A New Algorithm for Bi-Directional Evolutionary Structural Optimization*

XiaodongHUANG**, Yi Min XIE** and Mark Cameron BURRY***

In this paper, a new algorithm for bi-directional evolutionary structural optimization (BESO) is proposed. In the new BESO method, the adding and removing of material is controlled by a single parameter, i.e. the removal ratio of volume (or weight). The convergence of the iteration is determined by a performance index of the structure. It is found that the new BESO algorithm has many advantages over existing ESO and BESO methods in terms of efficiency and robustness. Several 2D and 3D examples of stiffness optimization problems are presented and discussed.

Key Words: Bi-Directional Evolutionary Structural Optimization, Stiffness Optimization, Optimal Design, Performance Index

1. Introduction

Topology optimization of structures has been extensively researched in the past two decades and many optimization methods have been developed(1)–(5). The evolutionary structural optimization (ESO) method has been received extensive attention because it can be easily implemented and linked to existing finite element analysis packages. Also, the resulting optimal design provides a clear profile of topology (with no “grey” area) and therefore easy to manufacture.

The ESO process begins with a full design domain and gradually removes underutilized material to achieve an optimal design. The amount of removed material is controlled by two parameters: rejection ratio (RR) and evolutionary rate (ER). These two parameters must be small. The theoretical basis of the ESO method has been explained mathematically by Tanskanen(6). To date, a wide range of structural optimization solutions for stiffness optimization, frequency, buckling etc. have been solved using the ESO method(7)–(9). Extended ESO methods and other techniques similar to the ESO method(10)–(13) have been developed in Japan recently and successfully applied to the design of real structures(11).

An alternative to ESO method is the addition evolu...

---

* Received 20th February, 2006 (No. 06-5027)
** School of Civil and Chemical Engineering, RMIT University, GPO Box 2476V, Melbourne 3001, Australia. E-mail: mike.xie@rmit.edu.au
*** Spatial Information Architecture Laboratory, RMIT University, GPO Box 2476V, Melbourne 3001, Australia
with no or low strain energy happens to be a part of the final topology. This is the reason that the evolutionary parameters must be very small or certain number of iterations is needed to assure that the BESO method produces a reasonable solution. Thus, the computational efficiency is limited. To overcome the deficiencies of the original BESO method, a new volume control and a stop criterion is developed in this study.

In this paper, a new algorithm for bi-directional evolutionary structural optimization (BESO) method is proposed. By the linear extrapolation of the sensitivity number in the surrounding elements, both adding and removing element material are realized according to a single parameter: the removal ratio of volume (or weight) (RRV).

Similar to the existing BESO method, the new BESO algorithm begins with a minimum initial design rather than a maximum. Examples in this paper show that the final optimal design is independent of the initial design. The similarity of the results from different initial designs will be discussed in details later. It shows that the new BESO method is more robust than traditional ESO and BESO methods.

In the new BESO method, a convergence criterion based on a performance index is introduced to monitor, and ultimately to terminate, the optimization process. The new BESO algorithm is very effective in significantly improving the performance index of structures.

2. Optimization Problem and Sensitivity Number

In many cases, the maximum stiffness of structures is pursued, in other words, the mean compliance of the structure should be minimized. When a linear structure is subjected to the external forces \( F \), the topology optimization problem for maximizing stiffness can be formulated with the volume constraint using elements as design variables.

Minimize \[ C = \frac{1}{2} [F]^T[K][u] \]  
Subject to \[ g = V^* - \sum_{i=1}^{m} V_i x_i = 0 \] \[ x_i \in \{0, 1\} \]  
where \([u]\) are the displacement vectors. \( V_i \) is the volume of an individual element and \( V^* \) the prescribed total structural weight. The binary design variable \( x_i \) declares the absence (0) or presence (1) of an element.

When the \( i \)th element is removed or added, the change of mean compliance is equal to the strain energy of \( i \)th element as(5):

\[ \Delta C_i = \frac{1}{2} [u'^T[K_i][u']] \]  
where \([u']\) is the nodal displacement vector of the \( i \)th element, \([K_i]\) is the element stiffness matrix. Therefore, the sensitivity number of \( i \)th element can be defined by its strain energy dividing its volume as

\[ \alpha_i = \frac{1}{2} [u'^T[K_i][u']] \frac{1}{V_i} \]  
which is defined as the sensitivity number of the \( i \)th element which indicates the contribute of an element removal or addition to the mean compliance.

Checkerboard pattern refers to the phenomena of alternating presence of solid and void elements ordered in a checkerboard like fashion. It is quite common in various fixed grid finite element based structural optimization methods. To eliminate the checkerboard, Li et al. (17) proposed a simple and effective smoothing algorithm. Firstly, the sensitivity number at each node can be calculated by averaging the element ones connecting to this node as

\[ \alpha_k = \sum_{i=1}^{m} \frac{V_i \alpha_i}{\sum_{i=1}^{m} V_i} \]  
where \( m \) denotes the total number of elements connected to the \( k \)th node, \( \alpha_i \) the sensitivity number of the \( i \)th connection element.

In order to obtain the sensitivity numbers for the added elements, it is necessary to extrapolate the sensitivity numbers for the nodes surrounding the structure. As shown in Fig. 1, it is assumed that the sensitivity numbers vary linearly along the coordinates, so that the unknown node sensitivity number is calculated by

\[ \alpha_k^\pm = 2\alpha_{k1} - \alpha_{k2} \]  
where \( \alpha_k^\pm \) denotes the sensitivity number of \( k \)th node which is calculated along the negative \( x \) direction. If possible, this extrapolation is repeated along other coordinate positive and negative directions. The final sensitivity number of the \( k \)th node is calculated by averaging all these values.

Finally, the smoothed sensitivity number of the candidate element (\( e \)) is calculated by averaging all nodal ones of this element as

\[ \alpha_e^* = \frac{1}{N} \sum_{n=1}^{N} \alpha_n \]  

Fig. 1 Extrapolation of sensitivity number
where $N$ is the total node number of the candidate element.

Thus, the sensitivity numbers for all elements within the structure as well as possible added elements surrounding the current structure are used to compare. Elements in the current structure can therefore be removed if they satisfy Eq. (7.a)

$$\alpha_e > \alpha_{th} \quad (7.a)$$

For these elements surrounding the current structure will be added if they satisfy Eq. (7.b)

$$\alpha_e \leq \alpha_{th} \quad (7.b)$$

where $\alpha_{th}$ is the threshold of the sensitivity number which is determined by the current removal ratio of volume ($RRV_i$). For example, if there are 1,000 elements in design domain and $\alpha_1 > \alpha_2 \cdots > \alpha_{1000}$ and $RRV_i$ corresponds to a design with 800 elements then $\alpha_{th} = \alpha_{800}$. This new element removal and addition scheme ranks all elements (void and solid) together, while in the original BESO methods$^{(14),(15)}$ elements for removal and those for addition are treated differently and ranked separated, which is a bit cumbersome and not very logical.

The cycle of finite element analysis and element removal and addition is repeated using the same value of $RRV_i$ until the solution is converged. Then an evolutionary rate ($ER$) is introduced and added to the removal ratio of volume, i.e.

$$RRV_{i+1} = RRV_i + ER, \quad i = 0, 1, 2, 3 \cdots \quad (8)$$

With this increased removal ratio of volume, the cycle of finite element analysis and element removal and addition takes place again until another optimum is obtained. The BESO procedure would be stopped when both the convergent criterion and final objective volume are satisfied.

3. Performance Index (PI) and Convergence Criterion

Performance index (PI) can be introduced to identify the performance of various optimal designs. Thus, the definition of the performance index should be directly related to the optimization objective. In the present case, the stiffness per unit volume or weight denotes the usage efficiency of the material. So the performance index is defined by

$$PI = \frac{1}{CV} \quad (9)$$

Where $C$ and $V$ are the mean compliance and volume (or weight) of the current design. While designs have a same volume, the one with a highest stiffness is the optimum which has the highest performance index. Thus, for a given removal ratio of volume, the structure should be gradually evolved to an optimum with a highest performance index by several removing and adding elements process.

Using the finite element analysis, the performance index of the structure can infinitely approach the theoretical one through the adjustment of the elements. When the structure satisfies the convergence criterion defined as Eq. (10), an optimum for the present removal ratio of volume is assumed to be obtained.

$$error_i = \frac{|PI_i - PI_{i-1}|}{PI_i} \leq error \quad (10)$$

where $error_i$ is the defined performance error for the $i$th iteration and $error$ is the maximum allowable error which is specified by the user.

Sometimes, an oscillation state occurs when elements are added in an iteration and the same elements are removed in the subsequent iteration. Thus, these elements will be added and removed in successive iterations. Normally, the occurrence of the oscillation state is caused by a low specified value of the maximum allowable error. When an oscillation occurs, the current design is assumed to be an optimum for the present removal ratio of volume although the convergence criterion has not been satisfied yet.

The proposed iterative procedure is similar to the conventional optimality criteria approach where if only marginal improvement in compliance over last design, the iterative procedure would be stopped. However, the designs of the conventional optimality criteria approach contain elements of different densities (or thicknesses) rather than elements of the same density in the BESO method.

4. Bi-Directional Evolutionary Structural Optimization Procedure

To explain better how the bi-directional evolutionary structural optimization can be implemented into a computer program, a step-by-step algorithm is given as follows:

Step 1: The maximum design domain is discretized with a densely finite element mesh.

Step 2: Define all boundary constraints, the design loads and nonlinear material properties.

Step 3: Carry out a linear finite element analysis of the structure.

Step 4: Output the results of analysis into BESO program.

Step 5: Calculate the sensitivity number for all existed elements as Eq. (3).

Step 6: Calculate the sensitivity number for all existed nodes as Eq. (4).

Step 7: Calculate the sensitivity number for all candidate nodes surrounding the structure as Eq. (5).

Step 8: Calculate the sensitivity number for all elements as Eq. (6)

Step 9: Remove and add elements which satisfy Eqs. (7.a) and (7.b).

Step 10: If the convergence condition Eq. (10) is satis-
fied, an optimum is reached and the optimization process skips Step 11. Otherwise, go to next step.

Step 11: If an oscillatory state is reached when a group of elements are removed and added back to the structure in the successive iterations. The optimum is reached. Otherwise, repeats step (3) to (11) until an optimum is reached.

Step 12: For the optimization process to continue, the removal ratio of volume is increased by Eq. (10). Repeat step (3) to (11) until another optimum is reached.

Figure 2 shows the flow chart which defines the logical steps for the new BESO method.

In the above procedure, the convergent optimums corresponding to various intermediate volumes are found. If only one optimum corresponding to the final volume, \( V^* \), is requested, the convergence condition may be checked only after the objective volume \( V^* \) is satisfied in step 10. In this way, the computation time can be saved significantly.

In order to present the final optimal topology with manufacturable boundary, an intuitive smoothing technique is applied. The coordinates of every node on the boundary are averaged by the coordinates of neighbor nodes. All topologies shown in the following sections are smoothed.

5. Examples and Discussion

5.1 Example 1: Michell type structure

The classic Michell type structure is fixed at both supports as shown in Fig. 3. The dimensions of the rectangular domain are 0.2 m by 0.1 m. The thickness of the plate is 0.001 m. A 100 N concentrated force is applied at the center of bottom edge. The linear material is used with Young’s modulus \( E = 210 \text{ GPa} \) and Poisson’s ratio \( \nu = 0.3 \). Because of symmetry, only half the structure is modeled with 100×100 four node elements. The BESO parameters are: \( RRV_0 = 0.01, \text{ } ER = 0.01 \) and \( error = 0.001 \).

![Fig. 3 Design domain of a Michell type structure](image)

**Fig. 3** Design domain of a Michell type structure

![Fig. 4 Optimal designs with various remained material ratio and performance index](image)

**Fig. 4** Optimal designs with various remained material ratio and performance index: (a) \( V_f = 80\% \) and \( PI = 2.93 \times 10^{-4} \text{ N}^{-1} \text{ mm}^{-4} \); (b) \( V_f = 60\% \), \( PI = 3.51 \times 10^{-4} \text{ N}^{-1} \text{ mm}^{-4} \); (c) \( V_f = 40\% \) and \( PI = 4.26 \times 10^{-4} \text{ N}^{-1} \text{ mm}^{-4} \); (d) \( V_f = 30\% \) and \( PI = 4.61 \times 10^{-4} \text{ N}^{-1} \text{ mm}^{-4} \); (e) \( V_f = 20\% \) and \( PI = 5.08 \times 10^{-4} \text{ N}^{-1} \text{ mm}^{-4} \); (f) \( V_f = 10\% \) and \( PI = 5.31 \times 10^{-4} \text{ N}^{-1} \text{ mm}^{-4} \).
The evolution history of the structure is shown in Fig. 4 (a) – (f) with various remained material volume, $V_f$. All structures consist of an arch and several spokes between the load and the top of the arch. However, the number and arrangement of spokes would be adjusted with the remained material volume. Figure 5 shows the history of the performance index of optimums during the whole evolution processes. The performance index increases continuously until the final remained volume/weight (say $V_f = 12\%$) arrives. The highest performance index of the design is $5.39 \times 10^{-4} \text{N}^{-1} \text{mm}^{-4}$. Thus, the performance of the design improves significantly comparing with the initial full design with $PI = 2.39 \times 10^{-4} \text{N}^{-1} \text{mm}^{-4}$.

In this way, all optimal designs with various remained material ratios are obtained. Sometimes, only one optimum with a certain volume/weight constraint, $V^*$, is needed. The convergence condition may be checked only after the objective volume $V^*$ is satisfied. In this way, the computation time can be saved significantly. To further save the computation time, we may start the BESO method from the initial guess design. In the following sections, we try two initial guess designs: one design is satisfied the volume/weight constraint and another is a minimum possible design.

The initial guess design with only $V_f = 40\%$ material of the design domain is shown in Fig. 6 (a). The BESO parameters are set to $RRV_0 = 0.60$, $ER = 0.0$ and $error = 0.001$. Thus, the location of elements is adjusted step by step and Fig. 6 (a) – (d) shows the history of the evolving process. The topology of the optimum (Fig. 6 (d)) is very close to the Fig. 4 (c). Its performance index is $4.23 \times 10^{-4} \text{N}^{-1} \text{mm}^{-4}$ which is a bit lower than the performance index of Fig. 4 (c), $4.26 \times 10^{-4} \text{N}^{-1} \text{mm}^{-4}$. These small differences can be attributed to the assigned convergence error in the later case. In the BESO starting from the full initial design, the topology after every iteration is very close to the next optimum. Thus its calculated convergence error is much lower than the assigned value and its results are more accurate than the BESO method starting from an initial guess whose topology is totally different from the optimum. It can be believed that much closer topology and performance index would be obtain if a smaller convergence error is specified in the later cases. Figure 7 shows the evolutionary histories of the performance index and structural volume/weight. It can be seen that the performance index increases steadily and the final optimum possesses the highest performance index in the whole optimization process.

Next we conduct the BESO method starting from the
initial minimum design as shown in Fig. 8 (a). The volume/weight constraint is set to the 20% of the full design domain. The BESO parameters are $RRV_0 = 0.80$, $ER = 0.0$ and $error = 0.001$. Figure 8(a)–(d) shows the evolving topology history using the present BESO method. Figure 9 shows the histories of the performance index and structural volume/weight. It can be seen that elements are added to the structure from 1st iteration to 16th iteration. Then, locations of elements are adjusted because the volume/weight constraint is satisfied. The structure is evolved to the optimum (95th iteration) when the specified convergence error is satisfied. The topology of the optimum (Fig. 8 (d)) is similar to the optimum obtained by the BESO starting from the initial full design (Fig. 4 (e)). The performance index of the optimum is $5.07 \times 10^{-4}$ N$^{-1}$mm$^{-4}$ which is also close to the performance index of Fig. 4 (e), $5.07 \times 10^{-4}$ N$^{-1}$mm$^{-4}$.

This well-known topology design problem has been widely reported in the literature. The above optimal topologies show comparable shapes with those generated with the original ESO method (Fig. 2.8 in Ref. (5)) and BESO method (Fig. 9 in Ref. (16)) as well as those proposed by classical methods like Michell’s layout theory (18). However, the present BESO method starting from initial guess design is more efficient than the original BESO method because only small portion of elements (no more than the objective volume due to $ER = 0.0$) is calculated in the finite element analysis.

5.2 Example 2: a 3D cantilever

Figure 10 shows the dimensions of the initial design domain and supports for a 3D cantilever. The applied force is 100 N downward at the middle of the free end. The mechanical properties of the material are Young modulus $E = 1$ GPa and Poisson ratio $\nu = 0.33$. The maximum design domain is meshed with 64000($80 \times 40 \times 20$) eight node solid elements.

Firstly, we start BESO from the full design. The BESO parameters are set to be $RRV_0 = 0.01$, $ER = 0.01$ and $error = 0.001$. Optimal designs with various remained volume/weight ratios are shown in Fig. 11 (a)–(f). Figure 12 shows the performance index of all optimal designs. The performance index increases gradually as the material is removed from the design domain step by step. The performance index of the optimal design with $V_f = 10\%$ is about 7.5 times than that of the initial full design.

As explained in the Example 1, the BESO method may starts from any initial guess design with any material in the design domain (without singularity). For exam-
Fig. 12 Evolutionary history of the performance indexes for all optimal designs

Fig. 13 Evolutionary history of the structural topology starting from the initial guess design: (a) initial guess design; (b) design after iteration 10; (c) design after iteration 15; (d) final optimal design

Fig. 14 Evolutionary history of the performance index and structure volume/weight

Fig. 15 Design domain for a bridge, where the solid section denotes the non-designable material

However, the BESO starting from the initial guess design saved the computation time significantly comparing with the BESO starting from the full design, although the performance index of the present design is little lower than that of the later method.

5.3 Example 3: an optimal bridge

Optimization for bridge type structure having the road on its upper part has been studied by Cui et al. The dimensions of the design domain, the loading and supporting conditions are given in Fig. 15. The mechanical properties of the material are assumed to be $E = 210$ GPa, and $\nu = 0.3$. In order to assure that the pressure is applied on the full top surface, a non-designable layer is defined with a thickness 1.5 m as shown in Fig. 15 with solid section. The design objective is to maximum the stiffness subject to the remained material ratio $V_f = 20\%$ comparing with the designable domain.

Due to symmetry, only quarter of the structure is modeled with $140 \times 10 \times 40$ eight node solid elements. The BESO method starts from the minimum design as shown in Fig. 16 (a). The BESO parameters are $RRV_0 = 0.8$, $ER = 0.0$ and $error = 0.0001$. After 121 iterations, the convergence criterion is satisfied and the evolutionary history of the structural topology is shown in Fig. 16 and Fig. 16 (d) and (e) shows the final optimal topology. Figure 17 shows the history of the performance index. It can be seen that the performance index has a big drop when one of supports breaks abruptly. Then a new structural topology is formed and the performance index increases steadily until
Fig. 16 Evolutionary history of the structural topology starting from the initial guess design: (a) initial design; (b) design after iteration 40; (c) design after iteration 80; (d) final optimal design; (e) rendered optimal design

Fig. 17 Evolutionary history of the performance index and structure volume/weight

Fig. 18 Design domain for a revised bridge

As shown in Fig. 18, the BESO method starting from the minimum design as shown in Fig. 16(a) is applied. The BESO parameters are $RRV_0 = 0.8$, $ER = 0.0$ and $error = 0.0001$ which are identical to the above case. Figure 19 shows the evolutionary history of the structural topology. After 74 iterations, the final optimum is obtained as shown in Fig. 19(d) and (e) which differs from the Fig. 16(d) and (e). Figure 20 shows the performance index history during the optimization process. The performance index of the final optimum is $3.51 \times 10^{-4} \text{N} \cdot \text{m}^{-4}$ which is lower than...
that of the previous case. It is reasonable because there is more constraint on the design domain in the later case.

Above two examples also demonstrate the efficiency of the present BESO method because the elements involved in finite element analysis is no more than 20% of total elements in the design domain. The convergence criterion used here is very strict, ensuring that the BESO algorithm really has converged to an optimum. The designs with the performance index just a few percent above the “optimal” one may be obtained using a large error such as 0.001 in approximately 50 iterations.

6. Conclusions

In this paper, a new algorithm for the bi-directional evolutionary structural optimization (BESO) has been developed. Several 2D and 3D examples have demonstrated the capability of the new BESO method to optimize a structure by adding and removing material. The developed BESO method has several advantages over traditional ESO and BESO methods:

1. The adding and removing of material is controlled by a single parameter: the removal ratio of volume (or weight ratio).
2. The final design is independent of the initial design, making the optimization algorithm more robust. The final topology and performance index are very close even when the initial designs are totally different.
3. The performance index provides a clear indication of structural efficiency of the resulting topologies and an effective termination criterion for the optimization process.
4. The proposed BESO method starting from initial guess design is more efficient than the original BESO method because only small portion of elements which is no more than the objective volume is calculated in the finite element analysis.

References