A Study on Leakage-Flow-Induced Vibrations*

(Fluid-Dynamic Forces Acting on the Walls of a One-Dimensional, Narrow Passage)

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In this paper, leakage-flow-induced vibrations are discussed in the case of a one-dimensional, narrow passage in which one of them is vibrating harmonically with a small amplitude. First, the fluid-dynamic forces acting on the walls of the tapered passage are analyzed. Second, the transfer matrix method is applied to the calculations of fluid-dynamic forces in the case of an arbitrary-shaped passage. The fluid-dynamic coefficients in the case of a tapered passage are calculated to check the accuracy of this method, and those in the case of the passage with a step are also calculated. It is found that if the cross-sectional area of the upstream passage is smaller than the downstream one, both negative fluid-dynamic stiffness and negative fluid-dynamic damping can occur in both the cases of a tapered passage and a passage with a step.

Key Words: Vibrations, Leakage Flow, Fluid-dynamic Forces, Self-excited Vibrations, Stability, Transfer Matrix

1. Introduction

Leakage-flow-induced vibrations have been observed in many fields of engineering and have been studied by many authors.

The fluid-dynamic forces for annular, parallel, and coaxial seals employed in pumps have been analyzed by Black⁽¹⁾, Hirs⁽²⁾ and Childs⁽³⁾⁽⁴⁾, and those for parallel, eccentric seals as well as tapered, coaxial seals have been analyzed by Yang-Iwatsubo⁽⁵⁾⁽⁶⁾. In an attempt to explain the observed behavior of the actual fuel assembly during on-load refueling of AGR's, the ways in which the gas flowing in a narrow annular passage with variable cross section cause self-excited vibrations of the inner cylinder are discussed by Hobson⁽⁷⁾. The stability of the inner of two concentric cylinders with a diffuser part has been analyzed by Spurr and Hobson⁽⁸⁾. About jet pumps of BWR's, the susceptibility of a cantilevered tube conveying water

to self-excitation by leakage flow through a slip joint is assessed experimentally by Mulcahy⁽⁹⁾⁽¹⁰⁾. Mateescu and Paidoussis⁽¹¹⁾ have obtained the unsteady flow forces in a narrow annular passage and analyzed the stability of the centerbody. From these studies, it has been suggested that negative, fluid-dynamic damping can occur when a narrow passage is divergent downstream or it includes a constriction at the inlet or a diffuser at the outlet. But there remain many points to be clarified as to the mechanism of the instability.

In this report, the fluid-dynamic forces acting on the walls of a one-dimensional, narrow, tapered passage are obtained analytically as the sum of two components, one proportional to the wall velocity and the other proportional to the wall displacement, from which the added mass coefficient, the added damping coefficient and the added stiffness coefficient have been determined, and the mechanism of instability generated from the flow through a narrow passage has been examined.

Further, the transfer matrix method is also applied to calculate the fluid-dynamic forces in the cases of arbitrary shapes of passage, where analytical solutions can not be obtained.

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Nomenclature

- C_e, C_w : added damping coefficients
 $\Delta h, e$: wall displacement and nondimensional wall displacement
 $F, \Delta f$: fluid force and unsteady fluid force
 H, H_0 : gap and gap at inlet
 i : $=\sqrt{-1}$
 K_a : added stiffness coefficient
 L : passage length
 M : transfer matrix for steady flow
 m : transfer matrix for unsteady flow
 M_a : added mass coefficient
 $O-YZ$: orthogonal coordinate system
 P, P^* : pressure and reference pressure
 P_{in}, P_{ex} : pressure just before the inlet and just after the outlet
 Q : flow rate per unit width
 q_w : nondimensional wall velocity
 R_e : Reynolds number [Eq. (9)]
 T : nondimensional time constant
 t, t_0 : time and reference time
 v, w : flow velocities in the directions of the Y - and Z -axes
 α, α' : area increment ratios in the case of a tapered passage and a stepped passage
 β : effective pressure loss factor of passage
 ζ_i, H_i^* : loss factor at the i -th step and the gap where ζ_i is defined
 λ : friction factor of the passage
 ν : kinematic viscosity
 ξ_{in}, ξ_{ex} : pressure loss factors at inlet and outlet
 ρ : fluid density
 τ : nondimensional time
 Ω, ω : angular frequency and nondimensional angular frequency

2. Basic Equations Governing the Flow through a Narrow Passage

2.1 Flow through a narrow passage and its coordinate system

As shown in Fig. 1, a flow through a narrow passage with pressure difference $P_{in}-P_{ex}$ between the inlet and the outlet is considered in this chapter. A coordinate system $O-YZ$ is fixed to the lower wall

Fig. 1 Tapered passage and the coordinate system

which is at rest, and the flow velocities in the directions of the Y - and Z -axes are denoted by v and w , respectively. The gap H is assumed to be sufficiently small compared with the passage length L . The upper wall is assumed to be vibrating harmonically relative to the lower wall with an angular frequency Ω and a sufficiently small amplitude compared with H . Under these conditions, the flow in the passage is almost parallel to the Y -axis, and we can assume $\partial P/\partial Z=0$. Such a flow is discussed in the following sections.

2.2 Basic equations of flow through a narrow passage

The equations of motion and continuity of fluid in the passage are written as

$$\frac{\partial v}{\partial t} + \frac{\partial v^2}{\partial Y} + \frac{\partial(vw)}{\partial Z} = -\frac{1}{\rho} \frac{\partial P}{\partial Y} + \nu \frac{\partial^2 v}{\partial Z^2} \quad (1)$$

$$\frac{\partial v}{\partial Y} + \frac{\partial w}{\partial Z} = 0 \quad (2)$$

The continuity of the normal velocity components at the upper wall is written as

$$w|_H - v|_H \cdot \frac{\partial H}{\partial Y} - \frac{\partial H}{\partial t} = 0 \quad (3)$$

Integrating Eqs. (1) and (2) with respect to Z from 0 to H , and taking Eq. (3) into consideration, we have

$$\begin{aligned} \frac{\partial Q}{\partial t} + \frac{\partial}{\partial Y} \left(\frac{Q^2}{H} \right) = & -\frac{H}{\rho} \frac{\partial P}{\partial Y} - \frac{1}{4} \lambda \frac{Q^2}{H^2} \\ & - H \sum_i \frac{\zeta_i Q_i^2}{2H_i^{*2}} \delta(Y - Y_i) \end{aligned} \quad (4)$$

$$\frac{\partial Q}{\partial Y} + \frac{\partial H}{\partial t} = 0 \quad (5)$$

where the following definition and approximation are employed :

$$Q = \int_0^H v dZ \quad (6)$$

$$\int_0^H v^2 dZ \approx \frac{1}{H} \left\{ \int_0^H v dZ \right\}^2 = \frac{Q^2}{H} \quad (7)$$

$$\nu \left(\frac{\partial v}{\partial Z} \Big|_H - \frac{\partial v}{\partial Z} \Big|_0 \right) = -\frac{1}{4} \lambda \frac{Q^2}{H^2} \quad (8)$$

$\delta(Y)$ denotes the delta function and the third term of the right-hand side in Eq. (4) represents the pressure losses at steps which may be included in the passage, which will be treated in chapter 4. $\lambda^{(12)}$ is the friction factor of the passage and given as a function of Reynolds number defined by

$$R_e = Q/\nu \quad (9)$$

Denoting the inlet and the outlet pressure loss factors as ξ_{in} and ξ_{ex} , respectively, the boundary conditions at the inlet and the outlet of the passage are expressed as

$$P(0) = P_{in} - \xi_{in} \frac{\rho Q^2(0)}{2H^2(0)} \quad (10)$$

$$P(L) = P_{ex} + \xi_{ex} \frac{\rho Q^2(L)}{2H^2(L)} \quad (11)$$

3. Fluid Forces Acting on the Walls of a Tapered Passage

3.1 Steady fluid forces in a tapered passage

First, the steady fluid forces, which are required to determine the equilibrium position, are obtained. Letting $\bar{Q}$ be the steady flow rate per unit width, Eq. (5) becomes $\partial\bar{Q}/\partial Y=0$, which means that $\bar{Q}$ is independent of Y . The nondimensional variable y and the nondimensional gap $h(y)$ are defined as

$$\left. \begin{aligned} y &= Y/L \\ h(y) &= \bar{H}/\bar{H}(0) = 1 + \alpha y \\ \alpha &= \{\bar{H}(L) - \bar{H}(0)\}/\bar{H}(0) \end{aligned} \right\} \quad (12)$$

where $\bar{H}$ is the gap at the steady state and α is the area increment ratio of the passage from the inlet to the outlet. When $\alpha > 0$, the passage is divergent, and when $-1 < \alpha < 0$, the passage is convergent. Rearranging Eq. (4), the steady pressure $\bar{P}$ is obtained from Eq. (13);

$$\frac{d\bar{P}}{dy} = \frac{\rho\bar{Q}^2}{H_0^2} \left\{ -\frac{1}{h} \frac{d}{dy} \left(\frac{1}{h} \right) - \left(\frac{\bar{\lambda}L}{4H_0} \right) \frac{1}{h^3} \right\} \quad (13)$$

where $\bar{\lambda}$ is the friction factor of the passage when the flow rate is $\bar{Q}$, and $\bar{H}(0)$ is replaced by H_0 . Integrating Eq. (13) and applying the boundary conditions, Eqs. (10) and (11), to the resultant equation, the flow rate $\bar{Q}$ and the pressure distribution $\bar{P}(y)$ at the steady state are expressed as

$$\frac{\bar{Q}}{H_0} = \sqrt{\frac{P_{in} - P_{ex}}{\rho \left\{ (\beta - \alpha) I_3(1) + \frac{\xi_{in}}{2} + \frac{\xi_{ex}}{2h^2(1)} \right\}}} \quad (14)$$

$$\frac{\bar{P}}{P^*} = \frac{P_{in}}{P^*} - \frac{\xi_{in}}{2} - (\beta - \alpha) I_3(y) \quad (15)$$

where P^* and β are defined as

$$\left. \begin{aligned} P^* &= \rho\bar{Q}^2/H_0^2 \\ \beta &= \bar{\lambda}L/4H_0 \end{aligned} \right\} \quad (16)$$

and the function $I_3(y)$ is given by putting $n=3$ in the following integration formulae

$$\left. \begin{aligned} I_n(y) &= \int_0^y \frac{dy}{h^n} = \frac{1}{(n-1)\alpha} \left(1 - \frac{1}{h^{n-1}} \right) \\ (n \text{ is an integer, not equal to unity}) \\ I_1(y) &= \ln(h)/\alpha \quad (n=1) \end{aligned} \right\} \quad (17)$$

The function $I_n(y)$ appears frequently in this paper. $I_n(y)$ is always positive, independent of whether α is positive or negative.

When the pressure $\bar{P}$ is integrated from the inlet to the outlet, the steady fluid force $\bar{F}$ acting on the walls is obtained as

$$\frac{\bar{F}}{LP^*} = \frac{P_{in}}{P^*} - \frac{\xi_{in}}{2} - \frac{(\beta - \alpha)}{2(1 + \alpha)} \quad (18)$$

3.2 Unsteady fluid forces in a tapered passage

Let us assume that the upper wall of a narrow passage is vibrating harmonically in the direction of the Z -axis with an angular frequency Ω and a sufficiently small amplitude ΔH . Then, the gap H during vibration is expressed as

$$H(Y, t) = \bar{H}(Y) + \Delta H \cdot \exp(i\Omega t) \quad (19)$$

As, in this paper, only the translational vibration is considered, ΔH is independent of the variable Y . The pressure P , the flow rate Q , the friction factor λ in the passage, and the fluid force F acting on the walls are assumed to change with the same angular frequency Ω as the wall vibration. They are expressed as

$$\left. \begin{aligned} P(Y, t) &= \bar{P}(Y) + \Delta P(Y) \cdot \exp(i\Omega t) \\ Q(Y, t) &= \bar{Q} + \Delta Q(Y) \cdot \exp(i\Omega t) \\ \lambda &= \bar{\lambda} + \left(\frac{d\lambda}{dQ} \right)_0 \Delta Q \cdot \exp(i\Omega t) \\ F &= \bar{F} + \Delta F \cdot \exp(i\Omega t) \end{aligned} \right\} \quad (20)$$

Equations (19) and (20) are substituted into Eqs. (4) and (5), and Eqs. (4) and (5) are then linearized with respect to the unsteady components. Introducing the following nondimensional variables and parameters,

$$\left. \begin{aligned} p &= \Delta P/P^*, \quad q = \Delta Q/\bar{Q}, \quad e = \Delta H/H_0 \\ \gamma &= 1 + \frac{\bar{Q}}{2\bar{\lambda}} \left(\frac{d\lambda}{dQ} \right)_0 \\ t_0 &= H_0 L / \bar{Q}, \quad \omega = \Omega t_0 \\ q_w &= i\omega e \end{aligned} \right\} \quad (21)$$

the linearized equations are expressed as

$$\frac{dp}{dy} = - \left\{ \frac{i\omega}{h} + \frac{2(\beta\gamma - \alpha)}{h^3} \right\} q - \frac{2}{h^2} \cdot \frac{dq}{dy} + \frac{3(\beta - \alpha)}{h^4} e \quad (22)$$

$$\frac{dq}{dy} = -q_w \quad (23)$$

Linearizing the boundary conditions, Eqs. (10) and (11), we have

$$p(0) = -\xi_{in} \{ q(0) - e \} \quad (24)$$

$$p(1) = \frac{\xi_{ex}}{h^2(1)} \left\{ q(1) - \frac{e}{h(1)} \right\} \quad (25)$$

Now, the nondimensional flow rate $q(y)$ is obtained from Eq. (23) as

$$q(y) = q(0) - q_w y \quad (26)$$

Substituting Eq. (26) into Eq. (22), integrating Eq. (22), and then taking the boundary condition of Eq. (24) into consideration, the nondimensional pressure $p(y)$ is obtained as

$$\begin{aligned} p(y) &= - \{ i\omega I_1(y) + 2(\beta\gamma - \alpha) I_3(y) + \xi_{in} \} q(0) \\ &\quad + \{ i\omega J_1(y) + 2(\beta\gamma - \alpha) J_3(y) + 2I_2(y) \} q_w \\ &\quad + \{ 3(\beta - \alpha) I_4(y) + \xi_{in} \} e \end{aligned} \quad (27)$$

where $J_n(y)$ is defined by

$$\begin{aligned} J_n(y) &= \int_0^y \frac{y}{h^n} dy = \frac{1}{\alpha} \{ I_{n-1}(y) - I_n(y) \} \\ (n \text{ is an integer}) \end{aligned} \quad (28)$$

In the following, we use I_n and J_n for $I_n(1)$ and $J_n(1)$, respectively, to simplify the expression.

Equation (27) shows that the pressure $p(y)$ depends on the three quantities, that is, the inlet flow rate $q(0)$, the wall velocity q_w , and the wall displacement e . In this problem, however, the inlet flow rate

$q(0)$ must be expressed as a function of q_w and e . When $y=1$ is substituted into Eq.(27) and the boundary condition Eq.(25) is considered, the flow quantity $q(0)$ is expressed as

$$q(0) = \frac{J_1}{I_1} \left(1 + \frac{B_1}{i\omega I_1 + r} \right) q_w + \frac{B_2}{i\omega I_1 + r} e \quad (29)$$

where B_1 , B_2 and r are given as

$$\left. \begin{aligned} B_1 &= 2(\beta\gamma - \alpha) \left(\frac{I_1 J_3}{J_1} - I_3 \right) \\ &+ 2 \frac{I_1 I_2}{J_1} - \xi_{in} - \frac{\xi_{ex}}{h^2(1)} \left(1 - \frac{I_1}{J_1} \right) \\ B_2 &= 3(\beta - \alpha) I_4 + \xi_{in} + \xi_{ex}/h^3(1) \\ r &= 2(\beta\gamma - \alpha) I_3 + \xi_{in} + \xi_{ex}/h^2(1) \end{aligned} \right\} \quad (30)$$

Equation(29) shows that the inlet flow rate $q(0)$ includes the components with 90° phase delay from q_w and e .

Integrating Eq.(27) from the inlet to the outlet, and rearranging the result, taking Eq.(29) into consideration, the fluid force ΔF acting on the walls is expressed as

$$\frac{\Delta F}{P^* L} = - \left\{ i\omega m + c_1 + \frac{c_2}{1 + i\omega T} \right\} q_w - \left\{ k_1 + \frac{k_2}{1 + i\omega T} \right\} e \quad (31)$$

When $N_1 \sim N_4$ and N_1^*, N_3^* are defined by

$$\left. \begin{aligned} N_n &= \int_0^1 I_n(y) dy = I_n - J_n, (n=1 \sim 4) \\ N_n^* &= \int_0^1 J_n(y) dy \\ &= \begin{cases} \frac{2+\alpha}{2\alpha^2} - \frac{1+\alpha}{\alpha^3} \ln(1+\alpha) & (n=1) \\ \frac{2+\alpha}{2\alpha^2(1+\alpha)} - \frac{1}{\alpha^3} \ln(1+\alpha) & (n=3) \end{cases} \end{aligned} \right\} \quad (32)$$

and B_3 is defined by

$$\left. \begin{aligned} B_3 &= 2(\beta\gamma - \alpha) \left(\frac{I_1 N_3}{N_1} - I_3 \right) \\ &- \xi_{in} \left(1 - \frac{I_1}{N_1} \right) - \frac{\xi_{ex}}{h^2(1)} \end{aligned} \right\} \quad (33)$$

then the coefficients of Eq.(31) are expressed as

$$\left. \begin{aligned} m &= \frac{J_1 N_1}{I_1} - N_1^* \\ c_1 &= 2(\beta\gamma - \alpha) \left(\frac{N_1 J_3}{I_1} + \frac{J_1 N_3}{I_1} - \frac{J_1 N_1 I_3}{I_1^2} \right. \\ &\quad \left. - N_3^* \right) + 2 \left(\frac{I_2 N_1}{I_1} - N_2 \right) \\ &\quad + \xi_{in} \frac{J_1}{I_1} \left(1 - \frac{N_1}{I_1} \right) + \frac{\xi_{ex}}{h^2(1)} \cdot \frac{N_1}{I_1} \left(1 - \frac{J_1}{I_1} \right) \\ c_2 &= \frac{J_1 N_1}{I_1^2} \cdot \frac{B_1 B_3}{r} \\ k_1 &= 3(\beta - \alpha) \left(\frac{N_1 I_4}{I_1} - N_4 \right) - \xi_{in} \left(1 - \frac{N_1}{I_1} \right) + \frac{\xi_{ex}}{h^3(1)} \cdot \frac{N_1}{I_1} \\ k_2 &= \frac{N_1}{I_1} \cdot \frac{B_2 B_3}{r} \\ T &= I_1 / r \end{aligned} \right\} \quad (34)$$

Especially, when the passage walls are parallel, that is, $\alpha=0$, then $I_n=1$, $J_n=N_n=1/2$ and $N_n^*=1/6$ for any integer n , so that Eqs.(34) become

$$\left. \begin{aligned} m &= \frac{1}{12} \\ c_1 &= \frac{\beta\gamma}{6} + \frac{\xi_{in} + \xi_{ex}}{4} \\ c_2 &= \frac{(\xi_{in} - \xi_{ex}) \{ 1 - (\xi_{in} - \xi_{ex})/4 \}}{2\beta\gamma + \xi_{in} + \xi_{ex}} \\ k_1 &= -\frac{\xi_{in} - \xi_{ex}}{2} \\ k_2 &= \frac{\xi_{in} - \xi_{ex}}{2} \cdot \frac{3\beta + \xi_{in} + \xi_{ex}}{2\beta\gamma + \xi_{in} + \xi_{ex}} \\ T &= 1/(2\beta\gamma + \xi_{in} + \xi_{ex}) \end{aligned} \right\} \quad (35)$$

Equations(31) show that the fluid force ΔF acting on the walls also includes the components with 90° phase delay from q_w and e , the quantities of which depend on ωT . When ωT is rewritten in the following form,

$$\begin{aligned} \omega T &= \frac{\Omega I_1}{r} \\ &= \frac{\rho L I_1 |i\Omega \Delta Q|}{\frac{\rho \bar{Q}}{H_0} \left\{ 2(\beta\gamma - \alpha) I_3 + \xi_{in} + \frac{\xi_{ex}}{h^2(1)} \right\} \Delta Q} \end{aligned} \quad (36)$$

it is seen that the numerator is the absolute value of the inertia force when the flow rate is vibrating with angular frequency Ω and amplitude ΔQ , and that the denominator is the increase in the fluid resistance when the flow rate increases by ΔQ . Therefore, ωT represents a kinematic Reynolds number for vibrating flow.

If $\omega T \ll 1$, the flow is dominated by the viscous force, and the quasi-steady fluid force is obtained. If $\omega T \gg 1$, the flow is dominated by the inertia force, and the inlet flow rate $q(0)$ responds only to the squeeze effect, that is, $q(0) = (J_1/I_1)q_w$, in which the delayed components again become extremely small.

If the first-order delay element in Eq.(31) is separated into real and imaginary parts, the fluid force ΔF for arbitrary frequencies is rewritten as

$$\frac{\Delta F}{P^* L} = -(i\omega M_a + C_w) q_w - (i\omega C_e + K_a) e \quad (37)$$

where

$$\begin{aligned} M_a &= \left(m - \frac{c_2 T}{1 + \omega^2 T^2} \right), \quad C_w = \left(c_1 + \frac{c_2}{1 + \omega^2 T^2} \right) \\ C_e &= -\frac{k_2 T}{1 + \omega^2 T^2}, \quad K_a = \left(k_1 + \frac{k_2}{1 + \omega^2 T^2} \right) \end{aligned} \quad (38)$$

Now, since the fluid force and the wall displacement have been assumed to be vibrating harmonically, that is,

$$\left. \begin{aligned} \Delta f &= \Delta F \cdot \exp(i\Omega t) \\ &= \Delta F \cdot \exp(i\omega \tau) \\ \Delta h &= (\Delta H/H_0) \cdot \exp(i\Omega t) \\ &= e \cdot \exp(i\omega \tau) \end{aligned} \right\} \quad (39)$$

Eq.(37) is then equivalent to the following expression

$$\frac{\Delta f}{P^* L} = -M_a \Delta \dot{h} - (C_w + C_e) \Delta \dot{h} - K_a \Delta h \quad (40)$$

where $\dot{}$ denotes the derivative with respect to the nondimensional time τ . It is seen from Eq.(40), that M_a stands for the added mass coefficient, $C_w + C_e$ the

added damping coefficient, and K_a the added stiffness coefficient. Even if both c_1 and c_2 in Eq.(31) are positive, C_e can become negative due to the fluid force delay, so that $C_w + C_e$ can also become negative; in other words, negative damping can occur. A certain passage shape causes K_a to be negative, in which the divergent-type instability can occur.

The advantage of obtaining the unsteady fluid forces ΔF as the sum of the two components proportional to q_w and e , respectively, as shown in Eq.(31), lies in the points described below. If the passage has a complicated shape, it is not possible to obtain the fluid force analytically, as in a tapered passage. Thus, we should calculate it numerically. If the fluid forces are obtained as the sum of the two components, one proportional to q_w and the other proportional to e , M_a is expressed by the amount of the imaginary part of the coefficient of q_w divided by ω ; $C_w + C_e$ is expressed by the sum of the real part of the coefficient of q_w and the amount of the imaginary part of coefficient of e divided by ω ; and K_a is expressed by the real part of the coefficient of e .

3.3 Fluid-dynamic force coefficients in a tapered passage

Assuming a laminar flow in the passage, the friction factor $\lambda^{(12)}$ is written in the form

$$\lambda = 48/R_e, (R_e < 1000) \quad (41)$$

In this case, γ , which is given by Eq.(21), is 0.5. The fluid-dynamic force coefficients in a tapered passage can then be calculated.

Figures 2(a)~(d) show the fluid-dynamic force coefficients M_a , C_w , C_e and K_a as a function of the area increment ratio α for three values of ωT , in which the inlet pressure loss factor, the outlet pressure loss factor, and the effective pressure loss factor of the passage are chosen as $\xi_{in}=1.5$, $\xi_{ex}=0$, and $\beta=1$, respectively.

In Fig. 2, the horizontal axis is taken on a log scale, so that unity shows the parallel passage, the right-hand side of unity shows divergent passages and the left-hand side of unity shows convergent passages. The added mass coefficient M_a becomes negative in the case that $\omega T \approx 0 \sim 1$ and the passage walls are nearly parallel. In this case, the fluid force causes the

(a) Added mass coefficient M_a

(c) Added damping coefficient C_e

(b) Added damping coefficient C_w

(d) Added stiffness coefficient K_a

Fig. 2 The relations between fluid-dynamic coefficients and area increment ratio α

natural frequency to rise. In the case that $\omega T \approx 0 \sim 1$ and the passage walls are not nearly parallel, and in the case that $\omega T \rightarrow \infty$, M_a is positive. The added damping coefficient C_w is always positive, but C_e becomes negative when α is above -0.53 in this calculation. C_e converges to null when $\omega T \rightarrow \infty$. The added stiffness coefficient K_a becomes negative when $\omega T = 1$ and α is above -0.06 and when $\omega T \rightarrow \infty$ and α is above -0.31 , but K_a is always positive when $\omega T \rightarrow 0$.

Second, the range where $C_w + C_e$ or K_a becomes positive or negative is shown on the α - β plane in Fig. 3. The solid line in Fig. 3 shows $C_w + C_e = 0$. Fluid force acts as negative damping in the range where $C_w + C_e < 0$. When $\omega T \rightarrow 0$, the point on $C_w + C_e = 0$ at which α becomes minimum is at $\beta = 1.3$. If β is less than 1.3, $\alpha \approx 1$. If β increases to the value greater than 1.3, α on which $C_w + C_e = 0$ also increases. If $\beta = 100$, then $\alpha = 7.9$. α on which $C_w + C_e = 0$ increases with increase of ωT . When $\omega T \rightarrow \infty$, no range exists where $C_w + C_e < 0$. The broken line in Fig. 3 shows the relation between α and β when $K_a = 0$. Fluid force acts as negative stiffness in the range where $K_a < 0$. When $\omega T \rightarrow 0$, K_a can be negative where β is above 15 or β is below 0.15. When $\omega T = 1$ and $\omega T \rightarrow \infty$, K_a is negative in all values of β if the passage is slightly convergent, and if the passage is divergent.

The relations between α and β for which $C_w + C_e = 0$ and $K_a = 0$ are shown in Fig. 4 in which ξ_{in} and ξ_{ex} are different from the values of Fig. 3. When ξ_{in} increases to 5, in the case of $\omega T \rightarrow 0$, a range exists where $C_w + C_e < 0$, but in the case of ωT increasing, no range exists where $C_w + C_e < 0$, as shown Fig. 4(a). In this case, the range where $K_a < 0$ decreases in size, but this range is almost equal to that of Fig. 3. When ξ_{ex} increases to 5 from the value of Fig. 3, in the case of $\omega T \rightarrow 0$, α on which $C_w + C_e = 0$ increases above 5.5,

Fig. 3 The range where added damping coefficient $C_w + C_e$ or added stiffness coefficient K_a becomes negative

and the range where $C_w + C_e < 0$ becomes narrow. When $\omega T \rightarrow 0$, K_a is negative in the case of convergent but almost parallel passages and in the case of divergent passages, so that the range where $K_a < 0$ becomes wider than that of Fig. 3. However, in the case of $\omega T = 1$ or $\omega T \rightarrow \infty$, α on which $K_a = 0$ is larger than that of Fig. 3, so that the range where $K_a < 0$ becomes narrower.

4. Calculations of Fluid-dynamic Forces in Arbitrary Passages by Transfer Matrix Method

4.1 Discretization of a passage

Numerical methods are required to calculate the hydrodynamic forces in arbitrary passages, where it is too hard to obtain analytical solutions as was obtained for a tapered passage. In this paper, the transfer matrix method is applied to the numerical calculations.

A narrow passage which is unity in nondimensional length is divided into N sections, as shown in

(a) The case of increased ξ_{in}

(b) The case of increased ξ_{ex}

Fig. 4 The range where added damping coefficient $C_w + C_e$ or added stiffness coefficient K_a becomes negative (ξ_{in} and ξ_{ex} are changed)

Fig. 5. Nondimensional lengths of the N sections are $l_1, l_2, \dots, l_N$, respectively. N nodal points 1, 2, ---, N are put at the center of each section. At the boundaries of each section, the boundary points 1', 2', 3', ---, N' , $(N+1)'$ are set, where 1' and $(N+1)'$ correspond to the inlet and the outlet of the passage, respectively. Nondimensional gap h_j is assumed to be constant between the nodal points $j-1$ and j . Let Q_j be the flow rate at the nodal point j , and P_j be the pressure at the boundary point j' , respectively. Let P_{in} and Q_0 be the pressure and flow rate just before the inlet of the passage, and P_{ex} and Q_{N+1} be those just after the outlet, as shown in Fig. 5. Nodal points are also set at steps, but the lengths of these sections are considered to be infinitesimally small because pressure changes are assumed to occur discontinuously at these points.

4.2 Steady fluid forces in an arbitrary passage

The basic equations of steady fluid flow in a narrow passage are given by

$$\frac{d\bar{P}}{dy} = \frac{\rho \bar{Q}^2}{H_0^2} \left\{ -\frac{1}{2} \frac{d}{dy} \left(\frac{1}{h^2} \right) - \left(\frac{\bar{\lambda} L}{4 H_0} \right) \frac{1}{h^3} - \sum_i \frac{\bar{\xi}_i}{2 h_i^{*2}} \delta(y - y_i) \right\} \quad (42)$$

where $\bar{\xi}_i$ is the loss factor at i -th step. Equation (42) is integrated from the boundary point j' to $(j+1)'$ and transformed into N difference equations. These difference equations as well as the inlet and the outlet boundary conditions of Eqs. (10) and (11) are expressed in matrix representation as

$$[\bar{P}_{j+1}/P^*, 1]^T = M_j [\bar{P}_j/P^*, 1]^T \quad (43)$$

$$M_j = \begin{bmatrix} 1 & M_j \\ 0 & 1 \end{bmatrix} \quad (44)$$

where

$$M_j = \begin{cases} -\frac{1}{2} \left\{ \left(\frac{1}{h_{j+1}^2} - \frac{1}{h_j^2} \right) + \beta l_j \left(\frac{1}{h_{j+1}^3} + \frac{1}{h_j^3} \right) \right\} \\ -\bar{\xi}_{in}/2 \quad (j=0) \\ -\bar{\xi}_{ex}/2 h_{N+1}^2 \quad (j=N+1) \end{cases} \quad (45)$$

T denotes the transpose of a matrix or a vector.

Fig. 5 Arbitrary-shaped passage and the coordinate system

M_0 expresses the transfer matrix at the inlet and M_{N+1} expresses the transfer matrix at the outlet. At step, $-\bar{\xi}_i/2 h_i^{*2}$ is added to M_j in Eq. (45). Multiplying M_j one by one from the inlet to the outlet, the transfer matrix of the whole passage is obtained, and the relation between the pressure just before the inlet and just after the outlet is expressed as

$$[P_{ex}/P^*, 1]^T = M [P_{in}/P^*, 1]^T \quad (46)$$

where

$$M = M_{N+1} M_N M_{N-1} \dots M_1 M_0 = \begin{bmatrix} 1 & -M \\ 0 & 1 \end{bmatrix} \quad (47)$$

Then, the steady flow rate $\bar{Q}$ is obtained as

$$\bar{Q}/H_0 = \sqrt{(P_{in} - P_{ex})/\rho M} \quad (48)$$

The steady pressure $\bar{P}_j$ of each section is also obtained from Eq. (43) by multiplying M_j one by one. Integrating pressure $\bar{P}_j$ numerically, the steady fluid force $\bar{F}$ is expressed as

$$\frac{\bar{F}}{L P^*} = \frac{1}{2} l_1 \frac{\bar{P}_1}{P^*} + \frac{1}{2} \sum_{j=2}^N (l_{j-1} + l_j) \frac{\bar{P}_j}{P^*} + \frac{1}{2} l_N \frac{\bar{P}_{N+1}}{P^*} \quad (49)$$

4.3 Unsteady fluid forces in an arbitrary passage

Let us assume that the upper wall is vibrating harmonically in the same manner as in section 3.2. The gap H during vibration is expressed as Eq. (19). The loss factor at step is also assumed to change with the same angular frequency Ω as the wall vibration, and it is then expressed as

$$\xi_i = \bar{\xi}_i + \frac{d\bar{\xi}_i}{d\left(\frac{h_{j+1}}{h_j}\right)} \left(\frac{1}{h_{j+1}} - \frac{1}{h_j} \right) \frac{h_{j+1}}{h_j} \times \frac{\Delta H}{H_0} \exp(i\Omega t) \quad (50)$$

Introducing the following parameters,

$$\eta_i = 1 - \frac{d\bar{\xi}_i}{2\bar{\xi}_i d\left(\frac{h_{j+1}}{h_j}\right)} \left(\frac{h_j^*}{h_{j+1}} - \frac{h_{j+1}^*}{h_j} \right) \left(\frac{h_{j+1}}{h_j} \right) \quad (51)$$

the basic equations are expressed as

$$\frac{dp}{dy} = - \left\{ \frac{i\omega}{h} + \frac{2\beta\gamma}{h^3} + \frac{d}{dy} \left(\frac{1}{h^2} \right) + \sum_i \frac{\bar{\xi}_i}{h_i^{*2}} \delta(y - y_i) \right\} q - \frac{2}{h^2} \frac{dq}{dy} + \left\{ \frac{3\beta}{h^4} + \frac{d}{dy} \left(\frac{1}{h^3} \right) + \sum_i \frac{\bar{\xi}_i \eta_i}{h_i^{*3}} \delta(y - y_i) \right\} e \quad (52)$$

$$\frac{dq}{dy} = -q_w \quad (53)$$

These equations are also transformed into N difference equations, as done in section 4.2, and are expressed in the matrix representation, together with the inlet and the outlet boundary conditions of Eqs. (24) and (25), as

$$[p_{j+1}, q_{j+1}, q_w, e]^T = m_j [p_j, q_j, q_w, e]^T \quad (54)$$

$$m_j = \begin{bmatrix} 1 - (i\omega A_j + B_j) & D_j & E_j \\ 0 & 1 & -G_j & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (55)$$

where

$$\begin{aligned}
 A_j &= \begin{cases} l_j \left(\frac{1}{h_j} + \frac{1}{h_{j+1}} \right) \\ 0 \quad (j=0, N+1) \end{cases} \\
 B_j &= \begin{cases} \beta \gamma l_j \left(\frac{1}{h_{j+1}^3} + \frac{1}{h_j^3} \right) + \frac{1}{h_{j+1}^2} - \frac{1}{h_j^2} \\ \xi_{in} \quad (j=0) \\ \frac{\xi_{ex}}{h_{N+1}^2} \quad (j=N+1) \end{cases} \\
 D_j &= \begin{cases} l_j \left(\frac{1}{h_j^2} + \frac{1}{h_{j+1}^2} \right) \\ 0 \quad (j=0, N+1) \end{cases} \\
 E_j &= \begin{cases} \frac{3}{2} \beta l_j \left(\frac{1}{h_{j+1}^4} + \frac{1}{h_j^4} \right) + \left(\frac{1}{h_{j+1}^3} - \frac{1}{h_j^3} \right) \\ \xi_{in} \quad (j=0) \\ \frac{\xi_{ex}}{h_{N+1}^3} \quad (j=N+1) \end{cases} \\
 G_j &= \begin{cases} (l_{j+1} + l_j)/2 \quad (j=1, \dots, N-1) \\ l_1/2 \quad (j=0) \\ l_N/2 \quad (j=N) \\ 0 \quad (j=N+1) \end{cases} \quad (56)
 \end{aligned}$$

and $\mathbf{m}_0$ expresses the transfer matrix at the inlet and $\mathbf{m}_{N+1}$ expresses the transfer matrix at the outlet. ξ_i/h_i^{*2} is added to B_j and $\xi_i \eta_i/h_i^{*3}$ is added to E_j in Eq. (56) at step.

If the inlet flow rate q_0 , q_w and e are given, the flow rates q_j and the pressure p_j are obtained from Eq. (54) by multiplying $\mathbf{m}_j$ one by one because the pressure perturbation just before the inlet, p_0 , is zero and A_j , B_j , D_j , E_j and G_j are known. Now, q_0 must be expressed as a function of q_w and e . The transfer matrix of the whole passage is obtained as

$$\begin{aligned}
 \mathbf{m} &= \mathbf{m}_{N+1} \mathbf{m}_N \mathbf{m}_{N-1} \dots \mathbf{m}_1 \mathbf{m}_0 \\
 &= \begin{bmatrix} 1 & -(\omega A + B) & \omega C + D & E \\ 0 & 1 & -G & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (57)
 \end{aligned}$$

Since the pressure perturbation just after the outlet, p_{N+1} , is also zero, the relation between the pressures

Fig. 6 Comparison of analytical and numerical solutions of C_w in tapered passages

just before the inlet and just after the outlet is expressed as

$$[0, q_{N+1}, q_w, e]^T = \mathbf{m}[0, q_0, q_w, e]^T \quad (58)$$

As a result, q_0 is obtained from first row of Eq. (58) as

$$q_0 = \frac{i\omega C + D}{i\omega A + B} q_w + \frac{E}{i\omega A + B} e \quad (59)$$

The coefficients of q_w and e in Eq. (59) include first-order delay elements, whose time constants are expressed as

$$T = A/B \quad (60)$$

The pressure p_j is also expressed as a function of q_w and e by substituting Eq. (59) into Eqs. (54) in the form

$$p_j = p_{wj} \cdot q_w + p_{ej} \cdot e \quad (61)$$

Integrating pressure p_j numerically, the fluid dynamic force ΔF acting on the walls is also obtained as a function of q_w and e as

$$\Delta F/LP^* = f_w \cdot q_w + f_e \cdot e \quad (62)$$

where

$$f_w = \frac{1}{2} \left\{ \sum_{j=2}^N p_{wj} (l_{j-1} + l_j) + p_{w1} l_1 + p_{wN+1} l_N \right\} \quad (63)$$

$$f_e = \frac{1}{2} \left\{ \sum_{j=2}^N p_{ej} (l_{j-1} + l_j) + p_{e1} l_1 + p_{eN+1} l_N \right\} \quad (64)$$

Now, f_w and f_e are generally complex numbers. Since ΔF and e are expressed by Eq. (39), Eq. (62) is equivalent to

$$\begin{aligned}
 \frac{\Delta f}{P^* L} &= \frac{\text{Im}(f_w)}{\omega} \Delta \dot{h} + \{ \text{Re}(f_w) + \frac{\text{Im}(f_e)}{\omega} \} \Delta \dot{h} \\
 &\quad + \text{Re}(f_e) \Delta h \quad (65)
 \end{aligned}$$

where Im and Re express the imaginary and the real parts of a complex number, respectively. From Eq. (65), the added mass coefficient, the added damping coefficients and the added stiffness coefficient are expressed as

$$\begin{aligned}
 M_a &= -\text{Im}(f_w)/\omega, \quad C_w = -\text{Re}(f_w) \\
 C_e &= -\text{Im}(f_e)/\omega, \quad K_a = -\text{Re}(f_e) \quad (66)
 \end{aligned}$$

4.4 Results of numerical calculations

First, the numerical solutions of the fluid-dynamic forces in tapered passages are compared with the analytical ones obtained in chapter 3. Comparison of both solutions of C_w is shown in Fig. 6. Close agreement between both solutions is obtained if the number of divisions is sufficiently large (say $N=10$).

Second, the fluid-dynamic forces in a passage with a step are calculated. The loss factor in a pipe flow is used as that at a step⁽¹³⁾, that is, when h_i and h_{ii} are gaps before and after the step, the loss factor at a sharp expansion is given as

$$\xi_1 \approx (1 - h_i/h_{ii})^2 = (1 - 1/h_{ii})^2 \quad (67)$$

where $h_i^* = h_i = 1$, and the loss factor at a sharp shrinkage of the passage is obtained from Table 1 for $h_i^* = h_{ii}$. The area increment ratio at step is defined by

$$\alpha' = (h_{ii} - h_i)/h_i = h_{ii} - 1 \quad (68)$$

Figure 7 shows the range where $C_w + C_e$ or K_a

Table 1 Loss factor at a sharp shrinkage of the passage⁽¹³⁾

h_2/h_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
ζ_1	0.41	0.38	0.34	0.29	0.24	0.18	0.14	0.089	0.036	0