Attitude Control of a Rhönrad by Internal Mass Motion

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Rhönrad is a gymnastic equipment that is interesting in terms of dynamics and control. The control method of the Spiral motion by the mass motion in the Rhönrad was clarified. First, the condition for a steady Spiral motion with arbitrary mass position was obtained. Then, by using the linearized equation of motion, the control method to control the attitude of the Rhönrad by the mass motion in it was derived. This control method allows the state of the Rhönrad to be changed from one steady Spiral motion to another steady motion.

Key Words: Rhönrad, Spiral, Steady Motion, Attitude Control, Mass Motion, Uncontrollable Mode

1. Introduction

Rhönrad is a gymnastic apparatus that is interesting in terms of dynamics and control. The Rhönrad was invented in Germany, and it was developed as a equipment for training, sports and games. In the Rhönrad game, there are two motions, ‘Straight-line’ and ‘Spiral’. ‘Straight-line’ is a rotary motion without any inclinations of the Rhönrad as seen in Fig. 1 (a). ‘Spiral’ is a rotary motion with an inclination of the Rhönrad as seen in Fig. 1 (b). The feature of these motions is that the Rhönrad is controlled by the motion of a man in it. We have already clarified the motion of the Rhönrad when a gymnast moves synchronously with the rotation of the Rhönrad and, the intrinsic characteristics of the Spiral motion without the motion of the gymnast. As the next step of our study, the control method of the attitude of the Rhönrad is clarified.

2. Nomenclatures

- $m$: mass of control mass
- $M$: mass of Rhönrad
- $R$: radius of Rhönrad
- $h$: width of Rhönrad
- $l$: a half of width of Rhönrad ($=h/2$)
- $I_p, I_d$: moment of inertia
- $\phi, \theta, \psi$: rotation angle
- $\omega, \Omega$: angular velocity
- $p_x, p_y, p_z$: displacement of control mass
- $F_r$: reaction force between Rhönrad and control mass
- $F_f$: frictional force from the ground
- $N_n$: normal reaction from the ground

![Fig. 1 Photograph of Rhönrad](image-url)

(a) Straight-line

(b) Spiral

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3. Analytical Model and Derivation of Basic Equations

Figures 2 and 3 show an analytical model and coordinate systems. In this model, gymnast is represented by an internal control mass. Point \( C \) represents the ground contact point of the Rh¨onrad. Point \( A \) represents the center of the Rh¨onrad. Point \( P \) represents the position of the control mass. \( O-XYZ \) are fixed coordinates. \( A-xyz \) are the posture coordinates of the Rh¨onrad whose y axis is horizontal. \( A-x'y'z' \) are moving coordinates fixed at the Rh¨onrad. The spin of the Rh¨onrad is represented by angle \( \phi \). The inclination of the Rh¨onrad is denoted by \( \theta \). The precession angle of the Rh¨onrad is represented by \( \psi \). The basic equations are derived on the basis of these coordinates (3).

The angular velocity of the Rh¨onrad is expressed as

\[
\omega = \dot{\phi} + \dot{o}j + \dot{\psi}k = (\dot{\phi} - \psi \sin \theta)i + \dot{o}j + \dot{\psi} \cos \theta k. \tag{1}
\]

The angular momentum of the Rh¨onrad \( H \) has components

\[
H = (I)\omega = I_\phi (\dot{\phi} - \psi \sin \theta)i + I_o \dot{o}j + I_\psi \psi \cos \theta k. \tag{2}
\]

The equations of translation of the control mass and Rh¨onrad are expressed as

\[
m(\dot{r}_{PA} + \dot{r}_{AO}) = F_i + mg(\sin \theta i - \cos \theta k) \tag{3}
\]

\[
M\ddot{r}_{AO} = F_j + N_n + Mg(\sin \theta i - \cos \theta k) - F_r \tag{4}
\]

where \( r_{AO} \) is a position vector from the point \( O \) to \( A \), \( F_i \) is the reaction force between the Rh¨onrad and the control mass, \( F_r \) is the frictional force from the ground, and \( N_n \) is the normal reaction from the ground. The equation of angular momentum of the Rh¨onrad is expressed as

\[
\frac{d}{dt} (H) = r_{PA} \times (-F_r) + r_{CA} \times (F_r + N_n). \tag{5}
\]

We assume that the Rh¨onrad rolls without sliding on the ground. Therefore, the constraint condition is expressed as

\[
r_{AO} = \omega \times r_{AC}. \tag{6}
\]

Moreover, the position vector of the control mass is expressed as

\[
r_{PA} = p_x i + p_y j + p_z k. \tag{7}
\]

Equations (3) to (7) each have three components. Therefore, the total number of equations of motion is 15. But by eliminating all the variables, except for \( \phi, \theta, \psi, p_x, p_y \) and \( p_z \), these equations are reduced to 3 components. If \( \phi \) and \( \psi \) are replaced with \( \omega \) and \( \Omega \) respectively, then these three equations can be written as

\[
f_i(\omega, \Omega, \phi, \psi, p_x, p_y, p_z) = 0. \tag{8}
\]

The analytical model is considered on the basis of the real Rh¨onrad and man. Thus, the radius of the Rh¨onrad is 1 m, the width of the Rh¨onrad is 0.5 m, the mass of the Rh¨onrad is 40 kg, and the mass of the control mass is 50 kg.

4. Condition for a Steady Spiral Motion

In the general Spiral motion, Rh¨onrad exhibits periodic vibration while rolling, as shown with the solid line in Fig. 4. But under a special condition, the Rh¨onrad does not vibrate, and its contacting point draws a circle without fluctuation, as shown with the dashed line in Fig. 4. In this section, we aim to derive the condition for a steady Spiral motion.

When the Rh¨onrad rotates steadily, \( \omega, \theta \) and \( \Omega \) are constant. Therefore

\[
\text{Fig. 4 The locus of a contacting point and the inclination angle } \theta \text{ at Spiral motion}
\]
from Eq. (11), the Rhönrad rolls without vibration, as
also varies.

Therefore  

where subscript 0 stands for the values at a steady motion. 
Since  \( p_{y0} = 0 \) as shown in the previous chapter,  

\[
p_y = \Delta p_y.  
\]

By performing Taylor expansion of Eq. (8) with respect to  
\( \omega_0, \theta_0, \Omega_0, p_{x0}, p_{y0}, \Delta p_0 = 0 \), and ignoring higher order 
terms of the perturbations, Eq. (8) can be linearized as follows.

\[
M_A \begin{bmatrix} \Delta \omega \\ \Delta \theta \\ \Delta \Omega \\ \Delta p_x \\ \Delta p_y \\ \Delta p_z \\ \Delta \theta_0 \\ \Delta \Omega_0 \\ \Delta p_{x0} \\ \Delta p_{y0} \\ \Delta p_{z0} \end{bmatrix} + M_B \begin{bmatrix} \Delta \dot{\omega} \\ \Delta \dot{\theta} \\ \Delta \dot{\Omega} \\ \Delta \ddot{p}_x \\ \Delta \ddot{p}_y \\ \Delta \ddot{p}_z \end{bmatrix} = 0, 
\]

\[
M_C \begin{bmatrix} \Delta p_x \\ \Delta p_y \\ \Delta p_z \end{bmatrix} = 0, 
\]

\[
: M_A \in R^{3x3}, M_B \in R^{3x10}, M_C \in R^{3x3} 
\]

We continue our discussion below assuming that the 
acceleration of the control mass, \( \dot{p}_x, \dot{p}_y, \) and \( \dot{p}_z \), can be 
arbitrarily controlled and using these as control inputs.

5. Linearization of Equations of Motion

Small perturbations of the Rhönrad from a steady motion 
are assumed to be in the following form:

\[
\begin{align*}
\omega &= \omega_0 + \Delta \omega \\
\theta &= \theta_0 + \Delta \theta \\
\Omega &= \Omega_0 + \Delta \Omega \\
p_x &= p_{x0} + \Delta p_x \\
p_y &= p_{y0} + \Delta p_y \\
p_z &= p_{z0} + \Delta p_z 
\end{align*}
\]

(13)

Fig. 5 Surface of condition for steady Spiral \((p_x = 0, p_y = 0)\)

\[
\omega = \dot{\theta} = \Omega = \dot{\theta} = 0.  
\]

Also, it is assumed that the control mass does not move in the Rhönrad. Therefore  

\[
\dot{p}_x = \dot{p}_y = \dot{p}_z = \dot{p}_y = \dot{p}_z = 0.  
\]

By substituting Eqs. (9) and (10) into Eq. (8), the condition 
for a steady Spiral motion can be obtained as shown below,

\[
p_y = \frac{D_{d1} \Omega^2 + D_{d2} \Omega}{D_{d3} \Omega},  
\]

(11)

where  \( D_{d1}, D_{d2} \) and  \( D_{d3} \) are

\[
\begin{align*}
D_{d1} &= -2 \cos 2\theta [(m + M) + l m p_x - m p, (R + p_z)] \\
&+ 2 \sin 2\theta \left[ -l_R + l_B - (m + M) (l^2 - R^2) + 2 l m p_x \\
&- m p_x^2 + m p (2 R + p_z) \right] \\
D_{d2} &= 2 \sin 2\theta \left[ l f_p - l f + (m + M) R^2 + m R p_z \right] \\
&+ 2 l (m + M) - m p_x \sin \theta \\
D_{d3} &= 2 \sin \theta (R + m R^2) + m R \sin \theta \\
&+ 2 l (m + M) - m p_x \sin \theta.
\end{align*}
\]

If the initial angle and angular velocity are determined 
from Eq. (11), the Rhönrad rolls without vibration, as 
shown with the dashed line in Fig. 4. Equation (11) can 
be represented as a three-dimensional curved surface with 
axes  \( \omega, \theta, \) and  \( \Omega \), as shown in Fig. 5 \((p_x = 0, p_y = 0)\). 
If the value of  \( p_x \) or  \( p_z \) varies, the shape of the curved surface 
also varies.

6. Attitude Control of a Rhönrad by 2-DOF Motion of 
Control Mass

In this section, we aim to control the attitude of the 
Rhönrad from one steady motion to another steady motion 
within the three dimensional curved surface of Fig. 5 
by motion of the control mass in the  \( x \) and  \( y \) directions, 
as shown in Fig. 6.

By substituting  \( \Delta p_z = \Delta p_\theta = \Delta \dot{p}_z = 0 \) into Eq. (15) 
and setting state variable vector  \( x \) and control input  \( u \) as follows,

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} \Delta \omega \\ \Delta \Omega \\ \Delta \theta \\ \Delta \dot{p}_x \\ \Delta \dot{p}_y \\ \Delta \dot{p}_z \\ \Delta \theta_0 \\ \Delta \Omega_0 \\ \Delta p_{x0} \\ \Delta p_{y0} \\ \Delta p_{z0} \end{bmatrix} \\
u &= \begin{bmatrix} \Delta \ddot{p}_x \\ \Delta \ddot{p}_y \\ \Delta \ddot{p}_z \end{bmatrix}^T.
\end{align*}
\]

the following state equation is obtained:

\[
\dot{x} = A x + B u \\
: A \in R^{8x8}, B \in R^{8x2}. 
\]

(17)

The output equation can be represented as follows if 
numbers of outputs are assumed to equal to the numbers of 
inputs.

\[
y = C x \\
y \in R^{2x8}, C \in R^{2x8}.
\]

Matrix  \( C \) is an appropriate 2 by 8 matrix. The rank of the 
controllable matrix  \( U_c \) of Eq. (17) equals 7. Therefore, 
this system is uncontrollable. Thus, in order to investigate 
the uncontrollable mode of this control system, Eqs. (17) and 
(18) are transformed according to the following theorem.

\[\text{[Theorem 1]}^{4)} \text{ For linear time-invariant system with} \]

\[
\begin{align*}
\dot{x}(t) &= A x(t) + Bu(t) \\
y(t) &= C x(t)
\end{align*} \\
: A \in R^{m \times m}, B \in R^{m \times n}, C \in R^{n \times n}.
\]
when matrices \((A, B)\) are uncontrollable and rank\((U_r) = r < n\), independent row vectors \(t_1 \sim t_r\) (in total) can be extracted from controllable matrix \(U_c\), and a square matrix \(T\) can be made as follows:

\[
T = [t_1, t_2, \cdots, t_r, W] \in \mathbb{R}^{r \times n}
\]

Matrix \(W\) is an appropriate matrix selected to achieve \([T] \neq 0\). By a transformation below

\[
x(t) = Tz(t),
\]

Eq. (19) is transformed as follows.

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\
z_2
\end{bmatrix} + \begin{bmatrix} B_1 \\
0
\end{bmatrix} u
\]

\[
y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} z_1 \\
z_2
\end{bmatrix}
\]

\[
\begin{align*}
A_{11} & \in \mathbb{R}^{r \times r}, A_{12} \in \mathbb{R}^{r \times (n-r)}, A_{22} \in \mathbb{R}^{(n-r) \times (n-r)} \\
B_1 & \in \mathbb{R}^{r \times m}, C_1 \in \mathbb{R}^{k \times r}, C_2 \in \mathbb{R}^{k \times (n-r)}
\end{align*}
\]

(22)

where transformed matrices \((A_{11}, B_1)\) are controllable, and the following equation stands true.

\[
C(sI - A)^{-1}B = C_1(sI - A_{11})^{-1}B_1
\]

(23)

By the coordinate transformation of Eq. (21), Eq. (17) is transformed as follows.

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\
z_2
\end{bmatrix} + \begin{bmatrix} B_1 \\
0
\end{bmatrix} u
\]

(24)

\[
y = \begin{bmatrix} C_1 \\
C_2 \end{bmatrix} \begin{bmatrix} z_1 \\
z_2
\end{bmatrix}
\]

(25)

where \(z_1\) is the uncontrollable mode, which is not affected by the input. By comparing Eq. (22) with Eq. (24), it is found that \(A_{22}\) is 0. In other words:

\[
\dot{z}_2 = 0 \Rightarrow z_2 = \text{constant}.
\]

(26)

For the above, the value of \(z_2\) does not change from the initial value. Therefore, \(z_2\) is uncontrollable but not unstable mode. From these results, we ignore this uncontrollable mode \(z_2\), and by using controllable mode \(z_1\), we design a servo system that makes output \(y\) follow the reference value\(^5\).

The top half of Eqs. (24) and (25) can be expressed as follows.

\[
\begin{align*}
\dot{z}_1 &= A_{11}z_1 + A_{12}z_2 + B_1u \\
y &= C_1z_1
\end{align*}
\]

(27)

(28)

By defining a reference of the output \(y\) as \(r\), the error of the output can be expressed as

\[
e = y - r.
\]

(29)

By differentiating Eqs. (27) and (29), these equations can be written as

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{e}
\end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\
e
\end{bmatrix} + \begin{bmatrix} B_1 \\
0
\end{bmatrix} u.
\]

(30)

The output matrix \(C\) have not been specifically defined yet, but now, matrix \(C\) is determined. The goal of this chapter is to control the Rhönrud's attitude. So for one output, the Rhönrud's inclination angle \(\Delta \theta\) is selected. For the other output, \(\Delta p_s\) is selected. This is because when the value of \(p_s\) changes, the curved surface of Fig. 5 also changes. Therefore, in order to change the state variables of the Rhönrud following the curved surface of Fig. 5, the control mass must ultimately return to the initial position. For these reasons, the output vector is set to \(y = [\Delta \theta, \Delta p_s]\). Also, matrix \(C\) and reference \(r\) are as follows.

\[
C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]

(31)

\[
r = [\Delta \theta, \Delta p_s]^T = [\Delta \theta, 0]^T
\]

(32)

When using this \(C\), the rank of the controllable matrix of Eq. (30) equals 9, so Eq. (30) is controllable.

When the following state feedback is applied to Eq. (30),

\[
u = -[F_1 F_2] \begin{bmatrix} \dot{z}_1 \\
e
\end{bmatrix} = -F_1\dot{z}_1 - F_2 e.
\]

(33)

Eq. (30) can be expressed as follows:

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{e}
\end{bmatrix} = \begin{bmatrix} A_{11} - B_1 F_1 & -B_1 F_2 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\
e
\end{bmatrix} = Q \begin{bmatrix} \dot{z}_1 \\
\dot{e}
\end{bmatrix}
\]

(34)

Because Eq. (30) is controllable, the feedback gain matrices \(F_1\) and \(F_2\) can be chosen such that the eigenvalue of matrix \(Q\) in Eq. (34) is in the left half plane. In other words, through this state feedback, \(e\) and \(z_1\) of Eq. (34) can be converged to 0.

By integrating Eq. (33), the control input \(u\) is derived as follows:

\[
u = -F_1z_1 - F_2 \int_0^t e \ dt
\]

(35)

By using this input \(u\), output \(y\) can follow the reference \(r\) without stationary deviation.

Then, by introducing the control input of Eq. (35) to Eq. (8), the attitude control of the Rhönrud is numerically

Fig. 6 Motion of the control mass (2-DOF)
simulated as shown in Fig. 7. In this simulation, the control mass is controlled in the following manner:

- **0–5 seconds**

  Applying the initial values obtained from steady Spiral condition Eq. (11), the Rhönrad is rotating steadily ($\Delta \omega = \Delta \theta = \Delta \Omega = \Delta p_x = \Delta p_y = \Delta p_z = 0$). No control is performed.

- **5 seconds or greater**

  The reference is set to $r = [\Delta \theta_r, \Delta p_{yr}]^T = [0.1, 0]^T$ such that the inclination angle increases and the control mass returns to the initial position.

As seen in Fig. 7 (a) and (b), the Rhönrad is controlled through control mass motion such that $\Delta \theta = 0.1$ rad and $\Delta p_z = 0$ after 5 seconds. Since no reference is set for $\Delta \omega, \Delta \Omega$, they converge to values other than 0. However, $\Delta p_y$ converges toward 0 even though no reference is set. The reason for this can be explained as follows.

First, coordinate transformation $T$ of Eq. (21) is comprised of the following components,

$$T = \begin{bmatrix}
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}$$

(36)

where “$*$” are constants obtained from Eq. (20). The vector $z$ of Eq. (21) is expressed as follows.

$$z = [z_{a1}, z_{a2}, z_{a3}, z_{a4}, z_{a5}, z_{a6}, z_{a7}, z_{a8}]^T.$$  

(37)

By substituting Eqs. (16), (36) and (37) into Eq. (21), the following relationship is obtained.

$$\Delta p_y = z_{a4}$$  

(38)

Thus, to investigate the value of $\Delta p_y$, we investigate the value of $z_{a4}$. Also, components of $A_{11}, A_{12}, B_1$ in Eq. (27) are as follows.

$$A_{11} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$A_{12} = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}$$

$$B_1 = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}$$

(39)

where $I_5$ is 5 by 5 identity matrix. Therefore, by substituting Eq. (39) into Eq. (27), the following relation is obtained.

$$\dot{z}_{a4} = z_{a4}$$  

(40)

The control system of this chapter is designed such that $\dot{z}$ converges toward 0, so $\dot{z}_{a4}$ converges toward 0. Therefore, according to Eq. (40), $z_{a4}$ also converges toward 0. Consequently, it is seen that, from Eq. (38), $\Delta p_y$ also converges toward 0. This is the reason that $\Delta p_y$ converges toward 0.

In this control, as seen in Fig. 7 (b), the maximum displacement of the control mass is roughly 0.04 m. It is notable that the attitude of the Rhönrad can be controlled through this extremely small motion. Figure 7 (c) shows the trajectory of the Rhönrad at its ground contact point. From this figure, we find that the radius of the ground contact point can be changed by controlling the inclination angle of the Rhönrad.

6.1 The effect of the uncontrollable mode

In the control of previous section, the Rhönrad was controlled such that the inclination angle $\Delta \theta$ and $\Delta p_y$ converge to the reference. In this section, in order to see what $\Delta \omega$ and $\Delta \Omega$ would be as the result of this control, the effect of the uncontrollable mode is investigated.
The value of the uncontrollable mode \( z_2(\in \mathbb{R}^{1 \times 1}) \) of state equation (24) can be obtained by an inverse transform of Eq. (21) as shown below

\[
 z_2 = a_1 \Delta \omega + a_2 \Delta \theta + a_3 \Delta \Omega + a_4 \Delta p_y + a_5 \Delta p_x \quad (41)
\]

where \( a_1, a_2, a_3, a_4 \) and \( a_5 \) are coefficients. The value of \( z_2 \) is steadily maintained as is shown in Eq. (26). Also, in the attitude control in the previous section, the control mass is controlled such that \( \Delta p_x = 0 \), and \( \Delta p_y = 0 \). Therefore, during steady motion, Eq. (41) can be expressed as:

\[
 a_1 \Delta \omega + a_2 \Delta \theta + a_3 \Delta \Omega = a_0
\]

(42)

where \( a_0 \) is the initial value of \( z_2 \). This equation can be represented as a three-dimensional plane with axes \( \omega, \theta \) and \( \Omega \), and the values of \( \omega, \theta \) and \( \Omega \) are constrained in this plane before and after the attitude control. In addition to this constraint, when the Rhönrad is in steady motion, the values of \( \omega, \theta \) and \( \Omega \) must lie on the three-dimensional curved surface of Fig. 5. Therefore, by the control of the previous section, the values of \( \omega, \theta \) and \( \Omega \) are estimated to shift along the intersection line of the plane of Eq. (42) and the curved surface of Fig. 5. The curved surface in Fig. 8 is the same three-dimensional curved surface of Fig. 5, and point \( A \) in Fig. 8 is the initial value of the simulation in Fig. 7. By using this initial value, the intersection line can be obtained as shown by the broken line in Fig. 8. Of course, this line passes through point \( A \). Point \( B \) in Fig. 8 is the end value of the simulation, and it is seen that it lies on this broken line. Consequently, it is possible to verify that the attitude control of the previous section cause the values of \( \omega, \theta \) and \( \Omega \) to move along this broken line. These results indicate that the initial inclination and initial angular velocity is important in actual Rhönrad Spiral games. No matter how much the gymnast inside the Rhönrad shifts his body, it is not possible to change a steady motion to another steady motion that would break the constraint of Eq. (42).

7. Attitude Control of a Rhönrad by 3-DOF Motion of Control Mass

When the control mass has two degrees of freedom on the \( x \) and \( y \) axes as previously described, the control system was uncontrollable. In this section, we investigate whether the system becomes controllable when the control mass has three degrees of freedom in \( x, y \) and \( z \) directions as shown in Fig. 9. For Eq. (15), when the state variable vector \( x \) and the control input \( u \) are set to

\[
x = [\Delta \omega, \Delta \Omega, \Delta \theta, \Delta p_x, \Delta p_y, \Delta p_z, \Delta \dot{p}_x, \Delta \dot{p}_y, \Delta \dot{p}_z]^T(U),
\]

(43)

the following state equation is obtained.

\[
x = Ax + Bu
\]

\[
A \in \mathbb{R}^{10 \times 10}, B \in \mathbb{R}^{10 \times 3}
\]

(44)

The rank of the controllability matrix of this system equals 9. Therefore, this system is uncontrollable. In other words, this system does not become controllable even when increasing the degrees of freedom for the control mass. Also, by transforming Eq. (44) as was done in the previous chapter, we can see that the uncontrollable mode is constant. This is the same result as the previous chapter. In the previous chapter, the number of outputs that can be followed to the reference equals that of the inputs. In this section, because the number of inputs increases from 2 to 3, the number of the controllable state variables also increases from 2 to 3. Therefore it can be expected that the attitude of the Rhönrad can be controlled with higher degrees of freedom. However, this is not the case. The reason for this is described below.

For control of 2-DOF motion of the control mass, when the Rhönrad is rotating steadily, the control mass lies on the \( x \) axis. Therefore, there is no relative motion between the Rhönrad and the control mass, and the control mass is fixed on the Rhönrad. However, when the control mass has three degrees of freedom and there exists a displacement \( p_z \) in the \( z \) axis under steady motion as seen in Fig. 10, the control mass must always be in relative motion to the Rhönrad. It is actually difficult to realize this motion. Thus, even in control systems where the control mass has 3 degrees of freedom, the control mass must be controlled such that \( p_z = 0 \). Therefore one of the reference...
should be $p_z = 0$. As a result, there are two remaining controllable state variables, and this is the same situation as the control by 2-DOF motion of the control mass. In other words, no benefit is obtained even if the degrees of freedom are increased from 2 to 3.

8. Conclusions

The condition for a steady Spiral motion was derived. Then, by means of linearized equations of motion, the control method of the attitude of the Rhönrad was derived. The results of this study are summarized as follows:

(1) To achieve a steady Spiral motion, the control mass must lie within the $x$–$z$ plane.

(2) When the attitude of the Rhönrad is controlled by 2-DOF motion of the control mass, the system is uncontrollable, but by designing a servo system after a transformation, it is possible to control the inclination of the Rhönrad.

(3) As the result of an uncontrollable mode, the values of spin angular velocity $\omega$, precession angular velocity $\Omega$, and inclination angle $\theta$ are limited to a three dimensional curve.

(4) There is no benefit even if the degrees of freedom of the control mass are increased from 2 to 3.

References


(4) Mita, T., Introduction to System Control, (in Japanese), (1979), JIKKYO SHUPPAN CO., LTD.