The $H^\infty$ Control for an Inverted-Double Pendulum System Consisting of Elastic Links Connected in Series*

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A stabilizing control technique for an inverted-double pendulum system consisting of two elastic links connected in series is developed, using a reduced-order model considering only the first vibration mode of the upper elastic beam, and applying an $H^\infty$ controller design method for a mixed sensitivity problem that can consider multiplicative perturbations at the input port and is characterized by a settling function. It is found that although the coupling effect of different natural frequencies between the two flexible beams makes the stabilization problem more difficult, it is possible to stabilize the system by a feedback control using only the system’s output signals that include the vibration signal of the upper elastic beam. Furthermore, it is concluded that the more different the lowest natural frequencies of the two elastic beams are, the more easily the system can be stabilized.

**Key Words:** Inverted-Double Pendulum, Double Elastic Links, $H^\infty$ Controller, Distributed-Parameter Modeling, Mixed Sensitivity Problem, Multiplicative Perturbation, Settling Function, Reduced-Order Model

1. Introduction

The stabilization control problem of an inverted pendulum system is suitable for verifying the performance of a developed control law. However, most studies on the problem are related to the stabilization and positioning control for rigid-link systems including multiple-inverted pendulums(1), while comparatively few studies deal with flexible structure systems with such unstable excitation factors as spillover(2). The stabilization problem of a flexible- and multiple-inverted pendulum system seems to be dealt with by only a few previous studies(3), since the stabilization of a multiple-inverted pendulum with flexible links leads to an even more difficult control problem with increasing complex and unstable factors.

In the present paper, an inverted-double pendulum system consisting of two elastic links connected in series is especially investigated to solve the stabilization control problem of a system having a large number of unstable factors, and to discuss such control effects as system stability, settling characteristics, and vibration controllability for induced vibration modes by noting the robustness of the $H^\infty$ control theory using a settling function(4) introduced by Nishimura et al. The $H^\infty$ vibration control method for a flexible-structure system is featured by the inclusion of higher-mode vibration components in the modeling error. However, there have not been many experiments on an inverted-double elastic pendulum system where higher-mode vibrations of each beam exist and are dynamically coupled. We believe that this study will be significant at verifying the robust stability performance of the control system designed via the $H^\infty$ control theory.

The stabilization control system is especially characterized by considering the higher-mode vibration components of the double elastic beams as the modeling error to control them using only strain vibration information on the upper elastic beam. By applying the mixed-sensitivity problem(4),(5) with perturbations at the input port, we verify both by simulation and by real system experiment that satisfactory settling characteristics and spillover suppression effects are obtained for both the beams.
2. Modeling

Figure 1 shows the schematic illustration for an inverted-double pendulum system consisting of two elastic links connected in series. We describe the equations of motion, using the distributed-parameters modeling method presented by Nishimura et al.\(^{(2)}\) The parameters used for the modeling are as follows:

- \(M_c\): Equivalent mass for the cart [kg]
- \(D_c\): Equivalent damping factor for the horizontal motion of the cart [Ns/m]
- \(G_e\): Equivalent gain constant for the horizontal-driving motion [N/V]
- \(l_k\): Full length of beam \(k\) [m]
- \(m_k\): Mass of beam \(k\) [kg]
- \(\rho_k\): Line density for beam \(k\) [kg/m]
- \(I_k\): Moment of inertia for beam \(k\) [m\(^2\)]
- \(E_k\): Young’s modulus of beam \(k\) [N/m\(^2\)]
- \(C_{int}\): Internal damping factor for beam \(k\) [Ns/m]
- \(d_k\): Thickness of beam \(k\) [m]
- \(w(r_{k}, t)\): Deflection of beam \(k\) [m]
- \(\mu_k\): Equivalent damping factor for each joint [Nms/rad]
- \(\theta_k\): Deflection angle of beam \(k\) [rad]
- \(\phi_i(r_k)\): Eigen-function for beam \(k\) [-]
- \(X_{k,i}(t)\): Mode variable for beam \(k\) [-]
- \(\epsilon_k\): Strain of beam \(k\) [-]
- \(S\): Horizontal displacement for the cart [m]
- \(u\): Control signal given as the horizontal-driving force [V]

\(x_{p1}\): Position vector from point 0 to small element \(dr_k\) in beam \(k\)

\((i_k, j_k)\): Orthogonal coordinate at joint \(k\)

\(w_k\): Deflection of beam \(k\) at distance \(r_k\)

Position vector \(x_{p1}\) to small element \(dr_k\) in each beam is expressed as

\[
\begin{align*}
x_{p1} &= (r_1 + S \sin \theta_1)\hat{x}_1 - (u_1 + S \cos \theta_1)\hat{j}_1, \quad (1) \\
x_{p2} &= r_2\hat{j}_2 + (S + l_1 \sin \theta_1)\sin \theta_2\hat{l}_2 - \cos \theta_2\hat{j}_2 \quad - l_1(1 - \cos \theta_1)(\cos \theta_2\hat{j}_2 + \sin \theta_2\hat{a}_2) - w_2\hat{j}_2. \quad (2)
\end{align*}
\]

Using these relations, taking into account the viscous friction factor \(\mu_k\), and performing the mode-separation by linear approximations, we have the following equations of motion:

For the cart system

\[
(M_c + m_1 + m_2)\ddot{S} + \left(\frac{1}{2}m_1l_1 + \frac{1}{2}m_2l_2 + m_2l_1\right)\dot{\theta}_1 \\
+ \frac{1}{2}m_2l_2\dot{\theta}_2 + \rho_1 \sum_{i=1}^{\infty} \xi_{1i}\dot{X}_{1i} + \rho_2 \sum_{i=1}^{\infty} \xi_{2i}\dot{X}_{2i} + D_c\dot{S} = G_e u,
\]

(3)

For the rigid mode in beam 1

\[
\left(\frac{1}{2}m_1l_1 + \frac{1}{2}m_2l_2 + m_2l_1\right)\ddot{\theta}_1 \\
+ \frac{1}{3}m_1l_1^2 + \frac{1}{3}m_2l_2^2 + m_1l_2 + m_2l_1\ddot{\theta}_1 \\
+ \frac{1}{3}m_2l_2\dot{\theta}_2 + \rho_1 \sum_{i=1}^{\infty} \xi_{1i}\dot{X}_{1i} \\
- \frac{1}{2}m_1gl_1 + m_2gl_1 + \frac{1}{2}m_2gl_2\dot{\theta}_1 \\
- \frac{1}{2}m_2gl_2\dot{\theta}_2 - \rho_1 \sum_{i=1}^{\infty} \xi_{1i}\dot{X}_{1i} \\
- \rho_2 \sum_{i=1}^{\infty} \xi_{2i}\dot{X}_{2i} + \mu_1\dot{\theta}_1 = 0,
\]

(4)

For the rigid mode in beam 2

\[
\frac{1}{2}m_2l_2\ddot{\theta}_1 + \left(\frac{1}{2}m_2l_2^2 + \frac{1}{2}m_2l_1l_2\right)\ddot{\theta}_1 \\
+ \frac{1}{3}m_2l_2^2\dot{\theta}_2 + \rho_2 \sum_{i=1}^{\infty} \xi_{2i}\dot{X}_{2i} - \frac{1}{2}m_2gl_2\theta_1 \\
- \frac{1}{2}m_2gl_2\dot{\theta}_2 - \rho_2 \sum_{i=1}^{\infty} \xi_{2i}\dot{X}_{2i} + \mu_2\dot{\theta}_2 = 0,
\]

(5)

For the elastic modes in beam 1

\[
\rho_1 \left(r_1\ddot{\theta}_1 + \sum_{i=1}^{\infty} \dot{\phi}_{1i}\dot{X}_{1i} + \ddot{S}\right) + C_{m1} I_1 \sum_{i=1}^{\infty} \nu_{1i}\phi_{1i}\dot{X}_{1i} \\
+ E_1 I_1 \sum_{i=1}^{\infty} \nu_{1i}^2\phi_{1i}\dot{X}_{1i} - \rho_1 g \theta_1 = 0,
\]

(6)
(For the elastic modes in beam 2)
\[
\rho_2 \left\{ r_2 (\ddot{\theta}_1 + \dot{\theta}_2) + \ddot{S} + l_1 \dot{\theta}_1 + \sum_{i=1}^{\infty} \phi_{2i} \ddot{X}_{2i} \right\} \\
+ C_{\text{ww}2} \sum_{i=1}^{\infty} \nu_i^2 \phi_{2i} \ddot{X}_{2i} + E_2 I_2 \sum_{i=1}^{\infty} \nu_i^4 \phi_{2i} \ddot{X}_{2i}
\]
\[= -\rho_2 g (\theta_1 + \theta_2) = 0, \tag{7}\]
with the relations
\[
\xi_{ki} = \int_{r_k}^{r_{ki}} \phi_{ki} (r) dr, \quad \eta_{ki} = \int_{r_k}^{r_{ki}} r_k \phi_{ki} (r) dr,
\]
\[\gamma_{ki} = \int_{r_k}^{r_{ki}} \phi_{ki}^2 (r) dr. \tag{8}\]
Taking the inner product between \(\phi_{ki} (r_k)\) and both sides in Eqs. (6) and (7), we obtain
\[
\beta_{ki} \ddot{S} + \varphi_{ki} \ddot{\theta}_1 + \ddot{X}_{1i} + 2 \xi_{ki} \omega_{1i} \dot{X}_{1i} + \omega_{1i}^2 \dot{X}_{1i} - \beta_{ki} g \theta_1 = 0, \tag{9}\]
\[
\beta_{ki} \ddot{S} + (\varphi_{ki} + l_1 \beta_{ki}) \ddot{\theta}_1 + \ddot{X}_{2i} + 2 \xi_{ki} \omega_{2i} \dot{X}_{2i} + \omega_{2i}^2 \dot{X}_{2i} - \beta_{ki} g (\theta_1 + \theta_2) = 0, \tag{10}\]
with
\[
2 \xi_{ki} \omega_{ki} = C_{ki} l_k \phi_{ki}^2 / \rho_k, \quad \beta_{ki} = \xi_{ki} / \gamma_{ki}, \tag{11}\]
Approximating the Eqs. (3) – (5), (9), and (10) with the nth mode equations and putting the state variable \(z\) as
\[
z = [S \ \theta_1 \ \theta_2 \ X_{11} \ \cdots \ X_{1n} \ X_{21} \ \cdots \ X_{2n}]^T, \tag{12}\]
we obtain the matrix differential equation of motion as
\[
\Theta \ddot{z} + \Gamma \dot{z} + \Phi \dot{z} = \Omega z, \tag{13}\]
where
\[
\Theta = \begin{bmatrix}
J_1 & J_2 & J_3 & O_{11} & \cdots & O_{1n} & O_{21} & \cdots & O_{2n} \\
J_2 & J_3 & J_5 & Q_{11} & \cdots & Q_{1n} & P_1 & \cdots & P_n \\
J_3 & J_5 & J_6 & 0 & \cdots & Q_{21} & \cdots & Q_{2n} \\
\beta_{11} \varphi_{11} & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\beta_{1n} \varphi_{1n} & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\beta_{21} \varphi_{21} & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\beta_{2n} \varphi_{2n} & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix}, \tag{14}\]
\[
\Gamma = \text{diag} \left[ D_2, \mu_1 \mu_2, 2 \xi_{11} \omega_{11}, \cdots \right] \\
2 \xi_{1n} \omega_{1n} + 2 \xi_{21} \omega_{21} + 2 \xi_{2n} \omega_{2n}, \tag{15}\]
\[
\Phi = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & K_1 & K_2 & S_{11} & \cdots & S_{1n} & S_{21} & \cdots & S_{2n} \\
0 & K_2 & K_2 & 0 & \cdots & 0 & S_{21} & \cdots & S_{2n} \\
0 & T_{11} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & T_{1n} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & T_{21} & T_{21} & 0 & \cdots & \omega_{21}^2 & 0 & \cdots & 0 \\
0 & T_{2n} & T_{2n} & 0 & \cdots & 0 & \omega_{2n}^2 & \cdots & 0 \\
\end{bmatrix}, \tag{16}\]
\[
\Omega = \begin{bmatrix}
G_x & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix}^T, \tag{17}\]
with the relations
\[
\begin{aligned}
J_1 &= M_x + m_1 + m_2, \\
J_2 &= (1/2)m_1 l_1 + (1/2)m_2 l_2 + m_2 l_1 \\
J_3 &= (1/2)m_2 l_2, \\
J_4 &= (1/3)m_1 l_1^2 + m_2 l_1 l_2 + m_2 l_1^2 + (1/3)m_2 l_2^2 \\
J_5 &= (1/3)m_2 l_2 l_1 + (1/3)m_2 l_1^2, \\
J_6 &= (1/3)m_1 l_2^2 \\
O_{ki} &= \rho_0 \xi_{ki}, \\
\rho_i &= \rho_0 (l_1 \xi_{2i} + \eta_{2i}), \\
Q_{ki} &= \rho_0 \eta_{ki}, \\
\beta_{ki} &= \varphi_{ki} / l_1, \\
K_1 &= -(1/2)m_1 g l_1 - m_2 g l_1 - (1/2)m_2 g l_2, \\
K_2 &= -(1/2)m_2 g l_2 \\
S_{ki} &= -\rho_0 g \xi_{ki}, \\
T_{ki} &= -\beta_{ki} g \\
(k = 1, 2, i = 1, 2, \cdots, n).
\end{aligned} \tag{18}\]
Then, by letting the state variable \(x\) be
\[
x = [z^T \ z^T] \in \mathbb{R}^{(6+4n)}, \tag{19}\]
the state variable differential equation is finally represented as
\[
\dot{x} = Ax + Bu, \tag{20}\]
where
\[
A = \begin{bmatrix}
0 & I \\
-\Theta^{-1} \Phi & -\Theta^{-1} \Gamma
\end{bmatrix} \in \mathbb{R}^{(6+4n)\times(6+4n)}, \tag{21}\]
\[
B = \Theta^{-1} \Omega \in \mathbb{R}^{(6+4n)}. \tag{22}\]
The output variables indicating cart position \(S\), its velocity \(S\), deflection angle \(\theta_{pk}\), and strain \(\varepsilon_i\) are expressed as
\[
y = Cx, \tag{23}\]
where
\[
C = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix}, \tag{24}\]
\[
\varepsilon_k = \sum_{i=1}^{n} \delta_{ki} X_{ki}, \quad \theta_{pk} = \sum_{i=1}^{n} \delta_{ki} X_{ki} + \theta_k \\
\delta_{ki} = \sum_{i=1}^{n} \frac{d \phi_i (r_k)}{dr} \bigg|_{r=r_k}, \\
\text{Signals for control} \\
\begin{aligned}
\delta_{ki} &= \sum_{i=1}^{n} \frac{d \phi_i (r_k)}{dr} \bigg|_{r=r_k} \\
\theta_{pk} &= \sum_{i=1}^{n} \delta_{ki} X_{ki} + \theta_k \\
k &= 1, 2. \tag{25}\end{aligned}\]
3. Control Law

In the present study, the elastic modes of beam 1 are not considered in the control system design. The mode considered is only the primary elastic mode of beam 2 for the reason explained in the section of experimental method. Denote the state variable for the controlled mode by \(x_a\), and the one for the residual modes by \(x_R\). The state equation (20) and the output equation (22) can be reduced in dimension by the following technique.

Using the transformation matrix \(T_{NR}\), we write

\[
x = T_{NR} \begin{bmatrix} x_N \\ x_R \end{bmatrix}, \quad T_{NR} \in \mathbb{R}^{(6+4n)(6+4m)}.
\] (26)

Then the transformed state equations are derived as

\[
\begin{bmatrix}
\dot{x}_N \\
\dot{x}_R
\end{bmatrix} =
\begin{bmatrix}
A_{NR11} & A_{NR12} \\
A_{NR21} & A_{NR22}
\end{bmatrix}
\begin{bmatrix}
x_N \\
x_R
\end{bmatrix} +
\begin{bmatrix}
B_{NR1} \\
B_{NR2}
\end{bmatrix} u
\]

\[
y_1 =
\begin{bmatrix}
C_{NR11} & C_{NR12} \\
C_{NR21} & C_{NR22}
\end{bmatrix}
\begin{bmatrix}
x_N \\
x_R
\end{bmatrix}
\] (27)

with

\[
x_N = [ S \, \theta_1 \, S \, \theta_1 \, \dot{x}_1 ]^T,
\]

\[
x_R = [ X_{11} \cdots X_{1n} \, \dot{x}_{21} \cdots \dot{x}_{2n} ]^T.
\] (28)

Since the steady-state value of the residual modes \(x_R\) is estimated as

\[
x_R = -A_{NR22}^{-1}A_{NR21}x_N - A_{NR22}^{-1}B_{NR2}u,
\] (30)

the following reduced-order equation is obtained:

\[
\begin{bmatrix}
\dot{x}_N \\
y_1
\end{bmatrix} =
\begin{bmatrix}
A_N & B_N \\
C_N & 0
\end{bmatrix} \begin{bmatrix}
x_N \\
x_R
\end{bmatrix}
\] (31)

where

\[
A_N = A_{NR11} - A_{NR12}A_{NR22}^{-1}A_{NR21} \in \mathbb{R}^{8 \times 8}
\]

\[
B_N = B_{NR1} - A_{NR12}A_{NR22}^{-1}B_{NR2} \in \mathbb{R}^{8 \times 1}
\]

\[
C_N = \begin{bmatrix}
C_{NR11} \\
C_{NR21}
\end{bmatrix}^T \in \mathbb{R}^{5 \times 8},
\]

\[
C_{N1} \in \mathbb{R}^{4 \times 8}, \quad C_{N2} \in \mathbb{R}^{1 \times 8}.
\] (32)

Also, according to the treatments by Nonami and Nishimura et al.,(4),(5), the mixed sensitivity problem with perturbations at the input port is described as follows. The design based on the \(H^\infty\) control scheme forms the control system that consists of the generalized plant and the controller as shown in Fig. 2, where \(P(s) = [A, B, C, 0]\) is the controller object including the residual modes, \(P_{\gamma}(s) = [A_N, B_N, C_N, 0]\) the reduced-order controlled object, and \(K(s) = [A_k, B_k, C_k, 0]\) the \(H^\infty\) compensator.

The section enclosed by broken lines is the generalized plant \(G_{wa}(s) = [\hat{A}, \hat{B}, \hat{C}, \hat{D}]\), \(u\) is the control input and \(w_1, w_2, w_3\) and \(w_4\) are the external inputs. Note that the external input \(w_1\) is prepared to avoid pole-zero cancellation between the controlled object and the controller.(5)

The external inputs \(w_2\) and \(w_3\) are introduced to make \(\hat{D}_{21}\) of row-full rank in the generalized plant \(G_{wa}(s)\). \(\hat{y}\)

is the observed output while \(\hat{z}_1\) and \(\hat{z}_2\) are the controlled variables. \(W_1(s) = [A_{w_1}, B_{w_1}, C_{w_1}, D_{w_1}]\) in Fig. 2 is the upper-limit function of the multiplicative error \(\Delta(s)\), while \(W_2(s) = [A_{w_2}, B_{w_2}, C_{w_2}, D_{w_2}]\) is a weighting function prescribing the vibration-control characteristics of \(P_N(s)\).

Regarding beam 2, taking the multiplicative error \(\Delta(s)\) between the transfer function \(\tilde{w}_2(u) = P_{\tilde{w}_2}(s)\) including the residual modes and the one \(\tilde{w}_2(u) = P_{\tilde{w}_2}(s)\) for the reduced-order model as

\[
\Delta(s) = P_{\tilde{w}_2}(s) - P_{\tilde{w}_2}(s),
\] (33)

the robust stability condition is given by

\[
||\Delta(s)N(s)||_{\infty} < 1,
\] (34)

where

\[
N(s) = K_{w_k}(s)I - K_{w_k}(s)P_{\tilde{w}_2}(s)^{-1}P_{\tilde{w}_2}(s),
\] (35)

is the transfer function from \(a\) to \(b\) in Fig. 2 with \(u/\tilde{v}_2 = K_{w_k}(s)\) being the transfer function from \(\tilde{v}_2\) to \(u\). Equation (34) is after all rewritten as

\[
||W_1(s)N(s)||_{\infty} < 1.
\] (36)

Next, we explain the solution of the \(H^\infty\) vibration control problem. In view of Fig. 2, the transfer function from \(w_1\) to \(\tilde{z}_2\) is expressed as

\[
M(s) = [I - P_N(s)K(s)]^{-1}P_N(s).
\] (37)

This function \(M(s)\) is called the settling function.(4) It is known that reducing the magnitude in a lower frequency domain moves the closed-loop poles to the left further in the complex plane and allows us to design a faster-responding regulator.(6)(7) This means that the above-mentioned weighting function \(W_2(s)\) is to be designed so that it has a high gain property in a lower frequency domain.

That is, the following relation is established:

\[
||W_2(s)M(s)||_{\infty} < \gamma.
\] (38)
Therefore, as the condition that Eqs. (36) and (38) are simultaneously satisfied we have
\[ \|G_{\text{op}}(s)\|_\infty = \left\| \begin{bmatrix} W_1(s)N(s) \\ \gamma^{-1}W_2(s)M(s) \end{bmatrix} \right\|_\infty < 1. \] (39)

The \( H^\infty \) controller \( K(s) \) satisfying the above condition is designed as follows. The generalized plant \( G_{\text{op}}(s) \) relating the input \((u_1, u_2, w_3, u)\) to the output \((\hat{z}_1, \hat{z}_2, \hat{y})\) is represented in the state variable form as
\[
\begin{bmatrix} \dot{\hat{z}} \\ \dot{\hat{y}} \end{bmatrix} = \begin{bmatrix} A_{n+2} & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{n+2} & B_{n+2}C_{N} & 0 & 0 & 0 \\ 0 & 0 & A_{N} & B_{N} & 0 & 0 \\ 0 & 0 & 0 & B_{N} & 0 & 0 \\ 0 & C_{n+2} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{N} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}.
\]

Together with the relations
\[
\dot{z} = [\hat{z}_1, \hat{z}_2]^T, \quad \dot{y} = [y_{N1}, y_{N2}]^T, \quad w = [u_1, u_2, w_3]^T.
\]

\[
\dot{C}_N = \begin{bmatrix} C_{N1} \\ G_{C_{N2}} \end{bmatrix}, \quad \dot{D}_{n1} = \begin{bmatrix} D_{n1} \\ 0_{1\times 4} \end{bmatrix}, \quad \dot{D}_{n2} = \begin{bmatrix} 0_{3\times 1} \\ D_{n2} \end{bmatrix}.
\]

There is also the relation \( D_{n1} = \sigma I \in \mathbb{R}^{4\times 4} \), where \( D_{n2} = \sigma \) is a minute disturbance and the coefficient \( G \) for the cart velocity \( \dot{S} \) is an adjusting factor used to improve the settling characteristics of the cart (The factor \( G \) is introduced because only the feedback signal \( S \) by itself cannot suppress the overshoot; here we put \( G = 1 \). Applying the analytical method by Glover and Doyle to the generalized plant equation (40), we have the following approximate solution:
\[
\begin{align*}
u &= K(s)y, \\
K(s) &= C_{k}(sI - A_{k})^{-1}B_{k}.
\end{align*}
\]

4. Experimental Method

First, in the numerical analysis, the elastic modes on both beams are finite-dimensionally approximated with \( n = 3 \) in Eq. (13). The modeling parameters employed here are \( M_1 = 4.4 \text{[kg]} \), \( D_1 = 19.2 \text{[Ns/m]} \), \( G_1 = 18.4 \text{[N/V]} \), \( l_1 = 0.27 \text{[m]} \), \( m_1 = 0.031 \text{[kg]} \), \( \rho_1 = 0.113 \text{[kg/m]} \), \( \mu_1 = 3.027 \times 10^{-4} \text{[Nms/rad]} \), \( I_1 = 1.25 \times 10^{-12} \text{[m^4]} \), \( E_1 = 206 \times 10^9 \text{[Nm/m]} \), \( C_{m1} = 1.001 \times 10^6 \text{[Ns/m]} \), \( d_1 = 0.001 \text{[m]} \), \( l_2 = 0.46 \text{[m]} \), \( m_2 = 0.027 \text{[kg]} \), \( \rho_2 = 0.059 \text{[kg/m]} \), \( \mu_2 = 5.109 \times 10^{-4} \text{[Nms/rad]} \), \( I_2 = 1.56 \times 10^{-15} \text{[m^4]} \), \( E_2 = 206 \times 10^9 \text{[Nm/m]} \), \( C_{m2} = 8.552 \times 10^7 \text{[Ns/m]} \), \( d_2 = 0.0005 \text{[m]} \).

The strain gauges are respectively attached at the flunk points in the first mode, which are located at \( r_{11} \approx 0.13 \text{[m]} \) on the lower beam and at \( r_{21} \approx 0.15 \text{[m]} \) on the upper beam.

Figure 3 shows the frequency responses to check the gain characteristics of each beam with respect to control \( u \). The solid lines represent the responses of the respective beams considered up to the third mode, while the broken line represents the response of beam 2, for the control system designed based on the model of Eq. (31). The resonant frequencies are \( f_{11} \approx 22.1 \text{[Hz]} \) and \( f_{12} \approx 152.6 \text{[Hz]} \) for beam 1 and \( f_{21} \approx 8.8 \text{[Hz]} \) and \( f_{22} \approx 32.1 \text{[Hz]} \) for beam 2. From Fig. 3 we see that the higher resonant frequencies on beam 1 are also reflected in the higher frequency ones on beam 2. Strictly speaking, however, the beam system has two uncertain portions with different vibration characteristics especially in the lower frequency range. Therefore, if the higher resonant modes (including all resonant modes of beam 1) of beam 2, whose resonant frequency of the primary mode is lower than that of beam 1, are taken as the multiplicative perturbation \( \Delta(s) \) given by Eq. (33), a high robust stability property may be achieved even for the parameter perturbation of beam 1. For the design of feedback compensation for the primary mode on beam 2, therefore, the weighting function \( W_1(s) \), which can be used to evaluate the robust stability performance, is selected so as to cover the multiplicative perturbation \( \Delta(s) \) as shown in Fig. 4.

Consequently, the following 4th-order weighting function
\[
W_1(s) = 15 \times \left( \frac{2\pi \times 8}{2\pi \times 22} \right)^{4} \times \frac{s^2 + (2 \times 0.7 \times 2\pi \times 22) s + (2\pi \times 22)^2}{s^2 + (2 \times 0.7 \times 2\pi \times 8) s + (2\pi \times 8)^2} , \] (45)
and the weighting function \( W_2(s) \) to suppress the vibration characteristics.
are employed, where each component is given as a 2nd-order rational function
\[ W_2(s) = \frac{k_{2i}}{s^2 + (2 \times 0.7 \times 2\pi \times 8)s + (2\pi \times 8)^2} \] (i = 1, ··· , 4).

Figure 5 shows the configuration of the experimental apparatus used in the experiment. The strain gauge on beam 1 is only for monitoring the vibration waveform. The actual control program is written in C language with the sampling period of 0.23 ms.

5. Results and Discussion

Figure 6 shows the simulation results on the control system with the designed $H^\infty$ controller for an applied fixed-value disturbance of $S = 0.3$ [m]. For comparison, it is confirmed that for the same disturbance the response by the LQ controller with high-frequency cutoff characteristics incurs the spillover instability probably due to the lack of robustness. Figure 6(I) shows the strain vibration $\varepsilon_1$ on beam 1, Fig. 6(II) the strain vibration $\varepsilon_2$ on beam 2, Fig. 6(III) the output angles $\theta_{p1}$ and $\theta_{p2}$ on both beams, and Fig. 6(IV) the horizontal displacement of the cart $S$. From these figures, it is easily seen that all outputs have satisfactory response speeds. Accordingly, it is confirmed that the weighting function $W_2(s)$ is appropriate for the settling function $M(s)$ formulated by Eq. (46).

Figure 7 shows the results obtained by using a Matlab function (Command impulse). Although in Fig. 7(I) and (II) excess vibrations occur on both beams, the vibrations of beam 1, which are not used for feedback, do not significantly affect the other responses (see Fig. 7(II) and (III)). As shown in Fig. 3, both the beams are dynamically coupled, and therefore the vibration mode of beam 1 significantly affects the motion of beam 2, causing parameter fluctuations but not generating excited (unstable) vibrations by spillover. Thus, the frequency weight, $W_1(s)$, which is given by Eq. (45) as a multiplicative perturbation so as to cover only the residual components of beam 2, can be valid for Eq. (35) of the transfer function $N(s)$.
Fig. 7 Simulation control for the impulsive response applied to the upper beam in the $H^\infty$ controller

Fig. 8 Real system control vs the cart-stepwise input using the $H^\infty$ controller

Fig. 9 Real system control vs impulsive disturbances applied to the upper beam in the $H^\infty$ controller

gles of both beams generate the hunting with amplitude of around 0.04 rad. Increasing the gains $k_{22}$ and $k_{23}W_2(s)$ in the weighting function Eq. (47) is expected to suppress this hunting to some extent, but may generate higher frequency vibrations (spillovers) due to high gain effects. In Fig. 8(IV) the beam hunting may be somewhat large, while the carriage displacement almost reaches the reference value of 0.3 m. When the cart almost reaches the reference value at $t = 2 \text{ s}$, a braking effect appears on the carriage system and produces a reaction force, which may make both the strain vibrations of Fig. 8(I) and (II) slightly larger.

Figure 9 shows the real system responses for an impulsively applied disturbance on beam 2. Both the strain vibrations take some time to converge. This may be largely due to the influence of higher vibrations on beam 1 (see Fig. 9(I)). In other words, the vibrations of beam 1, by propagating to beam 2, may amplify the parameter perturbation effect. It is experimentally confirmed that the beam vibration system tends to stably converge only when beam 2 is in a low-frequency vibration state. Therefore, for the system to be an easier controllable one it may be necessary to make beam 2 a little more flexible, and to separate more the natural frequencies of mode compo-
ments between the beams. This seems to confirm the eigen-

6. Conclusion

An inverted-double pendulum system consisting of
two elastic links connected in series has two different un-
certainties (resonant characteristics). By noting only the
vibration characteristics of beam 2 whose resonant fre-
frequency in the primary mode is lower, and by modeling the
uncertainties (residual modes) as a multiplicative pertur-
bation, it has been investigated from both numerical and
experimental points of view whether the spillovers due to
the residual modes on beam 1 can be suppressed.

(1) Although the dynamical coupling of vibration
modes of the two beams makes the stabilization control
difficult, a satisfactory stability property can be obtained
even when only the primary strain vibration of beam 2 is
used for feedback.

(2) The vibrations of both beams converge soon af-
after the vibration mode of beam 2 changes into a low-
frequency vibration state, while they are difficult to sup-
press when beam 2 is in a high-frequency vibration state.

(3) When beam 2, for which the vibration signal is
detected, is made more flexible for eigen-mode separation
between the beams, the stabilization control becomes easi-
er.

Consequently, the main purpose in the present paper
has been to experimentally verify the robust stability prop-
erty of the \( H^\infty \) controller designed for a flexible inverted-
double pendulum system. The authors will experimentally
verify the robust performance by improving the settling
characteristics of the beam deflection angles and by ana-
lyzing the spillover suppression problem in detail.

Appendix

The approximate solution (Eq. (44) in the text) for the
controller is at \( \gamma = 0.0039 \) given by

\[
K = \frac{1}{D(s)} [ Z_1(s) Z_2(s) Z_3(s) Z_4(s) Z_5(s) ], \quad \text{(A.1)}
\]

where

\[
D(s) = s^{20} + (4.23 \times 10^4)s^{19} + (2.35 \times 10^7)s^{18} + (6.57 \times 10^9)s^{17} + (1.20 \times 10^{12})s^{16} + (1.63 \times 10^{14})s^{15} + (1.71 \times 10^{16})s^{14} + (1.45 \times 10^{18})s^{13} + (1.01 \times 10^{20})s^{12} + (5.80 \times 10^{21})s^{11} + (2.77 \times 10^{23})s^{10} + (1.09 \times 10^{25})s^9 + (3.51 \times 10^{26})s^8 + (9.04 \times 10^{27})s^7 + (1.79 \times 10^{29})s^6 + (2.64 \times 10^{30})s^5 + (6.26 \times 10^{32})s^4
\]

\[
+ (1.09 \times 10^{33})s^3 + 6.20 \times 10^{32}, \quad \text{(A.2)}
\]

\[
Z_1(s) = -0.48s^{19} + (1.40 \times 10^4)s^{18} + (1.07 \times 10^7)s^{17} + (4.09 \times 10^9)s^{16} + (9.95 \times 10^{11})s^{15} + (1.72 \times 10^{14})s^{14} + (2.23 \times 10^{16})s^{13} + (2.25 \times 10^{18})s^{12} + (1.81 \times 10^{20})s^{11} + (1.18 \times 10^{22})s^{10} + (6.35 \times 10^{22})s^9 + (2.77 \times 10^{25})s^8 + (9.69 \times 10^{26})s^7 + (2.62 \times 10^{29})s^6 + (5.15 \times 10^{29})s^5 + (6.28 \times 10^{30})s^4 + (2.35 \times 10^{31})s^3 - (4.39 \times 10^{32})s^2 - (3.23 \times 10^{33})s - 2.42 \times 10^{33}, \quad \text{(A.3)}
\]

\[
Z_2(s) = 9.08s^{19} + (2.08 \times 10^5)s^{18} + (1.24 \times 10^6)s^{17} + (3.74 \times 10^10)s^{16} + (6.96 \times 10^{12})s^{15} + (8.64 \times 10^{14})s^{14} + (7.00 \times 10^{16})s^{13} + (3.00 \times 10^{18})s^{12} + (8.18 \times 10^{19})s^{11} - (2.70 \times 10^{22})s^{10} - (2.69 \times 10^{24})s^9 - (1.79 \times 10^{26})s^8 - (8.88 \times 10^{27})s^7 - (3.34 \times 10^{29})s^6 - (9.44 \times 10^{30})s^5 - (1.89 \times 10^{32})s^4 - (2.49 \times 10^{33})s^3 - (1.73 \times 10^{34})s^2 - (5.08 \times 10^{34})s - 3.58 \times 10^{34}, \quad \text{(A.4)}
\]

\[
Z_3(s) = -168.4s^{19} - (6.86 \times 10^6)s^{18} - (4.73 \times 10^9)s^{17} - (1.66 \times 10^{12})s^{16} - (3.79 \times 10^{14})s^{15} - (6.19 \times 10^{16})s^{14} - (7.62 \times 10^{18})s^{13} - (7.36 \times 10^{20})s^{12} - (5.71 \times 10^{22})s^{11} - (3.61 \times 10^{24})s^{10} - (1.87 \times 10^{26})s^9 - (7.92 \times 10^{27})s^8 - (2.69 \times 10^{29})s^7 - (7.16 \times 10^{30})s^6 - (1.43 \times 10^{32})s^5 - (2.00 \times 10^{33})s^4 - (1.76 \times 10^{34})s^3 - (8.42 \times 10^{34})s^2 - (1.80 \times 10^{35})s - 1.11 \times 10^{35}, \quad \text{(A.5)}
\]

\[
Z_4(s) = 0.01s^{19} + 485s^{18} + (3.24 \times 10^5)s^{17} + (1.11 \times 10^6)s^{16} + (2.45 \times 10^8)s^{15} + (3.86 \times 10^{10})s^{14} + (4.61 \times 10^{12})s^{13} + (4.32 \times 10^{14})s^{12} + (3.28 \times 10^{16})s^{11} + (2.05 \times 10^{18})s^{10} + (1.07 \times 10^{20})s^9 + (4.60 \times 10^{22})s^8 + (1.60 \times 10^{24})s^7 + (4.40 \times 10^{26})s^6 + (9.05 \times 10^{27})s^5 + (1.30 \times 10^{29})s^4 + (1.16 \times 10^{30})s^3 + (5.57 \times 10^{30})s^2 + (1.19 \times 10^{31})s + 7.41 \times 10^{30}, \quad \text{(A.6)}
\]

\[
Z_5(s) = -3.44s^{19} - (2.50 \times 10^7)s^{18}, \quad \text{(A.7)}
\]
\[-(9.2 \times 10^9)s^{17} - (2.17 \times 10^{12})s^{16} - (3.68 \times 10^{14})s^{15} - (4.69 \times 10^{16})s^{14} - (4.68 \times 10^{18})s^{13} - (3.74 \times 10^{20})s^{12} - (2.44 \times 10^{22})s^{11} - (1.30 \times 10^{24})s^{10} - (5.69 \times 10^{25})s^9 - (2.00 \times 10^{27})s^8 - (5.54 \times 10^{28})s^7 - (1.14 \times 10^{30})s^6 - (1.63 \times 10^{31})s^5 - (1.32 \times 10^{32})s^4 - (2.91 \times 10^{32})s^3 + (4.81 \times 10^{32})s^2 - (2.50 \times 10^{33})s - 2.81 \times 10^{33} \] 

(A.7)

References


