Stability Analysis of a Pipe Conveying Periodically Pulsating Fluid Using Finite Element Method*

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It is well known that the constant velocity of the fluid in a pipe which makes the lowest natural frequency zero is called critical velocity. The pipe becomes unstable when the fluid in a pipe is faster than the critical velocity. If the velocity of the fluid in a pipe varies with time, however, the instability of a pipe will occur even though the mean velocity of the fluid is below the critical velocity. In this paper, a new method for the stability analysis of a pipe conveying fluid which pulsates periodically is presented. The finite element model is formulated to solve the governing equation numerically. The coupled effects of several harmonic components in the velocity of fluid to a pipe is discussed. A new unstable region is shown in this paper which will not appear in the stability analysis of single pulsating frequency. The results of the stability analysis presented in this paper are verified by the time domain analysis.

Key Words: Stability Analysis, Parametric Resonance, Fluid-Structure Interaction, Periodically Pulsating Fluid, FEM

1. Introduction

The dynamic behavior of a pipe conveying fluid is a practical interest in the industrial field. The mass of fluid acts as a mass, and the velocity of fluid has effects on the stiffness and the damping of the pipe. The natural frequencies of the pipe become lower as the velocity of the fluid increases. The constant velocity of fluid which makes the lowest natural frequency of a pipe zero is called the critical velocity. The pipe becomes unstable if the fluid is faster than the critical velocity. However, when a pipe conveys pulsating fluid whose velocity varies with time, the pipe becomes unstable even though the mean velocity of the fluid is smaller than the critical velocity, which is called the parametric resonance. This phenomenon is owing to the fact that the damping and stiffness of a pipe vary with time due to the pulsating fluid.

There have been several studies undertaken to solve this problem. Ginsberg(1) derived the equation of motion of a pipe for any end condition and investigated the stability. Paidoussis(2),(3) developed numerical methods to check whether a point lies in the stable or the unstable region by calculating the determinant of a large matrix for every point in the parametric space. Paidoussis(4) also investigated the validity of the numerical method by comparing with experimental results. A method for large periodic excitation was studied by Ariaratnam(5). This method was based on the Floquet theory(6), which extended the stability boundaries of Bolotin(7) method. These studies(1),(3),(5) carried out the stability analysis by using assumed mode method, which could be only applied when the modes of a pipe were known. Seo(9) presented the finite element model to overcome the demerits of assumed mode method. But, these studies(1),(3),(5) assumed that the fluid in a pipe had only single frequency pulsation. However, in most real cases, the fluid in a pipe pulsates periodically, which means that the pulsations have several harmonics of fundamental period.

In this paper, a new method for the stability analysis of a pipe conveying periodically pulsating fluid is presented. The stability analysis for the fluid pulsating with
single frequency discussed by Seo\(^9\) is generalized to a fluid pulsating with several harmonics of fundament frequency. The numerical results of the stability analysis for the fluid with several harmonics are compared with those for the fluid with single frequency pulsation. The method presented in the paper is validated through the time domain analysis.

2. Finite Element Formulation

Figure 1 shows a model of a pipe conveying periodically pulsating fluid with period \(T\). If the diameter of a pipe is sufficiently smaller than the length of pipe, the behavior of a pipe can be assumed to satisfy the Euler’s beam theory which yields the following equation\(^9\):

\[
EI \frac{\partial^2 w}{\partial x^2} + \left( \frac{\partial A u(t)}{\partial x} - \left( (\rho A + m) g - \rho A \frac{du(t)}{dt} \right) \right) (L - x) \frac{\partial^2 w}{\partial x^2} + 2 \rho A u(t) \frac{\partial^2 w}{\partial x \partial t} + (\rho A + m) g \frac{\partial w}{\partial x} + c \frac{\partial w}{\partial t} + (\rho A + m) \frac{\partial^2 w}{\partial t^2} = p(x,t)
\]

(1)

where \(w(x,t)\) is the lateral displacement at position \(x\), \(p(x,t)\) is the distributed force, \(L\) is the length of pipe, \(u(t)\) is the velocity of the inner fluid, \(m\) is the mass of pipe per unit length, \(A\) is the area of internal cross-section of pipe, \(EI\) is the flexural rigidity of pipe, \(\rho\) is the density of fluid, \(c\) is the damping of pipe and \(g\) is the acceleration of gravity. Since the inner fluid has the pulsating velocity of \(u(t)\) with period \(T\), it can be expressed as follows:

\[
u(t) = u(t + T)
\]

(2)

The finite element formulation of Eq. (1) can be derived by using Galerkin’s method assuming the weighted functions are the same as trial functions in a weighted residual method. The discrete equation of motion can be expressed by the finite element method as follows\(^8\):

\[
(\rho A + m) [S_1] \{\ddot{w}(t)\} + [2 \rho A u(t) [S_3] + c [S_1] \{\dot{w}(t)\] 
\]

\[
[EI [S_2] - \left( \rho A u(t)^2 + \rho A L \frac{du(t)}{dt} + (\rho A + m) g L \right) [S_3] 
\]

\[
+ (\rho A + m) g [S_1] + \rho A L \frac{du(t)}{dt} \right] [S_1] + [S_1]) \{w(t)\} = \{f(t)\}
\]

(3)

The matrices \([S_i]\) could be calculated by the integration of shape functions\(^8\). To improve the computational efficiency in solving Eq. (3), the modal coordinate is introduced to reduce the degree-of-freedom. The displacement of pipe in physical coordinate, \([u(t)]\), can be transformed to the displacement in modal coordinate, \([\alpha(t)]\), using the relation given by

\[
[u(t)] = [\varphi][\alpha(t)]
\]

(4)

where, \([\varphi]\) is the eigenvector of the pipe. Substituting Eq. (4) into Eq. (3) and pre-multiplying \([\varphi]^T\) yields the following equation in modal coordinate.

\[
[M][\ddot{\alpha}(t)] + [C(t)][\dot{\alpha}(t)] + [K(t)][\alpha(t)] = \{F(t)\}
\]

(5)

The damping matrix \([C(t)]\) and stiffness matrix \([K(t)]\) vary with time while the inertia matrix\([M]\) is constant. Equation (5) shows the time-variant system. The time data of this ordinary differential equation could be solved by numerical methods. But, time domain analysis requires many computational efforts. The method to estimate the stability without solving the time data will be discussed in the next section.

3. Stability Analysis

When pulsating velocity of inner fluid varies with time, the pipe becomes unstable even though the mean velocity of fluid is smaller than the critical velocity. This phenomenon is called parametric resonance. The parametric resonance analysis for a pipe conveying fluid with single frequency pulsation was discussed by Paidoussis\(^8\) and Seo\(^9\) based on the Bolotin’s method\(^7\). In this paper, the stability analysis by Seo\(^9\) is generalized to a pipe conveying periodically pulsating fluid.
Since the fluid pulsates periodically with period $T = 2\pi/\omega$, the pulsation can be expressed by the Fourier series expansion as follows:

$$u(t) = \sum_{i=0}^{N} u_i \cos(i\omega t + \varphi_i) = \sum_{i=0}^{N} u_i \cos(\omega t + \varphi_i)$$  \hspace{1cm} (6)

Here, $N$ denotes the number of harmonics used to express the periodicity. The amplitude $u_i$ and phase $\varphi_i$ can be obtained easily from the Fourier series expansion. The Bolotin’s Method\(^1\) assumes the solution of parametric resonance as the Fourier-series expansion about $k/2$ times the pulsating frequency as follows:

$$\vec{\alpha}(t) = \sum_{k=0}^{M} \vec{\alpha}_k \sin \left(\frac{1}{2} k \omega t\right) + \vec{\beta}_k \cos \left(\frac{1}{2} k \omega t\right)$$  \hspace{1cm} (7)

Here, $M$ denotes the number of solutions used in Bolotin’s method to express the behavior of a pipe.

Substituting Eq. (7) into Eq. (5) and comparing the coefficients of sine and cosine yields an infinite set of algebraic equations given by

$$[G_{ij}] \left\{ \vec{\alpha}_i \right\} = \{0\}$$  \hspace{1cm} (8)

where,

$$\left\{ \vec{\alpha}_i \right\} = \{a_1, b_1, a_2, b_2, \ldots\}^T$$

If the number of modes in modal coordinate is $p$ and the number of solution in Bolotin method is $M$, the matrix $[G]$ becomes the square matrix of order $[(2M+1)\times p] \times [(2M+1) \times p]$. As the number $M$ in Eq. (7) goes larger, the size of matrix $[G]$ becomes larger while the solution of Eq. (8) goes more accurate and needs more computational time. The unstable region can be obtained by finding the frequency $\omega$ which renders the determinant of matrix $[G(\omega)]$ in Eq. (8) negative.

$$|G(\omega)| \leq 0 \text{ for unstable}$$  \hspace{1cm} (9)

In order to get the coefficient matrix $[G(\omega)]$ in Eqs. (8), (6) and (7) must be substituted into Eq. (5). During substitution process, there occur three nonlinear terms which are difficult to be dealt with as given by $u(t)\dot{\alpha}(t)$, $\frac{du(t)}{dt}\alpha(t)$, $u(t)^2\alpha(t)$. The time differentiation of pulsating velocity $u(t)$ given in Eq. (6) gives the followings:

$$\dot{u}(t) = \sum_{i=1}^{N} \left( -u_i \omega \sin(\omega t + \varphi_i) \right)$$  \hspace{1cm} (10)

Here, $\omega$ is integer multiple of $\omega$ because Eq. (10) is Fourier series of $u(t)$ whose period is $T = 2\pi/\omega$. And the time differentiation of assumed solution $\alpha(t)$ given in Eq. (7) gives the followings:

$$\dot{\alpha}(t) = \sum_{k=0}^{M} \frac{1}{2} \left( \omega \cos \left(\frac{1}{2} k \omega t\right) - \omega \sin \left(\frac{1}{2} k \omega t\right) \right)$$  \hspace{1cm} (11)

From Eqs. (6), (7), (10) and (11), the three nonlinear terms can be expressed as follows:

$$u(t)\dot{\alpha}(t) = \sum_{i=0}^{N} \sum_{k=0}^{M} u_i k \omega \alpha_i \frac{1}{4} \left( \cos \left(\frac{1}{2} k \omega t\right) + \cos \left(\frac{1}{2} k \omega t\right) \right)$$

$$\frac{du(t)}{dt}\alpha(t) = \sum_{i=0}^{N} \sum_{k=0}^{M} u_i \omega \alpha_i \frac{1}{2} \left( -\cos \left(\frac{1}{2} k \omega t\right) + \cos \left(\frac{1}{2} k \omega t\right) \right)$$

$$u(t)^2\alpha(t) = \sum_{i=0}^{N} \sum_{j=0}^{N} u_i u_j \frac{1}{4} \left( \sin \left(\frac{1}{2} k \omega t\right) - \sin \left(\frac{1}{2} k \omega t\right) \right)$$

Substituting Eqs. (12) ~ (14) into Eq. (5) and comparing the coefficients of trigonometric functions gives the constitution of $[G]$ matrix as shown in Fig. 2. The terms of $\omega_i + (\omega_i/2)$ in Eqs. (12) and (13), or the terms of $\omega_i + \omega_j + (\omega_i/2)$ in Eq. (14) can be considered as the form of $\Delta_{i\beta}/2$, which gives the clue that the coefficients of trigonometric functions should be inserted into the $\Delta_{i\beta}/2$ row in the matrix $[G]$. The coefficients of trigonometric functions in Eqs. (12)~(14) have the forms of $\Lambda_{ij}a_k$ or $\Lambda_{ij}b_k$. The coefficient $\Lambda_{i\beta}a_k$ should be inserted into $2\beta$ column and the coefficient $\Lambda_{i\beta}b_k$
Fig. 2 The constitution of $[G]$ matrix

should be inserted into $(2k+1)_{th}$ column. The coefficients of matrix $[G]$ in Fig. 2 or in Eq. (9) can be constituted as the same manner of the element assembly process in finite element method. Therefore, the stability analysis can be carried out by solving Eq. (9) regardless of the number of harmonic pulsations.

4. Numerical Results

The periodic pulsation can be expressed as the summation of several single pulsations. First, the stability analysis is applied to the case which a pipe has only one pulsating frequency of $\omega$ or $5\omega$ separately. Second, the stability analysis is applied to the case which pipe has two pulsating frequencies of $\omega$ and $5\omega$ simultaneously. The clamped-clamped straight pipe with length of 1 m is conveying harmonically pulsating fluid. The outer diameter of pipe is 0.008 m, and the thickness is 0.002 m. The mass of pipe per unit length is $2 \times 10^{11}$ N/m$^2$. The fluid density is $10^3$ kg/m$^3$. The average velocity of fluid is 21 m/s. The pipe is divided equally into 50 elements for the finite element analysis. And the comparison of the results of two cases is discussed.

4.1 Fluid with single frequency pulsation

The fluid in a pipe is assumed to have single harmonic pulsation in order to carry out the stability analysis. Figures 3 and 4 show the results of the stability analysis in the case that fluid has only single pulsating frequency of $\omega$ as given by

$$u(t) = u_0 + u_1 \cos(\omega t)$$ (15)  

or, pulsating frequency of $5\omega$ as given by

$$u(t) = u_0 + u_1 \cos(5\omega t)$$ (16)

The number of solution, $M$, in Eq. (7) was assumed to be 10. In Figs. 3 and 4, the area in black color denotes the unstable region, while the other region denotes the stable region. The dimensionless quantities are used in Figs. 3 and 4. The horizontal coordinate, $u_1/u_0$, is the ratio of the pulsating amplitude to the mean velocity. And the vertical coordinate, $\omega/\omega_n$, is the ratio of the pulsating frequency to the first natural frequency of the pipe structure. The parametric resonances occur over specific ranges of $\omega$ in the vicinity of $2\omega_n/k, k = 1, 2, 3, \ldots$, where $\omega_n$ is the $i_{th}$ natural frequencies of the pipe structure. In this example, the first and second natural frequencies of pipe are given by $\omega_{n1} = 22.37$ rad/s and $\omega_{n2} = 61.64$ rad/s. The $\omega/\omega_n$ in Fig. 3 is less than 2 with no pulsation because the natural frequency of pipe decreased due to the effects of the mass of internal fluid and damping. As the same reason, $\omega/\omega_n$ in Fig. 4 is also less than 0.4 with no pulsation, which means $5\omega/\omega_n$ is less than 2. It can be seen that the frequency range of instability becomes broader as the pulsating amplitude of fluid increases. In Fig. 3, the stable and unstable conditions are selected in order to verify the results of stability analysis. The symbol of triangle denotes stable condition ($u_1/u_0 = 0.4$, $\omega/\omega_n = 2.3$) and symbol of circle denotes unstable condition ($u_1/u_0 = 0.4$, $\omega/\omega_n = 2.3$).
\( \omega_1/\omega_n = 1.85 \) in Fig. 3. The differential equation given in Eq. (5) is solved in time domain by Runge-Kutta method. Figure 5(a) shows the time data for stable condition and Fig. 5(b) shows the time data for unstable condition, which gives the validation of the stability analysis presented in this paper.

4.2 Fluid with two harmonic pulsations

The fluid in a pipe has velocity with two harmonic pulsating frequencies of \( \omega \) and 5 \( \omega \) simultaneously as given by

\[
u(t) = u_0 + u_1 \cos(\omega t) + u_1 \cos(5\omega t).
\] (17)

Figure 6 shows the results of stability analysis when the inner fluid has two pulsating frequencies simultaneously. In Fig. 6, unstable region is denoted by the area in black color. It is necessary to check whether the unstable region for two harmonic pulsations given in Fig. 6 includes each unstable area of single frequency pulsation given in Figs. 3 and 4. It can be checked by overlapping Figs. 3 and 4 to Fig. 6. Figure 7 shows the new unstable region which does not appear in the stability results of single frequency pulsation. Because of this new unstable area, in the analysis of the pipes which have several harmonic pulsations, all pulsations must be taken into consideration at the same time, not each other.

In order to verify the results of stability analysis given in Fig. 7, the time domain analysis using Runge-Kutta method is carried out to check the validity of the stability results for the condition of \( u_1/u_0 = 0.4, \omega/\omega_n = 1.496 \) which is shown as symbol of circle in Fig. 7. This condition shows stable for the stability analysis of single frequency pulsation which can be easily seen from Figs. 3 and 4. The time data for this condition is shown in Fig. 5(b).
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(a) Unstable ($u = u_0 + u_1 \cos(\omega t) + u_1 \cos(5\omega t)$)

(b) Stable ($u = u_0 + u_1 \cos(\omega t)$)

(c) Stable ($u = u_0 + u_1 \cos(5\omega t)$)

Fig. 8 Time response ($u_1/u_0 = 0.4$, $\omega/\omega_n = 1.496$)

and 4, while it also shows unstable for the stability analysis of two pulsating frequencies simultaneously which can be seen from Fig. 7. The results of time domain analysis are shown in Fig. 8. This shows that there occur unstable conditions for the fluid with two harmonic pulsations which are not showed up in the analysis of single frequency pulsation, which validates the method presented in this paper.

5. Conclusions

(1) A new method for the stability analysis of the pipe conveying periodically pulsating fluid was presented based on the Bolotin’s method.

(2) In the stability analysis of a pipe conveying fluid with several harmonic pulsations, new unstable region appeared which did not appear in the analysis of single harmonic pulsation.

(3) Because of this new unstable region, pipe conveying several harmonically pulsating fluid have to be analyzed with taking into consideration of all pulsations at the same time, not each other.

(4) New unstable region presented in this paper was verified by the time domain analysis of the governing equation.

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