Lateral-Torsional Coupled Vibrations of a Rotating Shaft Driven by a Universal Joint*

(Derivation of Equations of Motion and Asymptotic Analyses)

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A driven shaft system consists of a flexible shaft with circular section and a rotor. The flexible shaft is supported by a universal joint at one end and a bearing at the other end. When a drive shaft rotates with a constant angular velocity, the driven shaft is subjected to the angular velocity fluctuation governed by a joint angle between the drive shaft and the driven one. However, when the driven shaft vibrates laterally and causes bending in itself, the joint angle is slightly changed and the rotating speed deviates. This is why the lateral-torsional coupled vibrations become unstable at the same time. These unstable vibrations occur when the drive shaft rotates at a speed of nearly half the sum of the natural angular frequency for the lateral vibration and the one for the torsional vibration.

Key Words: Vibration of Rotating Body, Universal Joint, Change of Joint Angle, Parametric Excitation, Asymptotic Solution, Unstable Vibrations Coupled by Bending and Torsion, Unstable Region

1. Introduction

There have been a number of investigations on unstable vibrations of a rotating shaft driven by a universal joint. Wehrli⁽¹⁾ and Saigo et al.⁽²⁾ treated instability of lateral vibration, and Zeman⁽³⁾ that of torsional vibration. Iida et al.⁽⁴⁾ studied vibrations coupled by the bending and torsion of a rotating shaft equipped with a gear.

The present paper analyzes the possibility that a flexible rotating shaft driven by a universal joint will make lateral-torsional coupled vibrations. The rotating shaft treated here has a uniform circular section and is supported by a universal joint at one end and a ball bearing at the other end. A balanced symmetrical rotor is mounted to the shaft. When a drive shaft rotates with a constant angular velocity, the driven shaft is subjected to an angular velocity fluctuation

the magnitude of which is determined by the joint angle between the drive shaft and the driven one. However, when the driven shaft laterally vibrates and causes bending in itself, the joint angle changes. Then the driven shaft comes to rotate with an angular velocity different from that of the shaft with no bending. This may make the rotating shaft unstable and generates lateral and torsional vibrations. From this viewpoint, equations of motion are derived by considering the influence of deflection on the torsional vibration. Bogoliubov - Mitropolsky's asymptotic method⁽⁵⁾ is used to analyze approximately the equations of motion for parametric vibrations caused by coupling of the bending and the torsion. The analyses revealed the stability of the driven shaft and showed that unstable vibrations occur when the drive speed nearly coincides with the arithmetic mean of two natural angular frequencies about bending and torsion.

2. Theory of Elasticity for a Shaft Undergoing Deformations of Bending and Torsion

A flexible shaft S_n with circular section is considered, in which support is at both ends A and B as

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shown in Fig. 1. A balanced symmetrical rotor D is attached at a point S of the flexible shaft S_h . The shaft is assumed to have no initial bending and no distributed mass. A universal joint gives a moment of force in z direction and also one in the direction perpendicular to z axis (the latter is called the secondary moment⁽⁶⁾). Thus, the shaft between A and S undergoes torsion as well as bending. To obtain the change of joint angle due to lateral vibration, we must derive an equation for a deflection curve of the shaft.

Let an axis of the rigid drive shaft be Az_{a1} , and that of the driven shaft without deflection be Az . These two axes intersect each other with an angle α_a at the center A of the universal joint. In the stationary rectangular co-ordinate system $O-xyz$, the xy plane coincides with the plane in which the point S moves when the shaft makes a small vibration. The direction of y axis is perpendicular to the $z_{a1}Az$ plane containing the joint angle α_a . In this connection, $\overline{AB}=l$, $\overline{AS}=a$, and $\overline{SB}=b$ are noted.

Figure 2 shows projections of forces and moments of force acting on the shaft AB to the xaz plane and the yaz plane, where x_a and y_a axes are parallel to x and y axes, respectively. At the point A, moments M_{ax} , M_{ay} , and M_{az} act in the directions x , y , and z . At the point S, forces F_{sx} and F_{sy} act in the directions x and y , and moments M_{sx} , M_{sy} , and M_{sz} in the directions x , y , and z . The point A is subjected to reaction forces R_{ax} and R_{ay} , and the point B reaction forces R_{bx} and R_{by} . Also, moments M_{bx} and M_{by} act on the point B when the shaft end B is fixed. Each of the arrows in Fig. 2 shows the positive sense of force or moment.

Fig. 1 Rotating shaft driven by universal joint

*** A shaft with circular section is treated here. Accordingly, each of the principal axes is assumed to have the same direction at every longitudinal shaft position when the shaft has no deformation.

The theory of elasticity⁽⁷⁾ gives expressions for moments M_1 , M_2 , and M_3 in the principal axes of an elastic shaft undergoing deformations of bending and torsion as follows:

$$\left. \begin{aligned} -\frac{M_1}{EI_0} &= \frac{\partial \theta}{\partial s} \sin \psi - \frac{\partial \phi}{\partial s} \sin \theta \cos \psi \\ -\frac{M_2}{EI_0} &= \frac{\partial \theta}{\partial s} \cos \psi + \frac{\partial \phi}{\partial s} \sin \theta \sin \psi \\ -\frac{M_3}{GI_z} &= \frac{\partial \psi}{\partial s} + \frac{\partial \phi}{\partial s} \cos \theta \end{aligned} \right\} \quad (1)$$

where θ , ϕ , and ψ are Eulerian angles which show angular positions of the principal axes^{***}, and ds is a minute length along the shaft center line. Notation EI_0 represents a rigidity of bending and GI_z that of torsion.

Let a rotating angle at the point A of shaft be θ_a . Then

$$\theta_t = \phi + \psi - \theta_a \quad (2)$$

is considered to be a small angle of torsion at a certain shaft position in relation to the point A. Equation (1) is rewritten by ignoring small quantities to the second order of θ and θ_t as follows:

$$\left. \begin{aligned} \frac{\partial \theta_x}{\partial z} &= -\frac{1}{EI_0} M_y \\ \frac{\partial \theta_y}{\partial z} &= \frac{1}{EI_0} M_x \\ \frac{\partial \theta_t}{\partial z} &= -\frac{1}{GI_z} M_z \end{aligned} \right\} \quad (3)$$

where M_x , M_y , and M_z are x , y , and z components of the vectorial sum of M_1 , M_2 , and M_3 . Variables θ_x and θ_y are projectional angles of θ to xaz and yaz planes. Deflections at a certain point on the shaft are given by the following equations

$$\left. \begin{aligned} x &= \int_{-a}^z \theta_x dz \\ y &= \int_{-a}^z \theta_y dz \end{aligned} \right\} \quad (4)$$

The origin O of z is made to coincide with the point S when the shaft has no deflection.

Fig. 2 Forces and moments of force acting on shaft

Integrating Eq. (3) under the prepared boundary conditions gives angles of deflection $\theta_x(z)$ and $\theta_y(z)$ and an angle of torsion $\theta_t(z)$. The functions $\theta_x(z)$, $\theta_y(z)$, and $\theta_t(z)$ of z include F_{sx} , F_{sy} , M_{sx} , M_{sy} , M_{sz} , M_{ax} , and M_{ay} as coefficients. As to the moments M_{bx} and M_{by} at the shaft end B, $M_{bx}=0$ and $M_{by}=0$ hold when the shaft end B is simply supported, while M_{bx} and M_{by} are determined by the conditions $\theta_x(b)=0$ and $\theta_y(b)=0$ when the shaft end B is fixedly supported.

3. Equations of Motion

3.1 Equations of motion for a rotor with variable rotating speed

The rotor has the mass m , the diametral moment of inertia I , and the polar moment of inertia I_p . Equations of deflection curve are used in order to derive translational and rotational motions of the rotor D. The shaft at the point S has deflections x_s and y_s , angles of deflection θ_{sx} and θ_{sy} , and an angle of torsion θ_{st} . Thus,

$$\left. \begin{aligned} x_s &= x(0), y_s = y(0), \\ \theta_{sx} &= \theta_x(0), \theta_{sy} = \theta_y(0), \\ \theta_{st} &= \theta_t(0) \end{aligned} \right\} \quad (5)$$

From Eq. (5), F_{sx} , F_{sy} , M_{sx} , M_{sy} , and M_{sz} are obtained as functions of x_s , y_s , θ_{sx} , θ_{sy} , and θ_{st} . Using these results^{**2}, we have equations of motion

$$\left. \begin{aligned} m\ddot{x}_s + c_1\dot{x}_s + \alpha x_s + \gamma\theta_{sx} &= \frac{3}{2a}M_{ay} \\ m\ddot{y}_s + c_1\dot{y}_s + \alpha y_s + \gamma\theta_{sy} &= -\frac{3}{2a}M_{ax} \\ I\ddot{\theta}_{sx} + c_2\dot{\theta}_{sx} + I_p\dot{\theta}_a\dot{\theta}_{sy} + \delta\theta_{sx} + \gamma x_s &= -\frac{1}{2}M_{ay} \\ I\ddot{\theta}_{sy} + c_2\dot{\theta}_{sy} - I_p\dot{\theta}_a\dot{\theta}_{sx} + \delta\theta_{sy} + \gamma y_s &= \frac{1}{2}M_{ax} \\ I_p\ddot{\theta}_{st} + c_3(\dot{\theta}_{st} + \dot{\theta}_a) + \delta_t\theta_{st} &= -I_p\dot{\theta}_a \end{aligned} \right\} \quad (6)$$

where

$$\delta_t = GI_z/a \quad (7)$$

Spring constants α , γ , and δ are given by

$$\left. \begin{aligned} \alpha &= 3EI_0l(a^2 - ab + b^2)/(a^3b^3) \\ \gamma &= 3EI_0l(a - b)/(a^2b^2) \\ \delta &= 3EI_0l/(ab) \end{aligned} \right\} \quad (8)$$

for fixed support at the shaft end B, and given by

$$\left. \begin{aligned} \alpha &= 3EI_0(4a^3 + b^3)/(a^3b^3) \\ \gamma &= 3EI_0(2a^2 - b^2)/(a^2b^2) \\ \delta &= EI_0(4a + 3b)/(ab) \end{aligned} \right\} \quad (9)$$

for simple support⁽⁸⁾. Viscous damping coefficients c_1 , c_2 , and c_3 are concerned with translation, inclination, and rotation of the rotor, respectively. Dot notation means differentiation with respect to time t .

3.2 Rotating speed of shaft

Equations of motion (6) are general equations governing the motion of a rotor attached on a flexible shaft with an angular velocity fluctuation. The driving condition for the shaft determines the rotating speed $\dot{\theta}_a$ and the moments M_{ax} and M_{ay} . We deal with the case in which a universal joint is used at the shaft end A. According to this condition, each of $\dot{\theta}_a$, M_{ax} , and M_{ay} should be determined as a function of the deflections x_s and y_s , the angles of deflection θ_{sx} and θ_{sy} , and the angle of torsion θ_{st} at the point S.

A cross-pin of the universal joint is shown in Fig. 3. The cross-pin consists of two pins, Ψ pin and Θ pin, rotating with the drive and driven yokes, respectively. A stationary rectangular co-ordinate system A - $x_{a1}y_{a1}z_{a1}$ inclines by an angle α_a about y_a axis from a co-ordinate system A - x_az . Time $t=0$ is decided at a moment when the Ψ pin, AP, overlaps with z_{a1} axis, and a rotating angle Ψ of the drive shaft is measured from this moment. The drive shaft is assumed to rotate with a constant angular velocity ω .

$$\Psi = \omega t \quad (10)$$

As shown in Fig. 3, let an angle of deflection at the point A be θ_a , an azimuth angle of θ_a be ϕ_a , and projectional angles of θ_a to the $x_{a1}z$ and $y_{a1}z$ planes be θ_{ax} and θ_{ay} . Therefore,

$$\left. \begin{aligned} \theta_{ax} &= \theta_x(-a) \\ \theta_{ay} &= \theta_y(-a) \end{aligned} \right\} \quad (11)$$

Fig. 3 Eulerian angles representing angular positions of cross-pin

^{**2} Forces and moments of force which act on the rotor are restoring forces and moments of force at the point S under the same deflection, angle of deflection, and angle of torsion as those caused by the forces F_{sx} and F_{sy} and the moments M_{sx} , M_{sy} , and M_{sz} , added to the point S. Thus, the rotor is subjected to forces $-F_{sx}$ and $-F_{sy}$ and moments $-M_{sx}$, $-M_{sy}$, and $-M_{sz}$.

hold. In Fig. 3, the plane NAQ' is perpendicular to z axis, the plane NAQ is perpendicular to a tangent line at the point A of the deflection curve. At a certain time t , the Θ pin rotates by Θ_{a1} from AN. Let unit vectors with the same directions and senses as x_a , y_a , and z axes be i , j , and k , respectively. Then a unit vector p in the direction $\overline{AP}$ is given by

$$p = \cos \alpha_a \cos \Psi i + \sin \Psi j + \sin \alpha_a \cos \Psi k \quad (12)$$

and a unit vector q in the direction $\overline{AQ}$ is given by

$$q = -(\cos \Theta_{a1} \sin \phi_a + \sin \Theta_{a1} \cos \theta_a \cos \phi_a) i \\ + (\cos \Theta_{a1} \cos \phi_a - \sin \Theta_{a1} \cos \theta_a \sin \phi_a) j \\ + \sin \Theta_{a1} \sin \theta_a k \quad (13)$$

A rotating angle Θ_a at the point A of shaft is

$$\Theta_a = \Theta_{a1} + \phi_a \quad (14)$$

If more small quantities than the second order of θ_{ax} and θ_{ay} are ignored, we have

$$q = -\sin \Theta_a i + \cos \Theta_a j \\ + (\theta_{ax} \sin \Theta_a - \theta_{ay} \cos \Theta_a) k \quad (15)$$

A rotating angle Θ_a of the shaft end A is determined by the orthogonal condition of the cross-pin:

$$p \cdot q = 0 \quad (16)$$

Here,

$$\left. \begin{aligned} \Theta_a &= \Theta_{a0} + \Delta_a \\ \tan \Theta_{a0} &= \sec \alpha_a \tan \Psi \end{aligned} \right\} \quad (17)$$

is considered. In Eq. (17), Δ_a is a small quantity of the same order as θ_{ax} or θ_{ay} . After Eq. (17) is substituted into Eq. (16), solving Eq. (16) for Δ_a yields the following relation:

$$\Delta_a = \frac{\sin \alpha_a \cos \Psi}{H(\Psi)} (\theta_{ax} \sin \Psi - \theta_{ay} \cos \alpha_a \cos \Psi) \quad (18)$$

where

$$H(\Psi) = 1 - \sin^2 \alpha_a \cos^2 \Psi \quad (19)$$

Definitely, Δ_a represents the angular velocity fluctuation caused by the shaft deflection. The universal joint treated here has no friction between the cross-pin and the yokes. Then, the universal joint transmits a moment of force

$$M_{a0} = M_{a0} p \times q \quad (20)$$

to the driven shaft⁽⁶⁾. The moment vector M_{a0} has the following components

$$\left. \begin{aligned} M_{ax} &= M_{a0} \cdot i \\ M_{ay} &= M_{a0} \cdot j \\ M_{az} &= M_{a0} \cdot k \end{aligned} \right\} \quad (21)$$

in x , y , and z directions. When the following equation for equilibrium of moment in the shaft part AS

$$M_{ax} = -M_{sz} \quad (22)$$

simultaneously holds with Eq. (5), each of θ_{ax} , θ_{ay} , M_{ax} , and M_{ay} appears as a function of x_s , y_s , θ_{sx} , θ_{sy} , θ_{st} , and Ψ . Substitution of these results and Eq. (17) into Eq. (6) brings out equations of motion for the rotating shaft driven by the universal joint.

4. Analyses of Vibration

4.1 Transformation to normal co-ordinate

Equations of motion (6) for the shaft driven by a universal joint cannot be easily integrated. Accordingly, in order to analyze Eq. (6) approximately, we prepare a small parameter ε_a defined by the following equation:

$$\varepsilon_a = \sin \alpha_a \quad (23)$$

Remember that $\varepsilon_a > 0$, namely, $0 < \alpha_a < \pi/2$. A natural angular frequency for the torsional vibration is put as $p_5 = \sqrt{\delta_t/I_p}$. Each of the deflection x_s/l and y_s/l , the angles of deflection θ_{sx} and θ_{sy} , the torsional angle θ_{st} , and the non-dimensional damping coefficients $\zeta_1 = c_1/\sqrt{m\alpha}$, $\zeta_2 = c_2/(l^2\sqrt{m\alpha})$, and $\zeta_3 = c_3/(l^2\sqrt{m\alpha})$ is assumed to be a small quantity of the order of ε_a^2 . Then, the following non-dimensional quantities are introduced.

$$\left. \begin{aligned} X &= x_s/(\varepsilon_a^2 l), Y = y_s/(\varepsilon_a^2 l), \\ \Theta_x &= \theta_{sx}/\varepsilon_a^2, \Theta_y = \theta_{sy}/\varepsilon_a^2, \\ \Theta_t &= \theta_{st}/\varepsilon_a^2, C_1 = \zeta_1/\varepsilon_a^2, \\ C_2 &= \zeta_2/\varepsilon_a^2, C_3 = \zeta_3/\varepsilon_a^2, \\ \delta' &= \delta/(al^2), \delta_t' = \delta_t/(al^2), \\ \gamma' &= \gamma/(al), I' = I/(ml^2), \\ I_p' &= I_p/(ml^2), a' = a/l, \\ \omega' &= \omega/\sqrt{a/m}, p' = p/\sqrt{a/m}, \\ \tau &= \sqrt{a/m} t \end{aligned} \right\} \quad (24)$$

Hereafter, prime in Eq. (24) is neglected. Furthermore, each of $M_{ax}/(al^2)$, $M_{ay}/(al^2)$, and Δ_a is divided in order by powers of ε_a . They contain $M_{ax}^{(1)}$, $M_{ay}^{(1)}$, and $\Delta_a^{(1)}$ as coefficients of ε_a^3 , and $M_{ax}^{(2)}$, $M_{ay}^{(2)}$, and $\Delta_a^{(2)}$ as those of ε_a^4 . Hence,

$$\left. \begin{aligned} M_{ax}^{(1)} &= \delta_t \Theta_t \cos^2 \Psi \\ M_{ay}^{(1)} &= (\delta_t/2) \Theta_t \sin 2\Psi \\ \Delta_a^{(1)} &= \cos \Psi \{ (3X - a\Theta_x) \sin \Psi \\ &\quad - (3Y - a\Theta_y) \cos \Psi \} / (2a) \\ M_{ax}^{(2)} &= 0, M_{ay}^{(2)} = 0 \\ \Delta_a^{(2)} &= \{ GI_z / (4EI_0) \} \Theta_t \cos^2 \Psi \end{aligned} \right\} \quad (25)$$

We here deal with unstable vibrations coupled by bending and torsion of the shaft. Therefore, the angular velocity ω of the drive shaft is assumed not to coincide with even submultiples of the natural angular frequency of the driven shaft. Under this assumption, the following change of variable is performed:

$$\Theta_t = \Theta_z - \frac{C_3 \omega}{\delta_t} + \frac{\omega^2}{p_5^2 - 4\omega^2} \sin 2\Psi \quad (26)$$

Then, Eq. (6) comes to

$$\left. \begin{aligned} \ddot{X} + X + \gamma \Theta_x &= \varepsilon_a F_x^{(1)} + \varepsilon_a^2 F_x^{(2)} \\ \ddot{Y} + Y + \gamma \Theta_y &= \varepsilon_a F_y^{(1)} + \varepsilon_a^2 F_y^{(2)} \\ I \ddot{\Theta}_x + I_p \omega \dot{\Theta}_y + \delta \Theta_x + \gamma X &= \varepsilon_a M_x^{(1)} + \varepsilon_a^2 M_x^{(2)} \\ I \ddot{\Theta}_y - I_p \omega \dot{\Theta}_x + \delta \Theta_y + \gamma Y &= \varepsilon_a M_y^{(1)} + \varepsilon_a^2 M_y^{(2)} \\ I_p \ddot{\Theta}_z + \delta_t \Theta_z &= \varepsilon_a M_z^{(1)} + \varepsilon_a^2 M_z^{(2)} \end{aligned} \right\} \quad (27)$$

where

$$\left. \begin{aligned}
F_x^{(1)} &= \{3/(2a)\} M_{ay}^{(1)} \\
F_y^{(1)} &= -\{3/(2a)\} M_{ax}^{(1)} \\
M_x^{(1)} &= -(1/2) M_{ay}^{(1)} \\
M_y^{(1)} &= (1/2) M_{ax}^{(1)} \\
M_z^{(1)} &= -I_p \ddot{\Delta}_a^{(1)} \\
F_x^{(2)} &= -C_1 \dot{X} \\
F_y^{(2)} &= -C_1 \dot{Y} \\
M_x^{(2)} &= -C_2 \dot{\Theta}_x - (I_p \omega/2) \dot{\Theta}_y \cos 2\Psi \\
M_y^{(2)} &= -C_2 \dot{\Theta}_y + (I_p \omega/2) \dot{\Theta}_x \cos 2\Psi \\
M_z^{(2)} &= -C_3 \dot{\Theta}_z - (C_3 \omega/2) \cos 2\Psi \\
&\quad + (I_p \omega^2/2) (\sin 2\Psi + \sin 4\Psi) \\
&\quad - I_p \ddot{\Delta}_a^{(2)}
\end{aligned} \right\} \quad (28)$$

with an accuracy of ε_a^2 . After Eq. (27), the dot notation means differentiation with respect to the non-dimensional time τ .

In the equations of motion (27), when $\varepsilon_a=0$, solutions for natural vibrations are given by the following equations.

$$\left. \begin{aligned}
X &= \sum_{j=1}^4 A_j \cos(p_j \tau + \Phi_j) \\
Y &= \sum_{j=1}^4 A_j \sin(p_j \tau + \Phi_j) \\
\Theta_x &= \sum_{j=1}^4 \kappa_j A_j \cos(p_j \tau + \Phi_j) \\
\Theta_y &= \sum_{j=1}^4 \kappa_j A_j \sin(p_j \tau + \Phi_j)
\end{aligned} \right\} \quad (29)$$

where A_j and Φ_j are arbitrary constants, κ_j is an amplitude ratio of the inclination to the deflection, and p_j is one of roots p of the following equation

$$\begin{vmatrix} 1-p^2 & \gamma \\ \gamma & \delta + I_p \omega p - I p^2 \end{vmatrix} = 0 \quad (30)$$

When these roots are allocated in the order of magnitude such as p_1, p_2, p_3 , and p_4 , the following equations

$$\left. \begin{aligned}
p_4 &< p_3 < 0 < p_2 < p_1 \\
|p_3| &< p_2, |p_4| < p_1
\end{aligned} \right\} \quad (31)$$

hold, provided that ω and γ are not zero. The amplitude ratio κ_j is given by

$$\begin{aligned}
\kappa_j &= -(1-p_j^2)/\gamma \\
&= -\gamma/(\delta + I_p \omega p_j - I p_j^2)
\end{aligned} \quad (32)$$

Considering Eqs. (29) and (30), we change the variables as follows:

$$\left. \begin{aligned}
X &= \sum_{j=1}^4 x_j + \varepsilon_a K_a \tau, \quad Y = -\sum_{j=1}^4 \frac{\dot{x}_j}{p_j} \\
\Theta_x &= \sum_{j=1}^4 \kappa_j x_j + \varepsilon_a K_b \tau, \\
\Theta_y &= -\sum_{j=1}^4 \frac{\kappa_j \dot{x}_j}{p_j}, \quad \Theta_z = X_5 \\
\dot{x}_j &= X_j (j=1 \sim 4)
\end{aligned} \right\} \quad (33)$$

The constants K_a and K_b should be determined so that X and Θ_x become periodic functions with respect to τ . After Eq. (33) is substituted into Eq. (27), the use of Eq. (32) yields the following equations of motion expressed by normal co-ordinates.

$$\dot{X}_j + p_j^2 X_j = \varepsilon_a Q_j^{(1)} + \varepsilon_a^2 Q_j^{(2)} \quad (j=1 \sim 5) \quad (34)$$

where

$$\left. \begin{aligned}
Q_j^{(1)} &= B_j^{(1)} + (\dot{F}_x^{(1)} E_{1j} + \dot{M}_x^{(1)} E_{3j} \\
&\quad + F_y^{(1)} E_{2j} + M_y^{(1)} E_{4j})/D \\
Q_j^{(2)} &= M_z^{(2)}/I_p \quad (j=1, 2) \\
B_j^{(1)} &= \{-(K_a + K_b) E_{1j} \\
&\quad - (\gamma K_a + \delta K_b) E_{3j} + I_p \omega K_b E_{4j}\}/D \\
B_j^{(2)} &= 0
\end{aligned} \right\} \quad (35)$$

in which D is a determinant

$$D = |e_{ij}| \quad (36)$$

with elements

$$\left. \begin{aligned}
e_{1j} &= 1, \quad e_{2j} = -1/p_j \\
e_{3j} &= \kappa_j (I p_j - I_p \omega)/p_j \\
e_{4j} &= -I \kappa_j/p_j
\end{aligned} \right\} \quad (37)$$

Because $p_j \neq p_k (j \neq k)$ generally holds, D does not come to zero. Notation E_{ij} represents a cofactor of e_{ij} .

4.2 Parametrically excited vibration of summed or differential type

4.2.1 Approximate analyses by asymptotic method We here investigate the case in which vibrations of deflection and torsion simultaneously occur. The defective vibration is assumed to resonate in a single mode, and a vibration for one of $p_i (i=1, 2, 3, 4)$ is considered.

In order to treat vibrations of frequencies ω_i and ω_5 nearly equal to natural angular frequencies p_i and p_5 for deflection and torsion,

$$p_k^2 = \omega_k^2 + \varepsilon_a d_k (k=i, 5) \quad (38)$$

is presented. In Eq. (38), $\varepsilon_a d_k$ is a small quantity which represents a detuning. By using Eq. (38), Eq. (34) is rewritten into the following form.

$$\left. \begin{aligned}
\dot{X}_k + \omega_k^2 X_k &= \varepsilon_a (Q_k^{(1)} - d_k X_k) \\
&\quad + \varepsilon_a^2 Q_k^{(2)} \\
\dot{X}_j + p_j^2 X_j &= \varepsilon_a Q_j^{(1)} + \varepsilon_a^2 Q_j^{(2)} \\
&\quad (k=i, 5; j \neq k)
\end{aligned} \right\} \quad (39)$$

Bogoliubov-Mitropolsky's method⁽⁵⁾ is used to analyze Eq. (39). A solution for resonance is considered in the following asymptotic expansion form

$$\left. \begin{aligned}
X_k &= A_k \cos \Psi_k \\
&\quad + \varepsilon_a U_k^{(1)}(A_i, \Phi_i, A_5, \Phi_5, \tau) + \dots
\end{aligned} \right\} \quad (40)$$

$$\left. \begin{aligned}
X_j &= \varepsilon_a U_j^{(1)}(A_i, \Phi_i, A_5, \Phi_5, \tau) + \dots \\
\Psi_k &= \omega_k \tau + \Phi_k
\end{aligned} \right\} \quad (41)$$

where A_k and Φ_k are given by

$$\left. \begin{aligned}
A_k &= \varepsilon_a R_k^{(1)}(A_i, \Phi_i, A_5, \Phi_5) \\
&\quad + \varepsilon_a^2 R_k^{(2)}(A_i, \Phi_i, A_5, \Phi_5) + \dots \\
\Phi_k &= \varepsilon_a S_k^{(1)}(A_i, \Phi_i, A_5, \Phi_5) \\
&\quad + \varepsilon_a^2 S_k^{(2)}(A_i, \Phi_i, A_5, \Phi_5) + \dots
\end{aligned} \right\} \quad (42)$$

Substitution of Eq. (40) into Eq. (39) yields the possibility that the driven shaft will make a parametrically excited vibration of the following summed or differential type.

$$\left. \begin{aligned}
(a) \quad 2\omega &= \omega_i + \omega_5 \\
(b) \quad 2\omega &= -\omega_i - \omega_5 \\
(c) \quad 2\omega &= \omega_i - \omega_5 \\
(d) \quad 2\omega &= -\omega_i + \omega_5
\end{aligned} \right\} \quad (43)$$

Considering Eq. (38), we can define the frequencies $\omega_k (k=i, 5)$ as

$$\left. \begin{aligned} (a) \quad \omega_k &= 2p_k\omega / (p_i + p_5) \\ (b) \quad \omega_k &= 2p_k\omega / (-p_i - p_5) \\ (c) \quad \omega_k &= 2p_k\omega / (p_i - p_5) \\ (d) \quad \omega_k &= 2p_k\omega / (-p_i + p_5) \end{aligned} \right\} \quad (44)$$

about each of (a), (b), (c), and (d) of Eq. (43). Remember that actual frequencies of vibrations generated are not ω_i and ω_5 but $\omega_i + \dot{\phi}_i$ and $\omega_5 + \dot{\phi}_5$.

According to normal procedure based on the asymptotic method, the conditions that $U_k^{(1)}, U_k^{(2)}, \dots$ do not have secular terms decide $R_k^{(1)}, S_k^{(1)}, U_k^{(1)}, R_k^{(2)}, S_k^{(2)}, \dots$ in succession. In every case of (a), (b), (c), and (d) of Eq. (43), K_a and K_b are decided by the conditions that X and Θ_x should be periodic with respect to time. That is, they satisfy the following equations

$$\left. \begin{aligned} \sum_{j=1}^4 (\bar{U}_j^{(1)} + \varepsilon_a \bar{U}_j^{(2)} + \dots) + K_a &= 0 \\ \sum_{j=1}^4 x_j (\bar{U}_j^{(1)} + \varepsilon_a \bar{U}_j^{(2)} + \dots) + K_b &= 0 \end{aligned} \right\} \quad (45)$$

where $\bar{U}_j^{(1)}, \bar{U}_j^{(2)}, \dots$ are constant values in $U_j^{(1)}, U_j^{(2)}, \dots$, respectively.

4.2.2 Vibration of $2\omega = \omega_i + \omega_5$ type Figure 4 shows the results for the stability criterion from Eq. (42) given by the second approximate solution when $i=2$, namely, $2\omega = p_2 + p_5$. An unstable region appears in this vicinity. Increase of damping coefficients ζ_1 and ζ_2 for the lateral vibrations enlarges the minimum joint angle α_a at which an unstable vibration occurs. When the joint angle α_a is larger than the critical joint angle, we have the unusual result that the unstable region becomes wider with increases of ζ_1 and ζ_2 ⁽⁹⁾. A growing rate of vibration is represented by a negative damping coefficient⁽¹⁰⁾ μ . The negative damping coefficient μ becomes smaller in the middle of the unstable region when ζ_1 and ζ_2 increase, whereas μ increases in the neighborhood of the boundary. Change of damping coefficient ζ_3 for the torsional vibration gives the usual effect to the unstable region and the negative damping coefficient. Thus, the increase of ζ_3 narrows the unstable region and makes the negative damping coefficient μ smaller. Accordingly, the summed parametrically excited vibration with $2\omega = \omega_2 + \omega_5$ type is effectively suppressed by enlarging the damping coefficient for the torsional vibration, but not always suppressed by enlarging those for the lateral vibrations.

4.2.3 Vibrations of other types For the vibration with $2\omega = -\omega_i - \omega_5$ or $2\omega = \omega_i + \omega_5$ type, the analytical result by the second approximation shows that $\dot{A}_5 < 0$ always holds. Therefore, the unstable vibrations with these types do not occur.

For the vibration with $2\omega = \omega_i - \omega_5$ type, the sta-

(a) Unstable region

(b) Negative damping coefficient

Fig. 4 Influence of damping coefficients ζ_1 and ζ_2 for lateral vibration on unstable vibration of $2\omega = \omega_2 + \omega_5$ type ($I = 2.275 \times 10^{-2}$, $I_p = 4.458 \times 10^{-2}$, $\gamma = 0.1459$, $\delta = 6.478 \times 10^{-2}$, $\delta_i = 5.293 \times 10^{-3}$, $a = 0.624$, $\zeta_3 = 4 \times 10^{-5}$)

Fig. 5 Negative damping coefficient for vibration of $2\omega = \omega_2 - \omega_5$ type (Same values as Fig. 4 are used in calculation but $\zeta_1 = 4 \times 10^{-3}$, $\zeta_2 = 4 \times 10^{-3}$, $\varepsilon_a = 0.259$)

bility cannot be immediately known as $2\omega = -\omega_i - \omega_s$ or $2\omega = -\omega_i + \omega_s$, but the numerical result shows that the unstable vibration with this type does not appear. Figure 5 shows the negative damping coefficient μ for $i=2$. In the neighborhood of this rotating speed, μ is always negative, and the stability becomes high with the smaller detuning.

5. Conclusions

The present study served to analyze the coupled vibrations of bending and torsion in the rotating shaft driven by a universal joint. The results may be summarized as follows.

(1) General equations of motion for a shaft rotating with angular velocity fluctuation are given by Eq. (8) with an accuracy of the first order of coordinates.

(2) When the drive shaft rotates with ω , which satisfies the relation $2\omega \doteq p_i + p_s$, the driven shaft becomes unstable and makes a summed parametric vibration. This unstable vibration has two frequencies. One is a frequency nearly equal to the natural angular frequency p_i for the lateral vibration, and the other is a frequency nearly equal to that p_s for the torsional vibration.

(3) A range in which this unstable vibration occurs, namely, the unstable region, becomes wider with the larger joint angle α_a . The unstable region becomes narrower with the larger viscous damping coefficient of the torsional vibration, whereas it does not always become narrower with that of the lateral vibration.

(4) When the drive shaft rotates with ω , which satisfies the relation $2\omega \doteq \pm p_i - p_s$ or $2\omega \doteq -p_i + p_s$, the driven shaft does not become unstable.

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