice of the desirable distribution affects the new parameter values modeled by the formula presented here. Our proposed method can be used in a pre-processing stage for subsequent studies (finite element analysis, computer graphics, etc.) by providing better input parameters for these processes. Experimental results are included to demonstrate the functionality of our method.

Key Words: Hexahedral Mesh, Statistics, Modeling, Mesh Quality Parameter, Finite Element Meshes, Space Mapping

1. Introduction

Finite element analysis (FEA) is an essential tool for engineers. Before analysis can begin, a mesh of the model is created. Mesh improvement is an almost obligatory step for obtaining a valid finite element mesh. Distorted elements can lead to some numerical problems during solution. In this case, the requirements of an automatic mesh generator can be weakened. The shapes of the elements in the mesh should be as similar as possible. Consequently, a square or regular hexahedron is a perfect choice for the element shape. In many cases hexahedral meshes provide advantages over tetrahedral meshes.

In spite of a flurry of activity in the field of mesh modification, this remains a difficult and computationally expensive problem. There are many improvement techniques for optimal point placement (see Ref. (1) and ref-

erences therein). Most of them are based on the idea of local optimization and require an improvement of such mesh quality parameters as aspect ratio (AR), area, angle, etc. Global mesh optimization has also been studied recently⁽²⁾. In spite of the well-known fact that local enhancement does not often provide reliable results, local methods are preferable in many applications with rather large set of polygons, because the computational time involved is much less than for global-based techniques. In fact, these methods can be called deterministic.

Known optimization techniques have targeted improvement of the average element quality or extreme element quality. Local mesh smoothing methods operate on one mesh vertex at a time to improve mesh quality in the neighborhood of that vertex. Each free vertex is geometrically repositioned according to information at the incident vertices, and the vertex's new position will improve the quality of the mesh according to some quality parameters such as the AR, angle, and element's area.

Let us here briefly overview methods of mesh optimization that are related to computer-aided design (CAD) and computer-aided engineering (CAE) applications.

There are two main ways of improving mesh quality: (1) modification of the mesh topology by inserting or deleting nodes, or by local reconnection (though sometimes it is necessary to minimize changes in the topology

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of a surface) and (2) smoothing technique. In Ref. (3) a smoothing technique is defined as a means of correcting poorly shaped elements; that is smoothing is an operation for producing elements that closely resemble theoretical elements.

Among the earliest of mesh improvement methods is the Laplacian smoothing technique^{(4),(5)} and its variations. In this technique, nodes are moved in areas of poorly conditioned elements of the mesh. The Laplacian algorithm locates an offending node and moves it to the centroid of the surrounding nodes so as to improve the shapes of the elements. In some non-convex domains, nodes can be pulled outside the boundary. An angle smoother in 2D that tends to mount torsion springs between nodes and minimize the system energy was proposed in Ref. (6). In optimization-based methods, instead of moving each node according to some geometric characteristic, as in the Laplacian method and angle-based smoothing, the so-called cost function is minimized. There are several kinds of cost functions: minimum/maximum angle^{(7),(8)}, aspect ratio⁽⁹⁾, and distortion metrics^{(10)–(12)}. While optimization-based methods are very effective for avoiding the invalid elements, the computational time involved is much higher than for the Laplacian and angle-based techniques. Therefore in Refs. (7) and (10) it was recommended that a combined Laplacian optimization-based approach should be applied. In physically based methods, it is assumed that nodes are moved under the influence of some forces so that the shape of the incident elements is improved. In Refs. (13) and (14), the authors consider a mesh as a system of springs that exert repulsive or attractive forces. In the methods proposed in Refs. (15)–(17), nodes are moved to attain equilibrium as centers of bubbles. Some physically based methods are classified as optimization-based techniques. For instance, in Ref. (18) the forces at each node are considered. Simulation of these forces and the minimum energy configuration are defined by using an optimization procedure.

In recent years, the computer graphics (CG) community has paid more attention to mesh smoothing based on a signal processing approach pioneered by Taubin in 1995⁽¹⁹⁾. Rational filters offer an approach to mesh smoothing⁽²⁰⁾ in which a Laplacian operator and a curvature operator are used to remove undesirable noise. Subdivision schemes are an alternative approach to the problem. Nevertheless, while such schemes⁽²¹⁾ are able to deal with arbitrary topology, they cannot handle arbitrary connectivity, as was mentioned by Kobbelt et al.⁽²²⁾.

Automatic generation of hexahedral meshes sometimes produces meshes with badly shaped and inverted elements. Hexahedral are highly flexible and can be distorted; there are also difficulties in generating hexahedral meshes for complex configurations^{(23),(24)}. In Ref. (24)

an optimization-based smoothing technique where each node is moved in order to maximize quality metric for improving all-hexahedral meshes is presented. A measure of node quality was chosen instead of element qualities for improving the quality of initial invalid meshes to acceptable level. Authors suggest using three algorithms and passing the control from one to another according to negative-metrics nodes. Finally, in some cases when prescribed time has passed, mesh quality still does not improved.

Our method can be interpreted as a kind of smoothing technique using distribution control of the mesh quality parameter. Recently, an implementation of the statistical approach for triangular meshes was discussed⁽²⁵⁾. That algorithm, proposed by the present authors, was extended to a theoretically proved universal and multi-purpose method for improving meshes of any kind by means of random variable modeling⁽²⁶⁾. This method can be used at a pre-processing stage for subsequent studies (finite element analysis (FEM⁽²⁷⁾), CG, etc.) based on meshes by providing better input parameters for them. Our main contribution in this paper is a new method for improving hexahedral meshes by quasi-statistical modeling of a mesh quality parameter values.

The rest of the paper is organized as follows. The background and algorithm are briefly introduced in section 2. In section 3, reducing flatness distortion is discussed. Section 4 demonstrates the results obtained by application of the algorithm. Section 5 concludes the paper.

2. Theory Aspects and Algorithm Description

In this section, we present our method. The problem of improving a mesh can be formulated as follows:

Given: Mesh elements generated by some existing algorithm.

Result: Mesh with higher-quality elements and uniform distribution of the mesh quality parameter values.

By “improvement of the mesh quality”, we understand a process of producing a mesh with a rather homogeneous distribution of the mesh elements. In fact, if a mesh is created for FEA, it is important to control the mesh gradation smoothness, since shape elements have a strong influence on the discretization error. When the mesh gradations are not smooth, the discretization error of the neighboring elements changes markedly.

Our main intention is to attain a fairly smooth change from one mesh element to another without creating any difference between the shapes of neighboring elements of the mesh. To achieve this, we produce elements with a Gaussian (normal) distribution of the mesh quality parameter values.

A real number (a value of the mesh quality parameter) is input in accordance with each element of the mesh. Such experimental data are grouped into special intervals

showing the number of elements that lie in such intervals of probabilistic space fragmentation. Since the empirical distribution of experimental data is discrete, we consider the frequency histogram as an analogue of the density function. All classical statistical parameters for discrete case of sampling, such as expectation, deviation, frequency histogram values, and probability values, are also defined in common way. From another point of view, a real number assigned to a mesh element is considered as a pseudo-random number, by analogy with the value derived from some given formula. The approach proposed in Ref. (26) is based on a modeling of random variables, whose values will be used in an intermediate stage of the method for predicting the mesh quality parameter value. Modeling and transformation of a random variable is the basis of all numerical methods, algorithms, approaches and models. The general classification of random variable transformations used for modeling was suggested in Ref. (28). The following aspects should be considered for implementation of quasi-statistical modeling.

2.1 Modeling formula

The distribution of the mesh quality parameter after the mesh improvement process is assumed to be Gaussian (uniform) one. The formula for obtaining new values of the mesh quality parameter is derived in polar coordinates. Finally, Eq. (1) is used to produce various ρ -values of a Gaussian random variable in polar coordinates from the given pseudo-random or random γ -numbers.

$$\rho = \sqrt{-2\ln(1-\gamma)}. \quad (1)$$

Equation (1) models a random variable with uniform distribution.

2.2 Deterministic pseudo-random numbers

We consider the mesh improvement process with respect to the mesh quality parameter. The definition of that parameter and classical statistics help us to make a choice of determined pseudo-random numbers, which are called quasi-random numbers. The choice of quasi-random numbers for the mesh quality parameter is to assign the number F_i to the i th element of the mesh. The number F_i is equal to the corresponding empirical distribution number (probability value) of the mesh quality parameter value a_i for the i th element of the mesh. In most applications, AR is used as the mesh quality parameter. The angle between adjacent edges of the elements is considered as well as a mesh quality parameter and denoted hereafter as α .

2.3 Formula interpretation

For each number F_i one can produce a new value a_i^* of a mesh quality parameter by the Eq. (2):

$$a_i^* = \sqrt{-2D\ln(1-F_i)} + \mu, \quad (2)$$

where μ is an ideal value for the given mesh quality parameter and D is the deviation (from the average value M of AR or angle).

Equation (2) is interpreted as a formula for modeling of new values of the mesh quality parameter for each element of the mesh. The ideal AR value $\mu = 2$, corresponding to a square element, has a ratio of maximum diagonal to its height equal to 2. If the angle is taken as a mesh quality parameter, μ is equal to $\pi/2$.

2.4 Construction of a potential form

Hexahedral element edges and faces must be orthogonal as far as possible. In the case of all-hexahedral mesh, a volume star is defined as the set of quadrilateral elements of a hexahedron sharing a node in 3D space (Fig. 1 (a)). For the ideal valency, which corresponds to eight hexahedral elements, the number of quadrilaterals in the star is twelve. The angle mesh quality parameter is found to be more appropriate for the algorithm than the aspect ratio parameter.

The coordinates of new centers of the star are then averaged. After all the stars have been considered in an arbitrary order, the result is achieved.

If a quadrilateral element in 3D space is folded, volume contained inside ABCD and dihedral angle (at the line of folding BD) (see Fig. 1 (b)) define the degree of distortion of element's flatness. Considering an element of the working star, only one of four triangles in the quadrilateral element becomes operational one. For example, while A is the center of the star, triangle BAD is taken, if B is a center, ABC is considered, and so on (Fig. 1 (b)). Thus, one quadrilateral element is considered in algorithm four times in four different stars. We call a triangle plane, in which the potential form is constructed in corresponding star, an operational plane. The potential form is determined for each quadrilateral element on the corresponding operational plane according to the new value of a_i^* (Fig. 1 (c) and (d)). The coordinates of new centers of the star are then averaged. After all the stars have been considered in an arbitrary order, the result is achieved.

2.5 Basic steps of the algorithm

The basic steps of the algorithm are as follows:

1. Calculation of the mesh quality parameter value (AR, angle), assignment of the parameter value to each mesh element in 3D space, and calculation of the initial distribution of these parameter values.
2. For each edge of the polygon (star), Eq. (2) uses to assign an improved quality measure to a quadrilateral to be constructed with a polygonal base.
3. Construct the potential forms of the elements of the star (new potential centers of the star).
4. Average the coordinates of all potential centers of the star and move the initial interior node of the star to the average location of the nodes heretofore constructed (the new point placement of the center of the star).
5. Repeat the process from step 2 to 4 for each node of the mesh.

Let us note that the order of selection of each inte-

Fig. 1 (a) Example of a volume star; (b) the operational planes; (c) construction of the potential form of the element. Dotted lines show the initial element form; (d) the volume star with the potential forms of elements

rior node of the mesh is produced according to the node's indices in the initial mesh generation.

3. Flatness Distortion

While considering a volume star, each quadrilateral element has more freedom to slide in 3D space than in 2D space. Thus, a geometrical realization of the modeled values can generate highly folded quadrilateral elements. Application of angle improvement, naturally leads to flatness distortion of quadrilateral elements of the volume star. To prevent (or at least to decrease) such distortion, we apply a space mapping approach to the centre of a star in accordance with the distances between the diagonals of quadrilateral elements, as shown in Fig. 2.

Space mapping in $\mathbf{R}^n$ defines a relationship between each pair of points in the original and deformed objects. Let an n -dimensional region $\Omega \subset \mathbf{R}^n$ of an arbitrary configuration be given, and let Ω contain a set of arbitrary control points $\{\mathbf{q}_i = (q_1^i, q_2^i, \dots, q_n^i) : i = 1, 2, \dots, N\}$ for a non-deformed object, and $\{\mathbf{d}_i = (d_1^i, d_2^i, \dots, d_n^i) : i = 1, 2, \dots, N\}$ for the deformed object, where N is the number of elements in a volume star. It is assumed that the points $\mathbf{q}_i$ and $\mathbf{d}_i$ are distinct and given. The goal of the construction of the deformed object is to find a smooth mapping function that approximately describes the spatial transformation applied to the centre of a star. The inverse mapping function that is needed to transform the centre of a volume star can be given in the form:

$$\mathbf{q}_i = f(\mathbf{d}_i) + \mathbf{d}_i,$$

where the components of the vector $f(\mathbf{d}_i)$ are volume splines interpolating displacements of initial points $\mathbf{q}_i$. Figure 2 presents a star. The small vectors mark these points.

We consider a mapping function as a thin-plate interpolation⁽²⁹⁾. For a three-dimensional arbitrary area Ω , thin-plate interpolation is a variational solution that defines a linear operator T when the following minimum condition is used:

$$\int_{\Omega} \sum_{|\alpha|=m} m! / a! (D^{\alpha} f)^2 d\Omega \rightarrow \min,$$

where m is a parameter of the variational functional and α

Fig. 2 Flatness distortion of the quadrilateral element

is a multi-index. It is equivalent to using the radial basis functions $\phi(r) = r^1$ or r^3 for $m = 2$ and 3 respectively, where r is the Euclidean distance between two points.

The volume spline $f(P)$ having values h_i at N points P_i is the function

$$f(P) = \sum_{j=1}^N \lambda_j \phi(P - P_j) + p(P), \quad (3)$$

where $p = v_0 + v_1x + v_2y + v_3z$ is a degree-one polynomial. To solve for the weights λ_j we have to satisfy the constraints h_i by substituting the right part of Eq. (3), which gives

$$h_i = \sum_{j=1}^N \lambda_j \phi(P_i - P_j) + p(P_i),$$

Solving for the weights λ_j and v_0, v_1, v_2, v_3 , it follows that in the most common case there is a double bordered matrix T , which consists of three blocks, square sub-matrices A and D of size $N \times N$ and 4×4 respectively, and B , which is not necessarily square and has the size $N \times 4$. Notice that, once we have the Gauss decomposition of T , we solve with three right-hand sides to obtain h -values for x, y, z -directional transformations applied to the centre of a volume star (for clarity Fig. 2 illustrates the operation on a "plain" star).

It is important to point out that to prevent a rigid motion of a volume star, boundary conditions as zero displacements of nodes adjustment to the centre of a star are used. Application of the above operation allows us to significantly reduce the distances between the diagonals of quadrilateral elements. For instance, for the model

shown in Fig. 5, the maximal deviation after applying QIA is 0.12, whereas after application it becomes 0.09.

4. Experimental Results

Figure 3 demonstrates a layer-by-layer the hexahedral solid mesh improvement process for a cube model.

Figures 4 and 5 present the initial and the final meshes of the cube model, the angle distribution histograms, and the results of eigenvalue analysis produced by NASTRAN 70.7 software. It can be seen that a more uniform mesh is generated after improvement based on quasi-statistical modeling. The range of the angle values is decreased. The exact solution, the regular mesh, and the mesh after improvement all have nearly eigenvalues. The final meshes are produced after two iterations of the presented technique.

Table 1 gives the results of the mesh quality before and after implementation of our method for the case in Fig. 4. After improving the histogram of angle distribution is smoother for the final mesh. Minimum angle value is increased by 80% and maximum angle value is decreased by 36%. The computational time of processing is 4 sec.

The result of mesh processing using to the ideal value $\mu = \pi/2$ is demonstrated in Fig. 4(d). In this case, Eq. (2) can be written:

$$\alpha_i^* = \pi/2. \quad (4)$$

This means that, in calculation of new values of the mesh quality parameter, the statistical approach need not be

Fig. 3 Hexahedral mesh of the cube model and inner layers: (a) the initial mesh; (b) the mesh after processing; (c) and (h) surface mesh; (d), (e), (f), (g) inner layers

used.

Figure 4(d) demonstrates that the range of the angle values is increased; the main statistical parameters are worse than the results parameters presented in Fig. 4(b). Thus, these results prove the efficiency of the quasi-statistical approach.

Figure 6 shows the results of improvement of the model (mechanical detail) with mesh fragments.

While the mesh of the test model in Fig. 6 has initially “good” shape elements, the distribution of the final mesh elements as it can be seen in Figure 6(d) is more uniform; the average angle value M is also improved. The computational time of processing is 60 sec.

All calculations were carried out on Microsoft Windows XP, Pentium 4 2.5 GHz, 1-GiB.

Fig. 4 Hexahedral mesh of the cube model, angle distribution histograms, and results of eigenvalue analysis. (a) The initial mesh; (b) the final mesh; (c) eigenvalue analysis by NASTRAN 70.7 (the x -axis denotes the number of mode, while the y -axis denotes the frequency); (d) mesh with the ideal μ (Eq. (4)) to assign an improved quality measure

Fig. 5 Cube model and angle distribution histograms. (a) Initial mesh of the model; (b) the mesh after improvement through 2 iterations

Table 1 Mesh quality statistics

	N el.	AR	D	Min. angle	Max. angle	Peak angle
Initial mesh	450	1.564	0.129	14 deg.	180 deg.	80 deg.
Final mesh	450	1.567	0.05	50 deg.	118 deg.	84 deg.
Ideal shape of elements	450	1.560	0.12	25 deg.	160 deg.	82 deg.

5. Concluding Remarks

The paper has presented a new approach to improving the quality hexahedral mesh based on quasi-statistical modeling. This technique can be used to improve the smoothness of a mesh (that is, to reduce the difference between adjacent elements) after one or a few iterations. Modeling of a random variable with the desirable distribution based on quasi-random numbers and on the initial topology of the mesh, which influences on the effectiveness of the geometrical realization, was discussed. The operation on the star proposed for incarnation of modeling random variable works almost perfectly on the convex stars. In the case of concave stars, topological modification of the initial mesh can be needed. Two-dimensional structures in which each node is shared by four elements are the best. In the 3D case, it is difficult to decide which the best topological structure for the hexahedral mesh is. Eigenvalue analysis by NASTRAN (See Fig. 4) shows that proposed approach allows obtaining a good mesh for FEM applications without inverted hexahedral elements. Until recently, as it was pointed out in Ref. (24) the Laplacian smoothing was applied to the hexahedral mesh. The drawback of Laplacian method is a possible inversion of elements near concave areas of the boundary and single one makes the whole mesh useless⁽²⁴⁾. In spite of that our ap-

Fig. 6 Mechanical detail. Number of elements: 18 000. (a) model; (b) distribution histograms of initial and final meshes; (c) the initial mesh, $M = 1.56$ (80 deg.), $D = 0.117$; (d) the mesh after processing, $M = 1.57$ (90 deg.), $D = 0.12$, 2 iterations

proach allows provide more gentle smoothing, it based also on averaging of predicted potential forms that does not guaranty improvement beyond imposed restrictions. The problem of fixing new positions without generation of inverted elements is under consideration. From the point of view of geometrical realization, our method can be used as a kind of a smoothing technique where the probabilistic direction of the point placement is calculated. The smoothness of each pair of neighbors in the mesh is guaranteed. If Eq. (2) is written as $a_i^* = \mu$, where μ is a value of the ideal shape element (square), our method can be used as an analogy to Laplacian smoothing. The technique discussed in section 3 makes it possible to reduce maximum flatness deviation by 30%. The results of applying the method and examples confirm the efficiency of the proposed improvement algorithm based on the quasi-

statistical modeling for 2D and 3D models.

Future research is development an efficient parallel algorithm for mesh improving.

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