LMI-Based Stability Criteria for Large-Scale Time-Delay Systems

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In this paper, the problem of asymptotic stability analysis for a class of linear large-scale systems with time delay in the state of each subsystem as well as in the interconnections is addressed in detail. By utilizing a model transformation and the Lyapunov stability theory, a delay-dependent criterion for stability analysis of the systems is derived in terms of some certain linear matrix inequalities (LMIs). A numerical example is given to illustrate that the proposed result is effective.

Key Words: Stability Criteria, Large-Scale Systems, Linear Matrix Inequality (LMI)

1. Introduction

In the large-scale dynamical system, time delay naturally arises in the processing of information transmission between subsystems and its existence is frequently the main cause source of instability and oscillation in many important systems. Many practical control applications can be often encountered including electrical power systems, nuclear reactors, chemical process control systems, transportation systems, computer communication, economic systems, etc. In the recent years, the problem of stability analysis for large-scale systems with or without delay has been extensively studied by a number of authors: see, for example, Refs. (3)–(12). Moreover, by depending on whether the stability criterion itself contains the delay argument as a parameter, stability criteria for systems can be usually classified into two categories, namely delay-independent criterion: see, for example, Refs. (3)–(6), (8), (11), (12) and delay-dependent criterion: see, for example, Ref. (10). In general, the latter ones are less conservative than the former ones.

However, to the best author's knowledge, few papers are considered to derive the delay-dependent and delay-independent stability criteria for a class of large-scale time-delay systems. Hence, we consider the following large-scale time-delay systems, which is composed of interconnected subsystems $S_i, i \in \mathbb{N}$. Each subsystem $S_i, i \in \mathbb{N}$, is described as

\[ \dot{x}_i(t) = A_i x_i(t) + \sum_{j=1}^{N} A_{ij} x_j(t-h_{ij}), \quad t \geq 0, \]  

where $t$ is the time, $x_i(t)$ is the state of each subsystem $S_i, i \in \mathbb{N}$, and $A_i, A_{ij}, i, j \in \mathbb{N}$, are known constant matrices with the appropriate dimensions, and $h_{ij}, i, j \in \mathbb{N}$, are nonnegative time delays in the state of each subsystem as well as in the interconnections. The initial condition for each subsystem is given by

\[ x_i(t) = \theta_i(t), t \in [-H,0], H = \max_{i,j} \{ h_{ij} \}, i, j \in \mathbb{N}, \]  

where $\theta_i(t), i \in \mathbb{N}$, is a continuous function on $[-H, 0]$.

In this paper, delay-independent and delay-dependent criteria for such systems can be derived to guarantee the asymptotic stability for large-scale systems with time delay in the state of each subsystem as well as in the interconnections. Appropriate model transformation of nominal original systems is useful for the stability analysis of the systems, and some tuning parameters which satisfy constraint on the LMIs can be easily obtained by the LMI technique. A numerical example is given to illustrate that the proposed result is useful.

The notation used throughout this paper is as follows. We denote the set of all nonnegative real numbers by $\mathbb{R}_+$, the transpose of matrix $A$ (resp., vector $x$) by $A^T$ (resp., $x^T$), the symmetric positive (resp., negative) definite by $A > 0$ (resp., $A < 0$). We denote identity matrix by $I$ and the set $\mathbb{N}$ by $\{1,2,\cdots,N\}$.

2. Stability Analysis

Before we develop a delay-dependent stability criterion, we give the model transformation and some lemmas which are necessary to derive the following results.
An important model transformation is constructed for large-scale time-delay systems of the form:

\[
d\frac{dx(t)}{dt} + \sum_{j=1}^{N} B_j(t) x_j(t) \sum_{j=1}^{N} x_j(s) ds = A_1 x(t) + \sum_{j=1}^{N} [B_j x_j(t) + (A_j - B_j) x_j(t - \tau_j)],
\]

where \( B_j, j \in \mathbb{N} \), are some matrices such that the matrix \( \check{A}_1 = A_1 + B_j, i \in \mathbb{N} \) is Hurwitz.

Lemma 1.1:

Consider an operator \( L_i : C_0 \rightarrow \mathbb{R}, i \in \mathbb{N} \), with \( L_i(x_i) = x_i(t) + \sum_{j=1}^{N} B_j(t) x_j(t) \sum_{j=1}^{N} x_j(s) ds \). For a given scalar \( \delta_i \) satisfying \( 0 < \delta_i < 1 \), the operator \( L_i(x) \) is stable if a positive definite symmetric matrix \( Q_i \) exists such that the following LMI holds:

\[
\begin{bmatrix}
-h_1^{1/2} \delta_i Q_i & 0 & 0 & B_i^T \Omega_i \\
0 & \ddots & \vdots & \vdots \\
0 & \ddots & \ddots & \vdots \\
-h_N^{1/2} \delta_i Q_i & 0 & 0 & -\Omega_i \\
\end{bmatrix} < 0,
\]

(3)

where \( C_0 \) is a continuous function and \( \Omega_i = \sum_{j=1}^{N} h_j Q_{ji} \).

Lemma 2.1:

The LMI

\[
\begin{bmatrix}
Q(y) & S(y) \\
S(y)^T & R(y)
\end{bmatrix} < 0,
\]

(4)

where \( Q(y) = Q(y)^T, R(y) = R(y)^T \), and \( S(y) \) depend affinely on \( y \).

Now, we present a flexible delay-dependent result for asymptotic stability of system (1).

Lemma 3.1:

If there exist symmetric and positive definite matrices \( P_i, R_i, T_{jk}, V_i, W_{jk}, X_{jk}, Y_{jk} \), and matrices \( U_i, i, j, k \in \mathbb{N} \), such that the following LMI conditions hold:

\[
\check{\Xi}_{ii} = \begin{bmatrix}
\check{\Xi}_{11} & \check{\Xi}_{12} & \check{\Xi}_{13} & \check{\Xi}_{14} & \check{\Xi}_{15} \\
\check{\Xi}_{12} & -\check{\xi}_{22} & 0 & 0 & 0 \\
\check{\Xi}_{13} & 0 & -\check{\Xi}_{33} & 0 & 0 \\
\check{\Xi}_{14} & 0 & 0 & -\check{\Xi}_{44} & 0 \\
\check{\Xi}_{15} & 0 & 0 & 0 & -\check{\Xi}_{55}
\end{bmatrix} < 0,
\]

(4)

\[
X_{jk} - 2P_j \begin{bmatrix}
U_j & U_j^T
\end{bmatrix} \check{\Xi}_{11} < 0,
\]

(5)

\[
Y_{jk} - 2P_j \begin{bmatrix}
U_j & U_j^T
\end{bmatrix} \check{\Xi}_{11} < 0,
\]

(6)

where

\[
\check{\Xi}_{11} = \begin{bmatrix}
\check{\psi}_1 & U_{11}^T + U_{12} & \cdots & U_{1N}^T + U_{1N+1} \\
U_{21} & U_{22} & \cdots & U_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
U_{N1} & U_{N2} & \cdots & \check{\psi}_N
\end{bmatrix},
\]

Furthermore, assume that \( B_j = P_j T_j \) satisfying the Lemma 1, such that \( \check{A}_i \) is Hurwitz. Then the system (1) is asymptotically stable for any constant time-delays \( \tau_j \) satisfying \( 0 < \tau_j < \tau_i, i, j \in \mathbb{N} \).

Proof:

From Lemma 2, the condition (4) is equivalent to the following result

\[
\begin{bmatrix}
\phi_1(\check{\nu}) & U_{12} & \cdots & U_{1N}^T + U_{1N+1} \\
U_{21} & U_{22} & \cdots & U_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
U_{N1} & U_{N2} & \cdots & \phi_N(\check{\nu})
\end{bmatrix} < 0,
\]

(7)

where \( \check{\nu} \) denotes \( \check{\nu}_{ij}, i, j \in \mathbb{N} \), and

\[
\phi_j(\check{\nu}) = A_j^T P_j + P_j A_j + U_j^T + \sum_{j=1}^{N} \check{h}_{ij} A_j^T U_j R_j^T U_j^T A_j,
\]

(8.a)

where

\[
V(x_i) = V_1(x_i) + V_2(x_i) + V_3(x_i),
\]

(8.b)
\[ L_i(x_i) = x_i(t) + \sum_{j=1}^{N} B_{ij} \int_{t-h_j}^{t} x_j(s) ds, \quad i \in \mathbb{N}, \]  
(8.b)

\[ V_2(x_i) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t-h_j}^{t} x_i(t) \left[ (s-t+h_j) \cdot R_{ij} + (A_{ij} - B_{ij})^{T} \cdot P_i (A_{ij} - B_{ij}) \right] x_j(s) ds, \]  
(8.c)

\[ V_3(x_i) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t-h_j}^{t} (s-t+h_j) \cdot x_i(t) (T_{ijk} + W_{ijk}) x_j(s) ds, \]  
(8.d)

is a Lyapunov functional candidate. The time derivatives of \( V_i(x_i), i = \mathbb{N}, \) along the trajectories of the system (2) is given by

\[ \dot{V}_i(x_i) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ h_{ij} \cdot R_{ij} + (A_{ij} - B_{ij})^{T} \cdot P_i (A_{ij} - B_{ij}) \right] x_j(s) ds, \]  
(8.e)

\[ \dot{V}_2(x_i) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ \int_{t-h_j}^{t} x_i(s) \left[ (s-t+h_j) \cdot R_{ij} + (A_{ij} - B_{ij})^{T} \cdot P_i (A_{ij} - B_{ij}) \right] x_j(s) ds, \right. \]  
(8.f)

\[ \dot{V}_3(x_i) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ (s-t+h_j) \cdot x_i(t) (T_{ijk} + W_{ijk}) x_j(s) ds, \right. \]  
(8.g)

Now, considering that

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ \int_{t-h_j}^{t} x_i(s) \left[ h_{ij} \cdot R_{ij} + (A_{ij} - B_{ij})^{T} \cdot P_i (A_{ij} - B_{ij}) \right] x_j(s) ds, \right. \]  
(8.h)

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ (s-t+h_j) \cdot x_i(t) (T_{ijk} + W_{ijk}) x_j(s) ds, \right. \]  
(8.i)

It is known fact that for any \( x, y \in \mathbb{R}^{n} \) and \( Z \in \mathbb{R}^{n \times n} > 0, \) the inequality \( 2x^{T}y \leq x^{T}Zx + y^{T}Z^{-1}y \) is true, we have

\[ \dot{V}_1(x_i) \leq \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ h_{ij} \cdot x_i(t) (A_{ij}^{T} P_i + P_i A_j) x_j(t) \right. \]  
(8.j)

\[ + \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ h_{ij} \cdot x_i(t) (A_{ij}^{T} P_i B_i R_{ij}^{T} B_{ij}^{T} P_i A_j) x_j(t) \right. \]  
(8.k)

\[ + \left. \int_{t-h_j}^{t} x_i(s) R_{ij} x_j(s) ds \right] \]  
(8.l)

\[ + \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ x_i(t) (B_{ij}^{T} P_i + P_i B_i) x_j(t) \right. \]  
(8.m)

\[ + \sum_{i=1}^{N} \sum_{j=1}^{N} \left. \sum_{j \neq j}^{N} \left[ h_{ij} \cdot x_i(t) (B_{ij}^{T} P_i B_i T_{ijk}^{T} B_{ijk}^{T} P_i B_i x_j(t) \right. \right. \]  
(8.n)

\[ + \left. \left. \int_{t-h_k}^{t} x_i(s) T_{ijk} x_j(s) ds \right] \right. \]  
(8.o)

\[ + \sum_{i=1}^{N} \sum_{j=1}^{N} \left. \left[ x_i(t) - h_{ij} (A_{ij} - B_{ij})^{T} P_j V_{ji}^{-1} P_i (A_{ij} - B_{ij}) \right. \right. \]  
(8.p)

\[ \cdot x_j(t - h_j) + x_i(t) (V_{ji} x_j(t)] \right. \]  
(8.q)

\[ + \sum_{i=1}^{N} \sum_{j=1}^{N} \left. \left[ h_{ij} \cdot x_i(t) (A_{ij} - B_{ij})^{T} P_j B_i W_{ijk}^{-1} B_{ijk}^{T} P_i B_i x_j(t) \right. \right. \]  
(8.r)

\[ - \left. \left. P_i (A_{ij} - B_{ij}) x_j(t - h_j) + \int_{t-h_k}^{t} x_i(s) W_{ijk} x_j(s) ds \right] \right. \]  
(8.s)
Assume that there exist symmetric and positive definite matrices $X_{ij}$, $Y_{ik}$, $i,j,k \in \mathbb{N}$, such that the following inequalities are satisfied
\[ X_{ij}^{-1} - B_{jk} T_{jk}^{-1} B_{jk}^T > 0, \tag{9} \]
and
\[ Y_{ik}^{-1} - B_{jk} W_{jk}^{-1} B_{jk}^T > 0. \tag{10} \]

Pre- and post-multiplying both sides of (9) and (10) by $P_j$ yield
\[ -P_j X_{ij}^{-1} P_j + U_{jk} T_{jk}^{-1} U_{jk}^T < 0, \]
and
\[ -P_j Y_{ik}^{-1} P_j + U_{jk} W_{jk}^{-1} U_{jk}^T < 0. \]

with the equality manipulation, we obtain
\[-(X_{ij} - P_j)^T Y_{ik}^{-1} (X_{ij} - P_j) + X_{ij} - 2P_j + U_{jk} T_{jk}^{-1} U_{jk}^T < 0, \]
and
\[ -(Y_{ik} - P_j)^T Y_{ik}^{-1} (Y_{ik} - P_j) + Y_{ik} - 2P_j + U_{jk} W_{jk}^{-1} U_{jk}^T < 0. \]

By using Lemma 2, it follows that
\[ \begin{bmatrix} X_{ij} - 2P_j & U_{jk} \\
                      U_{jk}^T & -T_{jk} \end{bmatrix} < 0, \]
and
\[ \begin{bmatrix} Y_{ik} - 2P_j & U_{jk} \\
                      U_{jk}^T & -W_{jk} \end{bmatrix} < 0. \]

Thus, we have
\[ \dot{V}(x_i) \leq X^T(t) \Xi \dot{X}(t), \tag{11} \]
where $X(t) = [x_1(t), \ldots, x_N(t)]^T$.

By Lemma 1, the condition (3) implies that the operator $L_c(x_i) = x_i(t) + \sum_{j=1}^{N} B_{ij} \int_{0}^{t} x_j(s) ds, i \in \mathbb{N}$, is stable [2, Theorem 9.3.5]. Hence, by Theorem 9.8.1 of Ref. (2) with Eqs. (7)–(11), we conclude that systems (1) and (2) are both asymptotically stable for any constant time delays $h_{ij}$ satisfying $0 \leq h_{ij} \leq \bar{h}_{ij}, i,j \in \mathbb{N}$.

Remark 1:

Notice that for any chosen matrix $B_{ij} = 0, i,j \in \mathbb{N}$, it represents that the delay term $B_{ij} \int_{0}^{t} x_j(s) ds, i,j \in \mathbb{N}$ have not been converted to the left side of the system (2). By the above Lemma 3, the corresponding matrices $R_{ij}$, $T_{jk}$, and $W_{jk}$, $i,j,k \in \mathbb{N}$ could be chosen as zero matrices, and LMI the condition in Eq. (4) are reduced by their corresponding elements.

Letting $B_{ij} = 0, i,j \in \mathbb{N}$, in Lemma 3, we achieve the following result that doesn’t depend on delay arguments.

Theorem 1:

The system (1) is asymptotically stable with $h_{ij} \in \mathbb{R}^+$, $i,j \in \mathbb{N}$, provided that $A_i$ is Hurwitz, and there exist $P_i$, and $V_{ij}, i,j \in \mathbb{N}$, such that the following LMI condition holds:
\[ \begin{bmatrix} \Xi_{11} & \Xi_{12} \\
                      \Xi_{12}^T & -\Xi_{22} \end{bmatrix} < 0. \tag{12} \]

where
\[ \Xi_{11} = \begin{bmatrix} \hat{\phi}_1 & \cdots & 0 \\
                           0 & \ddots & \ddots \\
                           \vdots & \ddots & \ddots \\
                           0 & \cdots & \hat{\phi}_N \end{bmatrix}, ~ \Xi_{12} = A_i^T P_i + P_i A_i + \sum_{j=1}^{N} V_{ij}, \\
\Xi_{22} = diag(\Lambda_1, \ldots, \Lambda_N), ~ \Lambda_i = [A_i^T P_i A_i + \sum_{j=1}^{N} V_{ij}]. \]

Proof: The proof directly follows that the proof of Lemma 3 with $B_{ij} = 0, i,j \in \mathbb{N}$.

Remark 2:

Suppose that for any chosen matrix $B_{ij} = A_{ij}, i,j \in \mathbb{N}$, it represents that the delay term $A_{ij} \int_{0}^{t} x_j(s) ds, i,j \in \mathbb{N}$ have been converted to the left side of the system (2). By the proof of Lemma 3, the corresponding matrices $V_{ij}$, and $W_{jk}, i,j,k \in \mathbb{N}$ could be chosen as zero matrices, and LMI the condition in Eq. (4) are reduced by their corresponding elements.

Simply choosing $B_{ij} = A_{ij}, i,j \in \mathbb{N}$, in Lemma 3, we achieve the following result that depend on delay arguments.

Theorem 2:

The system (1) is asymptotically stable for any constant time delays $h_{ij}$ satisfying $0 \leq h_{ij} \leq \bar{h}_{ij}, i,j \in \mathbb{N}$. Provided that $\hat{A}_i = A_i + A_{ii}$ is Hurwitz and there exist symmetric positive definite matrices $P_i$, $R_{ij}$, and $T_{jk}, i,j,k \in \mathbb{N}$, satisfying the following LMI condition holds:
\[ \begin{bmatrix} \Xi_{11} & \Xi_{12} \\
                      \Xi_{12} & -\Xi_{22} \end{bmatrix} < 0. \tag{13} \]

where
\[ \Xi_{11} = \begin{bmatrix} \hat{\phi}_1 & A_{i1}^T P_1 + P_1 A_{i2} + \cdots + A_{iN}^T P_N + P_N A_{iN} \\
                           A_{i1} & \ddots & \ddots \\
                           \vdots & \ddots & \ddots \\
                           A_{iN} & \ddots & \hat{\phi}_N \end{bmatrix}, \]
\[ \Xi_{12} = diag(\Gamma_1, \ldots, \Gamma_N), \]
\[ \Gamma_i = [\sqrt{h_{ij} A_{i1}^T P_1 A_{i1}} \cdots \sqrt{h_{ij} A_{iN}^T P_N A_{iN}}], \]
\[ \Xi_{22} = diag(\hat{\Gamma}_1, \cdots, \hat{\Gamma}_N), \]
\[ \hat{\Gamma}_i = diag(\Gamma_1, \cdots, \Gamma_N), \]
\[ \Pi_i = [\Pi_1(1, \ldots, \Pi_N(1)], \]
\[ \Pi_{kl} = \begin{bmatrix} \sqrt{h_{01}} A^T_{p1} P_{A1} \cdots \sqrt{h_{0N}} A^T_{pN} P_{A1} \end{bmatrix}, \]

\[ \Pi_{13} = \text{diag}(\hat{\Pi}_1, \cdots, \hat{\Pi}_N), \]

\[ \Pi_{M} = \text{diag}(T_{a1}, \cdots, T_{akh}), \]

Proof: The proof directly follows that the proof of Lemma 3 with \( B_j = A_{ij}, \ i, j \in \mathbb{N} \).

3. Example

Example. Large-scale system with time delays:

\[ \dot{x}_1(t) = \begin{bmatrix} -1.6 & 0 \\ 0 & -2 \end{bmatrix} x_1(t) + \begin{bmatrix} -1 & -0.2 \\ 0 & -1 \end{bmatrix} x_1(t-h_{11}) + \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix} x_2(t-h_{12}) \]

\[ \dot{x}_2(t) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} x_2(t) + \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix} x_1(t-h_{21}) + \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix} x_2(t-h_{22}). \]

Comparison of system (1) with system (14), we have \( N = 2 \). By Lemma 3 with \( \hat{h}_{11} = 2.255 \), and \( B_{12} = B_{23} = B_{22} = 0 \), the LMI (4) is satisfied with

\[ P_1 = \begin{bmatrix} 0.4527 & 0.5002 \\ 0.5002 & 6.6257 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.8702 & 3.3020 \\ 3.3020 & 24.4357 \end{bmatrix}, \]

\[ R_{11} = \begin{bmatrix} 0.0002 & -0.0043 \\ -0.0043 & 0.3577 \end{bmatrix}, \quad V_{11} = \begin{bmatrix} 0.4539 & 0.5008 \\ 0.5008 & 6.7107 \end{bmatrix}, \]

\[ V_{21} = \begin{bmatrix} 2.492 & 3.5960 \\ 3.5960 & 12.7234 \end{bmatrix}, \quad V_{12} = \begin{bmatrix} 0.0163 & 0.0273 \\ 0.0273 & 1.5300 \end{bmatrix}, \]

\[ V_{22} = \begin{bmatrix} 0.9196 & 1.5110 \\ 1.5110 & 12.4483 \end{bmatrix}, \quad X_{111} = \begin{bmatrix} 0.5723 & 0.4057 \\ 0.4057 & 5.5520 \end{bmatrix}, \]

\[ Y_{111} = \begin{bmatrix} 0.9050 & 1.0082 \\ 1.0082 & 12.6109 \end{bmatrix}, \]

\[ T_{111} = \begin{bmatrix} 0.0002 & -0.0042 \\ -0.0042 & 0.3506 \end{bmatrix}, \]

\[ W_{111} = \begin{bmatrix} 0.0002 & -0.0042 \\ -0.0042 & 0.3510 \end{bmatrix}, \]

\[ U_{11} = \begin{bmatrix} -0.0001 & 0.0004 \\ 0.0004 & -0.0251 \end{bmatrix}, \]

and \( B_{11} = P_1^{-1} U_{11} = \begin{bmatrix} -0.0004 & 0.0055 \\ 0.0004 & -0.0042 \end{bmatrix} \).

Hence, we conclude that systems (14) is asymptotically stable for \( 0 \leq h_{11} \leq 2.255 \), and \( h_{12}, h_{21}, h_{22} \in \mathbb{R}_+ \). The delay-independent criteria in Refs. (3), (4), (6), (8), (11) and (12) cannot be satisfied. The delay-dependent stability criteria of Ref. (10) cannot be applied for sufficiently large time delays, \( h_{12}, h_{21}, h_{22} \in \mathbb{R}_+ \).

4. Conclusions

A flexible delay-dependent stability criterion ex-

pressed in terms of some certain LMIs has been provided to guarantee asymptotic stability for a class of large-scale time delays systems. The effectiveness of the proposed result has been demonstrated through a numerical example.

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References


