Closed-Loop Displacement Control of a One-Link Flexible Arm with a Tip Mass*

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A theoretical and experimental study is presented for the displacement control of a one-link flexible arm with an end-point payload. The tip displacement of the arm is measured by a gap sensor fixed in space and controlled by a D. C. motor located at the other end of the arm, where the motor is driven by a feedback signal composed of the tip displacement and the velocity. As an example, the problem of shifting the end-point of the arm from its initial position to the commanded position by the amount of w^*_d is considered. Solutions derived by applying the method of the Laplace transform to the governing equations are calculated by the method of numerical inversion. Experimental results are obtained and compared with the theoretical ones, showing both results in good agreement for a wide range of parameters.

Key Words: Dynamics of Machinery, Robot, Flexible Arm, Application of Control

1. Introduction

The requirements for more precise positional accuracy, higher payload capacity, lighter members, and higher speeds make it a necessity to consider the flexibility of the links in the control of industrial robots. Several papers on this matter have been published during the past few years, Book et al,⁽¹⁾ Bolotnik and Gukasyan⁽²⁾ and Skaar and Tucker⁽³⁾ investigated the open-loop control of flexible arms. Further, Tsuchiya et al,⁽⁴⁾ and Sakawa and Matsuno⁽⁵⁾ studied the closed-loop control of flexible manipulators by applying the method of modal expansion. When the solution is assumed in the form of mode summation, one has to find the orthogonality relation of the mode functions prior to solving the problem. The proof, however, becomes quite difficult when the manipulator configuration is complicated. As a result, one encounters a situation in which the problem cannot be solved.

When one drives the end-point of the arm from its

initial position to the commanded position, it may be effective to use the tip displacement directly as a feedback signal. Such control becomes particularly useful when one moves the arm as synchronized with the motion of its object that is fluctuating with time. It appears, however, that no detailed work has been done on this type of control. As a related problem, Cannon and Schmitz⁽⁶⁾ studied a closed-loop, end-point control of a flexible pinned-free beam by the method of modal expansion. They obtained some experimental results on the arm response measuring the tip displacement by a photosensor, but no comparison has been made between the theoretical and the experimental results. To study the control of robot arms theoretically based on computer simulation, it is a necessity that the theoretical result corresponds well with the response of the real robot. To the authors' knowledge, no work has yet been done concerning the quantitative comparison of the theoretical with the actual results on the flexible arm control. It is in this context that the problem of the closed-loop, displacement control of a one-link flexible arm with an end-point payload is discussed in this paper. The tip displacement of the arm is measured by a gap-sensor

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fixed in space and controlled by a D.C. servomotor located at its other end, where the motor is driven by a feedback signal composed of the tip displacement and velocity. As an example, the problem of shifting the endpoint of the arm from its initial position to the desired position by the amount of w^*_d is considered. Solutions derived by applying the method of the Laplace transform to the governing equations are obtained by the method of numerical inversion. Experimental results are compared with the theoretical ones, from which it is shown that both results are in good agreement for a wide range of parameters.

2. Analysis

Figure 1 shows the flexible robot arm studied in this analysis. It consists of a motor/actuator, an arm of length L and an end-point payload. The three elements are related each other through the bending moment M and the shear force Q . The whole system is driven by the motor torque T . As shown in the figure the x -axis is placed along the axis of the arm and the y -axis placed normal to the x -axis. The z -axis is perpendicular to both the x and the y -axes.

It is assumed that the arm deforms only in the xy -plane. In this case denoting the lateral displacement of the arm by $w(x, t)$ the governing equation of motion may be written as

$$EI\left(1 + C\frac{\partial}{\partial t}\right)\frac{\partial^4 w(x, t)}{\partial x^4} + \rho A\frac{\partial^2 w(x, t)}{\partial t^2} = 0 \quad (1)$$

where E is the Young's modulus, ρ the mass density, I the moment of inertia, A the sectional area, C the internal damping coefficient of arm, and t the time. By assuming that the arm is resting statically at $t=0$, the above equation is transformed with respect to t as

$$EI(1 + Cs)\frac{\partial^4 W(x, s)}{\partial x^4} + \rho As^2 W(x, s) = 0 \quad (2)$$

where W is the Laplace transform of w . By introducing

$$\xi^4 = -\frac{\rho As^2}{EI(1 + Cs)} \quad (3)$$

Eq.(2) is simplified as

Fig. 1 Geometry of problem and coordinates

$$\frac{\partial^4 W(x, s)}{\partial x^4} - \xi^4 W(x, s) = 0 \quad (4)$$

The general solution of Eq. (4) is

$$W = \alpha \sin \xi x + \beta \cos \xi x + \gamma \sinh \xi x + \delta \cosh \xi x \quad (5)$$

where α through δ are unknowns to be determined from the boundary conditions. The derivatives of Eq. (5) with respect to x are

$$\frac{\partial W}{\partial x} = \xi(\alpha \cos \xi x - \beta \sin \xi x + \gamma \cosh \xi x + \delta \sinh \xi x) \quad (6)$$

$$\frac{\partial^2 W}{\partial x^2} = \xi^2(-\alpha \sin \xi x - \beta \cos \xi x + \gamma \sinh \xi x + \delta \cosh \xi x) \quad (7)$$

$$\frac{\partial^3 W}{\partial x^3} = \xi^3(-\alpha \cos \xi x + \beta \sin \xi x + \gamma \cosh \xi x + \delta \sinh \xi x) \quad (8)$$

Next, one considers the boundary conditions of the arm. At $x=0$ the arm is directly attached to the motor shaft. Hence, one has

$$w(0, t) = 0 \quad (9)$$

Further, the equilibrium of the moments around the motor shaft leads to

$$J_m \frac{\partial^3 w(0, t)}{\partial t^2 \partial x} + \epsilon \frac{\partial^2 w(0, t)}{\partial t \partial x} = EI\left(1 + C\frac{\partial}{\partial t}\right)\frac{\partial^2 w(0, t)}{\partial x^2} + T \quad (10)$$

where J_m is the polar moment of inertia of the motor shaft, ϵ the armature damping coefficient, and T the motor torque. At the endpoint of the arm, the equilibrium of moments gives

$$J_p \frac{\partial^3 w(L, t)}{\partial t^2 \partial x} = -EI\left(1 + C\frac{\partial}{\partial t}\right)\frac{\partial^2 w(L, t)}{\partial x^2} \quad (11)$$

where J_p is the moment of inertia of the payload around the z -axis. Further, the equilibrium of the lateral forces is

$$M_p \frac{\partial^2 w(L, t)}{\partial t^2} = EI\left(1 + C\frac{\partial}{\partial t}\right)\frac{\partial^3 w(L, t)}{\partial x^3} \quad (12)$$

where M_p is the mass of the payload. Equations (9) through (12) are the boundary conditions for the problem under consideration. Transformation of Eqs. (9), (11) and (12) with respect to time gives

$$\beta + \delta = 0 \quad (13)$$

$$J_p s^2 \xi(\alpha \cos \xi L - \beta \sin \xi L + \gamma \cosh \xi L + \delta \sinh \xi L) + EI(1 + Cs)\xi^2(-\alpha \sin \xi L - \beta \cos \xi L + \gamma \sinh \xi L + \delta \cosh \xi L) = 0 \quad (14)$$

$$M_p s^2(\alpha \sin \xi L + \beta \cos \xi L + \gamma \sinh \xi L + \delta \cosh \xi L) - EI(1 + Cs)\xi^3(-\alpha \cos \xi L + \beta \sin \xi L + \gamma \cosh \xi L + \delta \sinh \xi L) = 0 \quad (15)$$

Before transforming Eq.(10), one defines the motor torque T . As mentioned above, the torque is applied to the arm base to shift the endpoint from its initial position to the commanded position by the amount of

w_a . To this end, one compares w_a with the actual tip position $w(L, t)$ which is measured by the sensor fixed in space. The position error $w_a - w$ is then amplified and put into the motor to control the arm. In this case, the circuit equation governing the motor current i_a is

$$\frac{L_a di_a}{R_a dt} + i_a + \frac{K' \partial^2 w(0, t)}{R_a \partial x \partial t} = G_d \{w_a(L, t) - w(L, t)\} - G_v \frac{\partial w(L, t)}{\partial t} \quad (16)$$

where G_d is the displacement feedback gain, G_v the velocity feedback gain, K' the back electromotive force constant, L_a the motor inductance, and R_a the resistance of the circuit. The torque applied to the arm by the motor is $T = K_t i_a$, where K_t is the torque constant of the motor. Substituting this equation into Eq. (10) and further transforming the resulted equation give

$$(J_m s^2 + \epsilon s) \frac{\partial W(0, s)}{\partial x} - EI(1 + Cs) \frac{\partial^2 W(0, s)}{\partial x^2} = K_t I_a \quad (17)$$

One transforms Eq. (16) and obtains the Laplace transform of the armature current, I_a . I_a is then substituted into Eq. (17). By introducing Eqs. (5) through (8) the resulted equation is further modified as

$$\begin{aligned} & \left\{ \left(\frac{L_a}{R_a} s + 1 \right) (J_m s^2 + \epsilon s) + \frac{K_t K'}{R_a} s \right\} \xi (\alpha + \gamma) \\ & - \left(\frac{L_a}{R_a} s + 1 \right) EI(1 + Cs) \xi^2 (-\beta + \delta) \\ & + K_t (G_d + G_v s) (\alpha \sin \xi L + \beta \cos \xi L \\ & + \gamma \sinh \xi L + \delta \cosh \xi L) \\ & = K_t G_d W_d(L, s) \end{aligned} \quad (18)$$

Equations (13), (14), (15) and (18) constitute a system of simultaneous algebraic equations of the unknowns α, β, γ and δ . By using Eq. (13), these equations are further simplified in the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} K_t G_d W_d(L, s) \\ 0 \\ 0 \end{bmatrix} \quad (19)$$

where

$$\begin{aligned} a_{11} &= \left\{ \left(\frac{L_a}{R_a} s + 1 \right) (J_m s^2 + \epsilon s) + \frac{K_t K'}{R_a} s \right\} \xi \\ &+ K_t (G_d + G_v s) \sin \xi L \\ a_{12} &= 2 \left(\frac{L_a}{R_a} s + 1 \right) EI(1 + Cs) \xi \\ &+ K_t (G_d + G_v s) (\cos \xi L - \cosh \xi L) \\ a_{13} &= \left\{ \left(\frac{L_a}{R_a} s + 1 \right) (J_m s^2 + \epsilon s) + \frac{K_t K'}{R_a} s \right\} \xi \\ &+ K_t (G_d + G_v s) \sinh \xi L \\ a_{21} &= J_p s^2 \xi \cos \xi L \\ &- EI(1 + Cs) \xi^2 \sin \xi L \\ a_{22} &= -J_p s^2 \xi (\sin \xi L + \sinh \xi L) \\ &- EI(1 + Cs) \xi^2 (\cos \xi L + \cosh \xi L) \\ a_{23} &= J_p s^2 \xi \cosh \xi L \end{aligned}$$

$$\begin{aligned} &+ EI(1 + Cs) \xi^2 \sinh \xi L \\ a_{31} &= M_p s^2 \sin \xi L \\ &+ EI(1 + Cs) \xi^3 \cos \xi L \\ a_{32} &= M_p s^2 (\cos \xi L - \cosh \xi L) \\ &- EI(1 + Cs) \xi^3 (\sin \xi L - \sinh \xi L) \\ a_{33} &= M_p s^2 \sinh \xi L \\ &- EI(1 + Cs) \xi^3 \cosh \xi L \end{aligned}$$

The solution of Eq. (19) is

$$\begin{aligned} & \{\alpha, \beta, \gamma\} \\ &= \left\{ \frac{\Delta \alpha}{\Delta}, \frac{\Delta \beta}{\Delta}, \frac{\Delta \gamma}{\Delta} \right\} K_t G_d W_d(L, s) \end{aligned} \quad (20)$$

where

$$\begin{aligned} \Delta \alpha &= \begin{vmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} \\ \Delta \beta &= \begin{vmatrix} a_{11} & 1 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} \\ \Delta \gamma &= \begin{vmatrix} a_{11} & a_{12} & 1 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix} \\ \Delta &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{aligned}$$

Substituting Eqs. (13) and (20) into Eq. (5) one finally has the transformed input-output relation in the form

$$\begin{aligned} \frac{W(x, s)}{W_d(L, s)} &= \left\{ \frac{\Delta \alpha}{\Delta} \sin \xi x + \frac{\Delta \beta}{\Delta} (\cos \xi x - \cosh \xi x) \right. \\ &+ \left. \frac{\Delta \gamma}{\Delta} \sinh \xi x \right\} K_t G_d \end{aligned} \quad (21)$$

w_d , the desired arm tip position, is first stored in the computer memory through the keyboard. The arm control is then started by pushing a key. In this case, w_d is expressed as $w_d = W_d^* H(t)$, where $H(\cdot)$ is the Heaviside step function. Transforming this equation with respect to t gives

$$W_d(L, s) = \frac{W_d^*}{s}$$

Hence, in this case Eq. (21) becomes

$$\begin{aligned} \frac{W(x, s)}{W_d^*} &= \left\{ \frac{\Delta \alpha}{\Delta} \sin \xi x + \frac{\Delta \beta}{\Delta} (\cos \xi x - \cosh \xi x) \right. \\ &+ \left. \frac{\Delta \gamma}{\Delta} \sinh \xi x \right\} \frac{K_t G_d}{s} \end{aligned} \quad (22)$$

In the numerical examples that follow, one calculates the inverse integral of Eq. (22) by the numerical inversion method proposed by Weeks⁽⁷⁾.

3. Experiment

Some experiments are carried out to investigate the validity of the theoretical analysis. Figure 2 shows the whole experimental system. The flexible arm is composed of an aluminum beam of length L , width B and thickness H and a cylindrical payload fixed at its free end. The arm is directly attached to the vertical

motor shaft to avoid the drive system backrush and, at the same time, to achieve quick response with modest energy. The tip displacement of the arm is first measured by the gap sensor, and then the voltage signal from the sensor is amplified by the preamplifier (maximum ± 5 V) and put into the A/D converter. After that the digitized signal is put into the micro-computer. The processed signal from the computer, on the other hand, is transformed to a voltage signal through the D/A converter. The voltage signal is then converted to a current signal through the power amplifier (1 output ampere to 1 input voltage) and finally fed back to the motor armature. The physical parameters of the system measured are

$$\begin{aligned}\rho &= 2.67 \times 10^{-3} \text{ kg/m}^3 \\ B &= 11.94 \times 10^{-3} \text{ m} \\ H &= 1.999 \times 10^{-3} \text{ m} \\ E &= 6.565 \times 10^{10} \text{ Pa} \\ C &= 0.119 \times 10^{-3} \text{ s} \\ M_p &= 45.25 \times 10^{-3} \text{ kg} \\ J_p &= 4.509 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \\ J_m &= 4.9 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \\ \varepsilon &= 2.604 \times 10^{-3} \text{ kg} \cdot \text{m}^2/\text{s} \\ K_t &= 6.7647 \times 10^{-2} \text{ N} \cdot \text{m/A} \\ K' &= 1.772 \times 10^{-5} \text{ V/rad} \\ L_a &= 1.8 \times 10^{-3} \text{ H} \\ R_a &= 4.7 \Omega\end{aligned}$$

The Young's modulus E and the damping coefficient C were determined from the period and the logarithmic decrement of a freely vibrating cantilever beam with a tip mass. The moment of inertia of cylindrical payload J_p was calculated from $J_p = M_p r^2/4$, where M_p and r are the mass and the radius of the payload. The damping coefficient ε and the torque constant of motor K_t were determined from the response curve of a modified system, where the flexible beam was replaced by a rigid beam.

It is known that the effects of the motor inductance and the back electromotive force can be disregarded in the circuit when the motor torque is controlled

directly by the armature current⁽⁸⁾. Hence, in the following theoretical results the response curves have been obtained by putting $L_a = K' = 0$.

The tip displacement of the arm is measured by the gapsensor and the signal digitized through the A/D converter is sent to the computer with sampling period Γ . In this experiment Γ was selected as 3 ms. The velocity of the arm tip was not measured in the experiment, so it was evaluated from the next equation:

Digitized velocity signal V_v at time $i = \Delta_w/\Gamma$,

where

Δ_w = digitized displacement signal at time i - digitized displacement signal at time $(i-1)$,

Γ = sampling time (3 ms).

In the experiment, however, to simplify the calculation, Δ_w was used directly as the velocity signal and the effect of the sampling time Γ was later considered in relation to the feedback gain. The digitized feedback signal, V' , in the computer is

$$V' = f_{gd}(V_{wd} - V_w) - f_{gv}V_v$$

where V_w is the digitized displacement signal at time i , V_{wd} the commanded value of V_w , V_v the above mentioned digitized velocity signal, and f_{gd} and f_{gv} are the displacement and velocity feedback gains in the computer, respectively. V' is transformed into a current signal via the D/A converter and the power amplifier and then supplied to the motor. The relation between the theoretical and the experimental feedback gains is given by

$$G_d = f_{gd} \times 657.9,$$

$$G_v = f_{gv} \times 657.9 \times T.$$

Here, 657.9 (V/m) is the total gain of the gapsensor and the preamplifier.

4. Theoretical and Experimental Results

The results that follow are for the commanded tip position of $W^*_d = 3.8 \times 10^{-3}$ m.

Figures 3(a) and (b) are the experimentally obtained tip displacement showing the effectiveness of the feedback control. Figure 3(a) is the displacement when the motor shaft was rotated by the amount of W^*_d/L in the form of a step function. As can be seen, the time decrement of the curve is rather slow and the arm tip continues to fluctuate around its commanded position. Figure 3(b) shows the tip displacement when the arm is controlled by a feedback signal. It is clear that the feedback control is quite effective in bringing the endpoint rapidly to its final position.

Figure 4 shows the time variations of tip displacement w and the base angle θ , which was obtained by differentiating Eq. (22) with respect to x and putting $x = 0$. In the figure, W^*_d is the desired tip displacement, and $\theta^*_d (= W^*_d/L)$ the desired rotation angle. The

Fig. 2 Experimental system

Fig. 3 Tip displacement to step input (experimental results). $L=0.553$ m, $G_d=131.6$ A/m, $G_v=2.763$ As/m. (a) Without feedback control; (b) with feedback control

Fig. 4 Time variations of tip displacement and base rotation to step input (theoretical results). $G_d=131.6$ A/m, $G_v=2.763$ As/m

arm length is varied from 0.2 m through 0.5 m. When $L=0.2$ m, the displacement curve is similar to that of the rotation, which means that the elastic deformation is not noticeable and the arm is vibrating essentially as a rigid body. For the case of $L=0.3$ m, the response curve of the rotation angle is perturbed by the second, higher mode. The effect of the higher mode becomes more significant as the length of the arm is increased. For the case of $L=0.5$ m, the two response curves are quite different from each other, which means the tip location cannot be estimated from the base angle, i. e., the output of the encoder rotating with the motor shaft no longer represents the endpoint displacement.

Tip displacements of two arms are shown in Figs. 5 through 8 with the displacement gain G_d and the velocity gain G_v taken as parameters. Figures 5 through 7 are the results for an arm of $L=0.346$ m, and Fig. 8 those for $L=0.553$ m. In each figure, the response curves in the top figure show the theoretical results while the curves in the bottom figure are experimental results. As can be seen from the figure, the displacement approaches the desired value more rapidly as G_v increases. It was found, however, that the second higher mode becomes excited more significantly as G_v increases, and in some cases the convergence is interrupted by the second mode as is evident from Fig. 8. This means that there is an optimum value of G_v to achieve high-speed control of the flexible arm. As for the effect of G_d , the period of the predominant first mode decreases and the arm tip comes to the final position more rapidly with an

Fig. 5 Tip displacement to step input. $L=0.346$ m, $G_d=131.6$ A/m. Curve 1, $G_v=1.184$ As/m; Curve 2, $G_v=1.974$ As/m; Curve 3, $G_v=2.763$ As/m

Fig. 6 Tip displacement to step input. $L=0.346$ m, $G_d=263.2$ A/m. Curve 1, $G_v=1.184$ As/m; Curve 2, $G_v=1.974$ As/m; Curve 3, $G_v=2.763$ As/m

Fig. 8 Tip displacement to step input. $L=0.553$ m, $G_d=394.7$ A/m. Curve 1, $G_v=1.974$ As/m; Curve 2, $G_v=2.763$ As/m

Fig. 7 Tip displacement to step input. $L=0.346$ m, $G_d=394.7$ A/m. Curve 1, $G_v=1.974$ As/m; Curve 2, $G_v=2.763$ As/m

increase of G_d . An excess value of G_d , however, promotes the overshoot of the displacement and it interferes with the rapid convergence of the displacement. Hence, it is also said there is an optimum value of G_d to realize quick response of the flexible arm. Comparison of the theoretical results with the experimental ones shows that the total waveform and the frequencies of both the first and second modes are in

good agreement between the two response curves, but the time decrement of the experimental result is rather slow as compared with that of the theoretical one. This comes from the fact that the experimental system has various kinds of friction and the motor torque related to the velocity feedback mainly yields to the friction torque. As a result, the theoretical decrement of displacement with time cannot be achieved in the experiment.

5. Conclusions

A theoretical and experimental study has been presented for the closed-loop displacement control of a one-link flexible arm with an end-point payload. Results obtained are summarized as follows.

(1) Theoretical results are in good correspondence with the experimental ones. This means the optimum control problem of flexible arms can be discussed based on the theoretical results.

(2) The Laplace transform method combined with the method of numerical inversion is a powerful, theoretical method of analysis in dealing with the flexible arm control.

(3) There are optimum values of G_d (displacement feedback gain) and G_v (velocity feedback gain) to achieve quick response of the flexible arm. Those values can be determined by the numerical analysis based on the computer simulation.

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