The Dynamics of a Rotor System with a Shaft Having a Slant Crack*

(A Qualitative Analysis Using a Simple Rotor Model)

Masayuki ICHIMONJI** and Shunzo WATANABE**

In this study, a qualitative analysis of a transverse vibration of a rotor system with a crack that grows at an angle of 45 degrees toward the axis of the shaft is presented. Based on the assumption that the bending stiffness of the shaft changes synchronously with the opening/closing behavior of the crack caused by the torsional vibration of the shaft, the equation of motion of a simple rotor system with a shaft having a slant crack is represented by a differential equation with parametric excitation in the coordinate system rotating at the operating speed of the rotor. It is shown by the solution that the steady-state response of the rotor system with a slant crack on its shaft induced by imbalance contains the frequencies represented by $m\Omega + n\omega_\tau/2$; $m=1, 2, \dots$, and $n=0, 1, 2, \dots$, where Ω is the operating speed of the rotor, and ω_τ is the frequency of torsional vibration of the rotor system.

Key Words: Cracked Rotor, Vibration, Slant Crack, Parametric Excitation

1. Introduction

An increased level of reliability has been demanded for such rotating machinery as steam turbine generators. Consequently, various studies on the abnormal behavior of rotating machinery have been undertaken and systems for monitoring failure in the rotating machinery have also been developed. It is very important for improving the reliability of the rotating machinery to understand the dynamics of the rotor system with a cracked shaft; if allowed to progress, the crack may cause the eventual destruction of the shaft, resulting in the worst possible damage to the rotating machinery.

Through the previous studies^{(1)~(3)}, many useful results have been obtained regarding the dynamics of the rotor system with a cracked shaft. However, these studies have treated the case where a crack grows transversely toward the axis of the rotor, and show

that integer multiples, especially two and three times of the operating speed, exist in the vibration of the rotor. These frequencies are caused by the change in the stiffness of the shaft according to how opened or closed the crack is by the deflection of the shaft. This characteristic is applied to the monitoring system for detecting cracks in the rotating machinery.

The transversal crack of the shaft, as mentioned above, occurs from the fatigue of the shaft material due to an excessive bending moment. On the other hand, there exists a case where the crack grows at an angle of 45 degrees toward the axis on the surface of the shaft, occurring from the fatigue of the shaft due to the torsional moment. The stiffness of the shaft is thought to be closely related with the opening/closing behavior of the crack depending on the torsional vibration of the rotor system; hence, the rotor system with a slant crack occurring from torsional vibration may induce a different vibration from that of one with a transversal crack. Recently, use of the large rotating machines driven under operating systems which are apt to induce torsional vibration, such as daily start-and-stop operations, or thyristor motor driving, has increased. This requires the operators to monitor the

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** Heavy Apparatus Engineering Laboratory, Toshiba Corporation, 2-4 Suehiro-cho, Tsurumi-ku, Yokohama, 230, Japan

torsional vibration of the rotor, and to detect slant as well as transversal cracks. In this study, the authors investigate qualitatively the transverse vibration of the rotor system with a slant crack on its shaft by considering the opening/closing behavior of the crack in terms of the torsional vibration.

2. Theory

It is thought that slant cracks change the stiffness of the shaft just as transversal cracks do. However, the slant crack occurs from the torsional vibration of the shaft, therefore the opening/closing behavior of the crack depends on torsional vibration. Then, we investigate, first, the relationship between the stiffness of the shaft with a slant crack and the torsional vibration of the shaft. After this investigation, applying the characteristic of the shaft with a slant crack to a simple rotor model, the transverse vibration of a rotor system with a slant crack is analyzed qualitatively.

2.1 The torsional vibration and the stiffness of the shaft

Torsional moment M_z applied to a shaft causes on its surface the primary stresses σ_1 and σ_2 lying orthogonally to each other at an angle of 45 degrees toward the axis of the shaft as shown in Fig. 1, so that the primary stresses shape the spiral lines on the surface of the shaft. Applying a torsional vibratory moment to the shaft under the conditions of smaller amplitude over a longer period of time, a slant crack grows along the primary stress line as shown in Fig. 2.

The positive torsional moment shown in Fig. 2(a) causes tension in such a way as to open the crack, on the other hand the negative torsional moment shown in Fig. 2(b) causes compression in a such a way as to close the crack. The moment of inertia of this section of the shaft will be higher when the negative moment is applied, as opposed to when the positive one is applied. As the stiffness of the shaft is proportional to the inertia of the section of the shaft, we expect that the stiffness of the shaft with the slant crack changes according to the opening/closing condition of the crack. As the crack continues opening and closing alternatively with the same period as the torsional vibration of the shaft, the stiffness of the

Fig. 1 The primary stress directions of a shaft under torsional moment

shaft will change synchronously with the torsional vibration. On the assumption that the change in the stiffness is proportional to the torsional deflection angle of the shaft, the stiffness of the shaft is represented as follows (see Fig. 3):

$$k = k_0 - \frac{\Delta k}{2}(1 + \cos \omega_r t) \quad (1)$$

where k_0 and $k_0 - \Delta k$ are the stiffness values of the shaft when the crack closes and opens, respectively, ω_r is the circular frequency of the torsional vibration, and t is time.

2.2 The equation of motion of the cracked rotor

A simple rotor model⁽⁴⁾ is considered consisting of a rigid disk and a shaft with a slant crack supported by two rigid bearings, as shown in Fig. 4, using fixed coordinate system $(x, y-z)$ and a rotating coordinate system $(\zeta, \eta-z)$. The z -axis originates from the disk along the line passing through the centers of each bearing, and the x - and y -axes lie horizontally and vertically, respectively. The rotating coordinate system $(\zeta, \eta-z)$ rotates around the z -axis at the speed

Fig. 2 The opening/closing behavior of the slant crack

Fig. 3 Bending stiffness of the shaft and torsional vibration

Fig. 4 The simple rotor model with a shaft having a slant crack

as same as the operating speed Ω of the rotor. The equation of motion in the rotor is expressed in the fixed coordinate system as

$$\begin{aligned} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \frac{k_\zeta - k_\eta}{2m} \begin{bmatrix} -\cos 2\Omega t & -\sin 2\Omega t \\ -\sin 2\Omega t & \cos 2\Omega t \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \\ + \frac{k_\zeta + k_\eta}{2m} \begin{Bmatrix} x \\ y \end{Bmatrix} = \varepsilon \Omega^2 \begin{Bmatrix} \cos(\Omega t + \varphi) \\ \sin(\Omega t + \varphi) \end{Bmatrix} + \begin{Bmatrix} 0 \\ g \end{Bmatrix} \quad (2) \end{aligned}$$

where m is the mass of the disk, k_ζ and k_η are the stiffness of the shaft in the ζ and η directions, respectively, ε is the eccentricity of the center of the gravity, G , of the disk, ϕ is the phase angle of G , and g is the acceleration of gravity. The relationship between the rotating and fixed coordinate systems is

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos \Omega t & -\sin \Omega t \\ \sin \Omega t & \cos \Omega t \end{bmatrix} \begin{Bmatrix} \zeta \\ \eta \end{Bmatrix} \quad (3)$$

Substituting Eq. (3) into Eq. (2) leads to

$$\begin{aligned} \begin{Bmatrix} \ddot{\zeta} \\ \ddot{\eta} \end{Bmatrix} + \begin{bmatrix} 0 & -2\Omega^2 \\ 2\Omega^2 & 0 \end{bmatrix} \begin{Bmatrix} \zeta \\ \eta \end{Bmatrix} \\ + \begin{bmatrix} k_\zeta/m - \Omega^2 & 0 \\ 0 & k_\eta/m - \Omega^2 \end{bmatrix} \begin{Bmatrix} \zeta \\ \eta \end{Bmatrix} \\ = \varepsilon \Omega^2 \begin{Bmatrix} \cos \varphi \\ \sin \varphi \end{Bmatrix} - g \begin{Bmatrix} \sin \Omega t \\ \cos \Omega t \end{Bmatrix} \quad (4) \end{aligned}$$

Assuming k_ζ and k_η take the form of Eq. (1), they can be written as follows:

$$\begin{aligned} k_\zeta &= k_0 - \frac{\Delta k_\zeta}{2} (1 + \cos(\omega_\tau t + \theta_\zeta)) \\ k_\eta &= k_0 - \frac{\Delta k_\eta}{2} (1 + \cos(\omega_\tau t + \theta_\eta)) \end{aligned} \quad (5)$$

where θ_ζ and θ_η are the respective phases of bending stiffness, but are assumed to be negligible. Hence, we consider the case of

$$\theta_\zeta = \theta_\eta = 0 \quad (6)$$

and introduce the following parameters:

$$\omega_0 = \sqrt{k_0/m} \quad (7)$$

$$\mu = \frac{\Delta k_\zeta - \Delta k_\eta}{\Delta k_\zeta + \Delta k_\eta} \quad (8)$$

$$\nu = \frac{\Delta k_\zeta + \Delta k_\eta}{2k_0} \quad (9)$$

where ω_0 is the circular natural frequency of the transverse vibration of the rotor when the crack closes, μ is the anisotropic ratio of the shaft, and ν is the crack depth ratio. Substituting Eqs. (5)–(9) into Eq. (4), and rearranging, we obtain

$$\begin{aligned} \begin{Bmatrix} \ddot{\zeta} \\ \ddot{\eta} \end{Bmatrix} + \begin{bmatrix} 0 & -2\Omega^2 \\ 2\Omega^2 & 0 \end{bmatrix} \begin{Bmatrix} \zeta \\ \eta \end{Bmatrix} + (\omega_0^2 - \Omega^2) \begin{Bmatrix} \zeta \\ \eta \end{Bmatrix} \\ - \nu \omega_0^2 \begin{bmatrix} 1-\mu & 0 \\ 0 & 1+\mu \end{bmatrix} \begin{Bmatrix} \zeta \\ \eta \end{Bmatrix} - \nu \omega_0^2 \cos \omega_\tau t \begin{bmatrix} 1-\mu & 0 \\ 0 & 1+\mu \end{bmatrix} \begin{Bmatrix} \zeta \\ \eta \end{Bmatrix} \\ = \varepsilon \Omega^2 \begin{Bmatrix} \cos \varphi \\ \sin \varphi \end{Bmatrix} - g \begin{Bmatrix} \sin \Omega t \\ \cos \Omega t \end{Bmatrix} \quad (10) \end{aligned}$$

or

$$\ddot{\xi} + H \dot{\xi} + \{(\omega_0^2 - \Omega^2)I - \nu \omega_0^2 A\} \xi - \nu \omega_0^2 \cos \omega_\tau t \cdot A \xi = f \quad (11)$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 0 & -2\Omega \\ 2\Omega & 0 \end{bmatrix}, A = \begin{bmatrix} 1-\mu & 0 \\ 0 & 1+\mu \end{bmatrix},$$

$$\xi = \begin{Bmatrix} \zeta \\ \eta \end{Bmatrix}, f = \varepsilon \Omega^2 \begin{Bmatrix} \cos \varphi \\ \sin \varphi \end{Bmatrix} - g \begin{Bmatrix} \sin \Omega t \\ \cos \Omega t \end{Bmatrix}$$

2.3 The transverse vibration of the rotor with a slant crack^{(5),(6)}

Since Eq. (11) includes parametric excitation, there exists a periodic solution which can be obtained by setting the right-hand side equal to zero:

$$\ddot{\xi} + H \dot{\xi} + \{(\omega_0^2 - \Omega^2)I - \nu \omega_0^2 A\} \xi - \nu \omega_0^2 \cos \omega_\tau t \cdot A \xi = 0 \quad (12)$$

The solution of Eq. (12) is of the form

$$\xi = e^{\lambda t} \sum_{n=-\infty}^{+\infty} \hat{\xi}_n e^{jn \cdot \omega_\tau / 2 \cdot t} \quad (13)$$

Substituting Eq. (13) into Eq. (12) leads to

$$\begin{aligned} e^{\lambda t} \sum_{n=-\infty}^{+\infty} \left\{ (\Lambda_n^2 I + \Lambda_n H + B) \hat{\xi}_n - \frac{\nu \omega_0^2}{2} A \hat{\xi}_{n-2} \right. \\ \left. - \frac{\nu \omega_0^2}{2} A \hat{\xi}_{n+2} \right\} e^{jn \cdot \omega_\tau / 2 \cdot t} = 0 \quad (14) \end{aligned}$$

where

$$\Lambda_n = \lambda + j \cdot n \frac{\omega_\tau}{2}$$

$$B = (\omega_0^2 - \Omega^2)I - \nu \omega_0^2 A$$

Equation (14) is equivalent to the following pair of equations:

$$\begin{bmatrix} \Lambda_{-2}^2 I + \Lambda_{-2} H + B & -\frac{\nu \omega_0^2}{2} A & & \\ -\frac{\nu \omega_0^2}{2} A & \Lambda_0^2 I + \Lambda_0 H + B & -\frac{\nu \omega_0^2}{2} A & \\ & -\frac{\nu \omega_0^2}{2} A & \Lambda_2^2 I + \Lambda_2 H + B & \\ & & & \ddots \end{bmatrix} \begin{Bmatrix} \hat{\xi}_{-2} \\ \hat{\xi}_0 \\ \hat{\xi}_2 \\ \vdots \end{Bmatrix} = 0 \quad (15)$$

$$\begin{bmatrix} \ddots & & & \\ & -\frac{\nu\omega_0^2}{2}\mathbf{A} & \Lambda_{-1}^2\mathbf{I} + \Lambda_{-1}\mathbf{H} + \mathbf{B} & -\frac{\nu\omega_0^2}{2}\mathbf{A} \\ & & & \\ & -\frac{\nu\omega_0^2}{2}\mathbf{A} & \Lambda_{+1}^2\mathbf{I} + \Lambda_{+1}\mathbf{H} + \mathbf{B} & -\frac{\nu\omega_0^2}{2}\mathbf{A} \\ & & & \ddots \end{bmatrix} \begin{Bmatrix} \vdots \\ \hat{\xi}_{-1} \\ \vdots \\ \hat{\xi}_{+1} \\ \vdots \end{Bmatrix} = 0 \quad (16)$$

Non trivial solutions are possible when

$$\begin{vmatrix} \ddots & & & \\ & \Lambda_{-2}^2\mathbf{I} + \Lambda_{-2}\mathbf{H} + \mathbf{B} & -\frac{\nu\omega_0^2}{2}\mathbf{A} & \\ & -\frac{\nu\omega_0^2}{2}\mathbf{A} & \Lambda_0^2\mathbf{I} + \Lambda_0\mathbf{H} + \mathbf{B} & -\frac{\nu\omega_0^2}{2}\mathbf{A} \\ & & -\frac{\nu\omega_0^2}{2}\mathbf{A} & \Lambda_{+2}^2\mathbf{I} + \Lambda_{+2}\mathbf{H} + \mathbf{B} \\ & & & \ddots \end{vmatrix} = 0 \quad (17)$$

$$\begin{vmatrix} \ddots & & & \\ & -\frac{\nu\omega_0^2}{2}\mathbf{A} & \Lambda_{-1}^2\mathbf{I} + \Lambda_{-1}\mathbf{H} + \mathbf{B} & -\frac{\nu\omega_0^2}{2}\mathbf{A} \\ & & & \\ & -\frac{\nu\omega_0^2}{2}\mathbf{A} & \Lambda_{+1}^2\mathbf{I} + \Lambda_{+1}\mathbf{H} + \mathbf{B} & -\frac{\nu\omega_0^2}{2}\mathbf{A} \\ & & & \ddots \end{vmatrix} = 0 \quad (18)$$

Equations (17) and (18) both have infinite solutions.

Introducing the notations

$$\lambda = \lambda_r + j\lambda_i \quad (19)$$

$$\hat{\xi}_n = \begin{Bmatrix} \hat{\xi}_n \\ \hat{\eta}_n \end{Bmatrix} \quad (20)$$

and using Eq.(3), Eq.(13) is transformed to the fixed coordinate system;

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{cases} 2e^{\lambda_r t} \sum_{n=-\infty}^{+\infty} \text{Re}[(\hat{\xi}_n + j\hat{\eta}_n)e^{j(\lambda_i + \Omega + n\omega_r/2)t}] \\ 2e^{\lambda_r t} \sum_{n=-\infty}^{+\infty} \text{Re}[(\hat{\eta}_n - j\hat{\xi}_n)e^{j(\lambda_i + \Omega + n\omega_r/2)t}] \end{cases} \quad (21)$$

Equation (21) shows that the transient response of the rotor with a slant crack contains the frequencies represented by

$$\omega_n = \lambda_i + \Omega \pm n\frac{\omega_r}{2}; n=0, 1, 2, \dots \quad (22)$$

and that the sign of λ_r determines the stability of the system;

$$\begin{aligned} \lambda_r > 0 & \dots\dots \text{unstable} \\ \lambda_r \leq 0 & \dots\dots \text{stable} \end{aligned} \quad (23)$$

On the other hand, the steady-state response of the cracked rotor system will be obtained by neglecting the gravitational term of the right-hand set in Eq.(11), and assuming the solution as

$$\hat{\xi} = \sum_{n=-\infty}^{+\infty} \hat{\xi}_n e^{jn\omega_r/2 \cdot t} \quad (24)$$

Then, the following pair of equations are obtained:

$$\begin{bmatrix} \ddots & & & \\ & -\omega_r^2\mathbf{I} - j\omega_r\mathbf{H} + \mathbf{B} & -\frac{\nu\omega_0^2}{2}\mathbf{A} & \\ & -\frac{\nu\omega_0^2}{2}\mathbf{A} & \mathbf{B} & -\frac{\nu\omega_0^2}{2}\mathbf{A} \\ & & -\frac{\nu\omega_0^2}{2}\mathbf{A} & -\omega_r^2\mathbf{I} + j\omega_r\mathbf{H} + \mathbf{B} \\ & & & \ddots \end{bmatrix} \begin{Bmatrix} \vdots \\ \hat{\xi}_{-2} \\ \vdots \\ \hat{\xi}_0 \\ \vdots \\ \hat{\xi}_{+2} \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \vdots \\ 0 \\ \vdots \\ \varepsilon\Omega^2 \\ \vdots \\ 0 \\ \vdots \end{Bmatrix} \quad (25)$$

$$\begin{bmatrix} \ddots & & & \\ & -\frac{\nu\omega_0^2}{2}\mathbf{A} & -\left(\frac{\omega_r}{2}\right)^2\mathbf{I} - j\left(\frac{\omega_r}{2}\right)\mathbf{H} + \mathbf{B} & -\frac{\nu\omega_0^2}{2}\mathbf{A} \\ & & & \\ & -\frac{\nu\omega_0^2}{2}\mathbf{A} & -\left(\frac{\omega_r}{2}\right)^2\mathbf{I} + j\left(\frac{\omega_r}{2}\right)\mathbf{H} + \mathbf{B} & -\frac{\nu\omega_0^2}{2}\mathbf{A} \\ & & & \ddots \end{bmatrix} \begin{Bmatrix} \vdots \\ \hat{\xi}_{-1} \\ \vdots \\ \hat{\xi}_{+1} \\ \vdots \end{Bmatrix} = 0 \quad (26)$$

Table 1 The Numerical Values of the model used in the calculation

natural frequency of the normal rotor ω_0	1005[rad/s] (160[Hz])
frequency of torsional vibration ω_T	157[rad/s] (25[Hz])
operation speed of the rotor Ω	377[rad/s] (60[Hz])
eccentricity of the disk e	1.0 [mm]
gravitational acceleration g	9800.0 [mm]
external damping ratio γ	0.01

Fig. 5 The steady-state response of rotor with isotropic cracked shaft

centered at the operation speed Ω . However, in the numerical results shown in Figs. 5 and 6, we can determine the frequencies represented by

$$\omega_n = \Omega \pm n\omega_r : n=0, 1, 2, \dots \quad (31)$$

They satisfy the even terms of n in Eq. (28), but not the odd terms. As the excitation force in Eq. (26) is equal to zero, the solutions are represented by

$$\hat{\xi}_n = 0 : n=\dots, -3, -1, 1, 3, \dots \quad (32)$$

The magnitude of vibration becomes larger with an increase in ν , on the other hand the ratios of the magnitude of the subharmonic vibrations to that of the synchronous vibration also become larger. The system becomes unstable when the amplitude of vibration continues to increase as ν approaches 0.5, but returns to a stable state by means of damping.

The anisotropy of the stiffness of the shaft also induces the subharmonic vibrations. Increasing μ , a frequency twice of the operating speed Ω is produced in the vibration accompanying the subharmonic frequencies at intervals of ω_r (see Fig. 6). In Eq. (11), the steady-state response of the gravitational force is of the form

$$\xi = e^{j\Omega t} \sum_{n=-\infty}^{+\infty} \hat{\xi}_n e^{jn \cdot \omega_r/2 \cdot t} \quad (33)$$

Consequently, the frequencies in the fixed coordinate system are represented by

$$\omega_n = 2\Omega \pm n\frac{\omega_r}{2} : n=0, 1, 2, \dots \quad (34)$$

However, the even terms of n in Eq. (34) only appear in the numerical results and the odd terms are missing as well as the steady-state response of an imbalance. The system becomes unstable when μ approaches 0.45 for $\nu=0.3$.

Conclusions

The transverse vibration of the rotor system with a shaft having a slant crack has been investigated qualitatively using a simple rotor model under the assumption that the bending stiffness of the shaft changes synchronously with the opening/closing behavior of the crack caused by the torsional vibration, and the following conclusions have been derived.

(1) It is assumed that the slant crack continues to open and close with the frequency ω_r of the torsional vibration, so that the bending stiffness of the shaft will oscillate synchronously with the torsional vibration. The motion of the simple rotor system with a slant crack is represented by a differential equation

with parametric excitation in the coordinate system rotating at the operating speed Ω of the rotor as follows:

$$\ddot{\xi} + H\dot{\xi} + \{(\omega_0^2 - \Omega^2)I - \nu\omega_0^2 A\}\xi - \nu\omega_0^2 \cos \omega_r t A \xi = f$$

(2) This equation has a periodic solution represented by

$$\xi = e^{(\lambda_r + j\Omega)t} \sum_{n=-\infty}^{+\infty} \hat{\xi}_n e^{jn \cdot \omega_r/2 \cdot t}$$

Consequently, the transient vibration of the cracked rotor system contains the frequencies in the fixed coordinate system of the form

$$\omega_n = \lambda_i + \Omega \pm n\frac{\omega_r}{2} : n=1, 2, \dots$$

(3) The frequencies of the steady-state response of the cracked rotor system in the fixed coordinate system are represented by

$$\omega_{m,n} = m\Omega \pm n\frac{\omega_r}{2} : m=1, 2, n=0, 1, 2, \dots$$

However, only even terms of n appear in the numerical results and odd terms don't appear.

(4) Increasing the depth of the crack or the anisotropy of the shaft by the crack, the system becomes unstable. However, the damping is effective in returning the system to a stable state.

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