Design of a Multivariable Servomechanism and Realization of System Type $l_j, j=1, 2, \dots, m$ by Using Sliding-Mode Control*

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A method for designing a servomechanism for a nonlinear multivariable system by sliding-mode control is proposed. The derived servomechanism is completely decoupled during sliding motion, so that a set of servomechanisms with single input-single output is obtained. The input-output relations are given by transfer functions of which the poles are arbitrarily assigned. Servomechanisms with an arbitrary system type can also be realized. The advantageous properties of sliding-mode control, such as insensitivity to parameter variations and disturbances, are preserved in this servomechanism. An example is given in order to illustrate the design procedure.

Key Words: Control Theory, Control System Design, Servomechanism, Sliding-Mode Control, Variable Structure System

1. Introduction

We have proposed a basic method⁽¹⁾ for designing a servomechanism for a linear single input-single output system by using sliding-mode control⁽²⁾⁻⁽⁴⁾. This control method does not require a tracking model, being totally different from the methods⁽⁵⁾⁻⁽⁸⁾ which achieve output tracking by a model-following structure. We can easily realize a system type in this servomechanism corresponding to reference signals. Furthermore, it has been shown that insensitivity⁽⁹⁾ with respect to parameter variations and disturbances, which is one advantageous property in sliding-mode control, is preserved.

In this paper, we show that a sliding-mode servomechanism can be constructed for a nonlinear multivariable system in which there are internal interactions. We also propose a design method for a servomechanism with system type $l_j, j=1, 2, \dots, m$, where m denotes the input numbers.

2. Input-Output Relation

Consider the following time-varying system.

$$y_j^{(n_j)} = \sum_{i=1}^m A_{ji}(t) Y_i + f_j(Y_1, Y_2, \dots, Y_m; t)$$

$$+ b_j(t) u_j + d_j(t), j=1, 2, \dots, m \quad (1 \cdot a)$$

where u_j and $y_j, j=1, 2, \dots, m$ are the inputs and the outputs. $d_j(t)$ are the disturbances. Y_j and A_{ji} are

$$Y_j = [y_j \ y_j \dots y_j^{(n_j-1)}]^T, j=1, 2, \dots, m \quad (1 \cdot b)$$

$$A_{ji}(t) = [a_{ji1}(t) \ a_{ji2}(t) \dots a_{jin_j}(t)],$$

$$i, j=1, 2, \dots, m \quad (1 \cdot c)$$

which denote the coefficients of linear terms. The varying ranges of a_{jip} and b_j are assumed to be known together with the maximum values of $|d_j|, j=1, 2, \dots, m$. We also assume without loss of generality that $b_j(t) > 0, j=1, 2, \dots, m$. The nonlinear terms f_j are assumed to be polynomially bounded.

System (1) can be described as a state space form

$$\dot{x}_{ji} = x_{j,i+1}, j=1, 2, \dots, n_j-1$$

$$\dot{x}_{j,n_j} = \sum_{i=1}^m A_{ji}(t) X_i + f_j(X_1, X_2, \dots, X_m; t)$$

$$+ b_j(t) u_j(t), j=1, 2, \dots, m \quad (2 \cdot a)$$

$$X_j = [x_{j1} \ x_{j2} \dots x_{jn_j}]^T \quad (2 \cdot b)$$

where $y_j = x_{j1}, j=1, 2, \dots, m$. Define

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$$X \triangleq (X_1^T \ X_2^T \cdots X_m^T)^T \quad (3 \cdot a)$$

$$|X| \triangleq (|X_1|^T |X_2|^T \cdots |X_m|^T)^T \quad (3 \cdot b)$$

where

$$|X_j| \triangleq (|X_{j1}| |X_{j2}| \cdots |X_{j,n_j}|)^T \quad (3 \cdot c)$$

Since f_j are polynomially bounded, there are polynomials $F_j(X; t)$, $j=1, 2, \dots, m$ such that

$$|f_j(X; t)| \leq F_j(|X|; t), \quad j=1, 2, \dots, m \quad (4)$$

We can always choose the coefficients of $F_j(X; t)$ to be continuously differentiable and positive functions of time.

We introduce the following first order systems which are connected to system (2).

$$\dot{x}_{j0} = a_{j0}x_{j0} + (x_{j1} - v_j), \quad j=1, 2, \dots, m \quad (5)$$

where x_{j0} , $j=1, 2, \dots, m$ are the state variables and v_j , $j=1, 2, \dots, m$ the inputs of these first order systems. a_{j0} , $j=1, 2, \dots, m$ are constant gains which are considered later. Equation (5) represents a set of first order lag systems to which the inputs are the differences between the system outputs and v_j , $j=1, 2, \dots, m$.

Let the equation of switching hyperplane in sliding mode be

$$\sigma_j = \sum_{i=0}^{n_j} c_{ji} x_{ji}, \quad j=1, 2, \dots, m \quad (6)$$

where c_{ji} , $i=1, 2, \dots, n_j$, $j=1, 2, \dots, m$ are constants. In particular, c_j , $n_j=1$. Equation (6) can be written as follows

$$\sigma_j = \sum_{i=0}^{n_j} (c_{ji} - a_{j0}c_{j,i+1})x_{j0}^{(i)} + \sum_{i=1}^{n_j} c_{vi}u_j^{(i-1)}, \quad j=1, 2, \dots, m \quad (7)$$

from Eqs. (2) and (5), where $c_{j,n_j+1}=0$. Therefore, if $\sigma_j=0$, $j=1, 2, \dots, m$ are attained by sliding-mode control, the relation $v_j \rightarrow x_{j0}$ can be obtained as follows.

$$x_{j0}(s) = \frac{\sum_{i=0}^{n_j} c_{ji} s^{i-1}}{\sum_{i=0}^{n_j} (c_{ji} - a_{j0}c_{j,i+1})s^i} v_j(s) \quad (8)$$

where s denotes the Laplace-transformation variable. Since

$$x_{j1}(s) = (s - a_{j0})x_{j0}(s) + v_j(s) \quad (9)$$

from Eq. (5),

$$y_j(s) = \frac{C_0}{\sum_{i=0}^{n_j} (c_{ji} - a_{j0}c_{j,i+1})s^i} v_j(s), \quad j=1, 2, \dots, m \quad (10)$$

are derived from Eqs. (8), (9) and $y_j(s) = x_{j1}(s)$. Consequently, it is found that when $\sigma_j=0$, $j=1, 2, \dots, m$ are obtained by sliding-mode control, the controlled system is decoupled, then y_j can be controlled by only v_j . The poles of the transfer functions (10) which determine the relations $v_j \rightarrow y_j$, $j=1, 2, \dots, m$ can be arbitrarily assigned by suitable design of the coefficients of the switching hyperplane equation (6). In this case, a_{j0} , $j=1, 2, \dots, m$ can be chosen to be arbitrary real numbers. Furthermore, since there is no system parameter or disturbance term in Eq. (10), we

can obtain a robust servomechanism being insensitive with respect to these parameters.

3. Derivation of Control Law

We investigate a control law which realizes the sliding mode. Differentiating σ_j , $j=1, 2, \dots, m$ along a solution of the augmented system given by Eqs. (2) and (5), we obtain

$$\begin{aligned} \dot{\sigma}_j &= c_{j0}a_{j0}x_{j0} + \sum_{i=1}^{n_j} c_{ji}x_{ji} + \sum_{i=1}^{n_j} \sum_{p=1}^{n_j} a_{jip}x_{ip} \\ &\quad + f_j(X; t) + b_j u_j - c_{j0}v_j + d_j \end{aligned} \quad (11)$$

The k -th term of $F_j(X; t)$ can be described as

$$F_{jk} = a_{jk}(t) \prod_{i=1}^m \prod_{p=1}^{n_j} (x_{ip})^{\nu(i,p,j,k)}, \quad j=1, 2, \dots, m \quad (12)$$

as is explained in ref. (8), where $a_{jk}(t)$ are positive functions of time and $\nu(i,p,j,k)$ represents the power of x_{ip} in F_{jk} . We choose u_j as follows.

$$u_j = \Psi_{j0}x_{j0} + \sum_{i=1}^m \sum_{p=1}^{n_j} \Psi_{jip}x_{ip} + \sum_k u_{jk}(X; t) + \Phi_j, \quad j=1, 2, \dots, m \quad (13)$$

where Ψ_{j0} and Ψ_{jip} , $p=1, 2, \dots, n_j$, $i,j=1, 2, \dots, m$ are switched gains. $u_{jk}(X; t)$ are given by the same forms as Eq. (12), i.e.

$$u_{jk}(X; t) = \gamma_{jk}(X; t) \prod_{i=1}^m \prod_{p=1}^{n_j} (x_{ip})^{\nu(i,p,j,k)}, \quad j=1, 2, \dots, m \quad (14)$$

where $\gamma_{jk}(X; t)$ are switched gains.

The switched control algorithm is given as follows so that $\lim_{\sigma_j \rightarrow 0} \sigma_j \dot{\sigma}_j < 0$, $j=1, 2, \dots, m$ are always satisfied.

$$\begin{cases} \Psi_{j0} \leq \min_{b_j} \left(-\frac{c_{j0}a_{j0}}{b_j} \right) & \text{if } x_{j0}\sigma_j > 0 \\ \Psi_{j0} \geq \max_{b_j} \left(-\frac{c_{j0}a_{j0}}{b_j} \right) & \text{if } x_{j0}\sigma_j < 0 \end{cases} \quad (15)$$

$$\begin{cases} \Psi_{jip} \leq \min_{a_{jip}, b_j} \left(-\frac{c_{i,p-1}\delta_{ij} + a_{jip}}{b_j} \right) & \text{if } x_{ip}\sigma_j > 0 \\ \Psi_{jip} \geq \max_{a_{jip}, b_j} \left(-\frac{c_{i,p-1}\delta_{ij} + a_{jip}}{b_j} \right) & \text{if } x_{ip}\sigma_j < 0, \\ p=1, 2, \dots, n_i, i=1, 2, \dots, m \end{cases} \quad (16)$$

where δ_{ji} denotes the Kronecker's delta.

$$\begin{cases} \gamma_{jk}(X; t) \leq -\frac{a_{jk}(t)}{b_{j\min}} & \text{if } \prod_{i=1}^m \prod_{p=1}^{n_j} (x_{ip})^{\nu(i,p,j,k)} \sigma_j > 0 \\ \gamma_{jk}(X; t) \geq \frac{a_{jk}(t)}{b_{j\min}} & \text{if } \prod_{i=1}^m \prod_{p=1}^{n_j} (x_{ip})^{\nu(i,p,j,k)} \sigma_j < 0 \end{cases} \quad (17)$$

$$\begin{cases} \Phi_j = \frac{c_{j0}}{b_{j\max}} v_j - k_j \operatorname{sgn} \sigma_j & \text{if } c_{j0}v_j \sigma_j > 0 \\ \Phi_j = -\frac{c_{j0}}{b_{j\min}} v_j - k_j \operatorname{sgn} \sigma_j & \text{if } c_{j0}v_j \sigma_j < 0 \end{cases} \quad (18)$$

where

$$k_j > \max_{b_j, d_j} |d_j|/b_j \quad (19)$$

Notations $b_{j\min}$ and $b_{j\max}$, $j=1, 2, \dots, m$ represent the minimum and the maximum values of b_j . Equations (15) and (19) can easily be obtained by naturally

extending the results derived in ref.(1). The control law (17) guarantees that the following nonlinear terms in $\sigma_j \dot{\sigma}_j$ are always nonpositive.

$$\left\{ f_j(X; t) + b_j \sum_k \gamma_{jk} \prod_{i=1}^m \prod_{p=1}^{n_j} (x_{ip})^{\nu(i,p,j,k)} \right\} \sigma_j \quad (20)$$

In the case that there are terms in $f_j(X; t)$ which can be directly represented by Eq.(12) and the variation ranges of $\alpha_{jk}(t)$ can be known, $\gamma_{jk}(X; t)$ can be chosen as

$$\left\{ \begin{array}{l} \gamma_{jk}(X; t) \leq \min \left(-\frac{\alpha_{jk}(t)}{b_j} \right) \\ \quad \text{if } \prod_{i=1}^m \prod_{p=1}^{n_j} (x_{ip})^{\nu(i,p,j,k)} \sigma_j > 0 \\ \gamma_{jk}(X; t) \geq \max \left(-\frac{\alpha_{jk}(t)}{b_j} \right) \\ \quad \text{if } \prod_{i=1}^m \prod_{p=1}^{n_j} (x_{ip})^{\nu(i,p,j,k)} \sigma_j < 0 \end{array} \right. \quad (17)'$$

This result can be derived as the same way as the derivation of Eqs.(15) and (16). It is not always necessary that $\alpha_{jk}(t)$ are positive.

It is found that the control law shown above also satisfies the reaching condition which guarantees that a state trajectory always moves towards a switching hyperplane.

4. Realization of System Type $l_j, j=1, 2, \dots, m$

As discussed in chapter 2, the total system obtained by the control law proposed in this paper is completely decoupled. The input-output relations $v_j \rightarrow y_j$ are given by transfer functions(10). Therefore, a servomechanism with any system type can easily be realized by further compensation for a steady-state error. If we choose c_{j0} as

$$c_{j0} = a_{j0} c_{j1}, j=1, 2, \dots, m \quad (21)$$

in Eq.(10), then

$$y_j(s) = \frac{a_{j0} c_{j1}}{s \sum_{i=1}^{l_j} (c_{ji} - a_{j0} c_{j,i+1}) s^{i-1}} v_j(s), \quad j=1, 2, \dots, m \quad (22)$$

Since Eq.(22) is already system type 1, the following feedforward dynamic compensators of order l_j-1 are required in order to obtain a servomechanism with system type $l_j, j=1, 2, \dots, m$.

$$\begin{aligned} \dot{z}_{ji} &= z_{j,i+1}, i=1, 2, \dots, l_j-2 \\ \dot{z}_{j,l_j-1} &= r_j - y_j \\ v_j &= \sum_{i=1}^{l_j-1} g_{ji} z_{ji} + g_{j,l_j} (r_j - y_j), \\ j &= 1, 2, \dots, m \end{aligned} \quad (23)$$

where $q_{ji}, i=1, 2, \dots, l_j, j=1, 2, \dots, m$ are constant gains. The relations $r_j \rightarrow y_j, j=1, 2, \dots, m$ are given by

$$y_j(s) = \frac{a_{j0} c_{j1} \sum_{i=1}^{l_j} g_{ji} s^{i-1}}{s^{l_j} \sum_{i=1}^{n_j} (c_{ji} - a_{j0} c_{j,i+1}) s^{i-1} + a_{j0} c_{j1} \sum_{i=1}^{l_j} g_{ji} s^{i-1}} \times r_j(s) \quad (24)$$

which are obviously the transfer functions of a servo-

mechanism with system type l_j . The poles of Eq.(24) can be arbitrarily assigned. Assuming desirable characteristic equations

$$A_j(s) = \sum_{i=1}^{n_j+l_j-1} \mu_{ji} s^i, \text{ where } \mu_{n_j+l_j-1} = 1 \quad (25)$$

the following relations are derived.

$$\left\{ \begin{array}{l} c_{ji} = \mu_{j,l_j+i-1} + a_{j0} c_{j,i+1}, i=1, 2, \dots, n_j-1 \\ g_{ji} = \frac{1}{a_{j0} c_{j1}} \mu_{j,i-1}, i=1, 2, \dots, l_j \end{array} \right. \quad (26)$$

Therefore, since $c_{j,n_j}=1, c_{ji}$ can be determined for arbitrary nonzero a_{j0} . Furthermore, $g_{ji}, i=1, 2, \dots, l_j$ can easily be determined. In case that $c_{ji}=0$ is derived, it is necessary that the value of a_{j0} is altered since $c_{j1} \neq 0$ is required.

Consequently, a servomechanism with system type $l_j, j=1, 2, \dots, m$ can be derived according to the following design procedure. The values of $\mu_{ji}, j=1, 2, \dots, m$ are determined by giving the desirable system type l_j and the transfer functions of $r_j \rightarrow y_j$. The values of $c_{ji}, i=1, 2, \dots, n_j-1$ and $g_{ji}, i=1, 2, \dots, l_j$ are derived from Eq. (25) for arbitrarily fixed values of $a_{j0}, j=1, 2, \dots, m$. $c_{j0}, j=1, 2, \dots, m$ are also derived from Eq.(21). Therefore, the feedforward compensators(23) and switching hyperplane(6) are determined. The switched control law explained in chapter 3 is obtained in order that the trajectory moves towards the switching hyperplane and continue on it. When we choose the values of $a_{j0}, j=1, 2, \dots, m$, it is necessary to take account of the physical feasibility of the first order system(5).

5. Example

Consider the following system

$$\begin{aligned} \ddot{\theta}_1 &= 3\dot{\theta}_1 + \theta_2^2 + 2\theta_1 \dot{\theta}_2 \cos \theta_2 + u_1 \\ \ddot{\theta}_2 &= \theta_1^3 - (\cos \theta_1) \dot{\theta}_2 + u_2 \end{aligned} \quad (27)$$

Assume that θ_1 and θ_2 are required to track second order functions of time without steady-state error. Therefore, system type 3 is required for both cases. Assume that the denominators of the transfer functions of $r_1 \rightarrow y_1$ and $r_2 \rightarrow y_2$ are

$$\sum_{i=0}^4 \mu_{ji} s^i, i=1, 2 \quad (28)$$

where $\mu_{j4}=1, j=1, 2$.

Obviously, $m=2, n_1=n_2=2$ and $l_1=l_2=3$. Since $x_1 = [\theta_1 \dot{\theta}_1]^T$ and $x_2 = [\theta_2 \dot{\theta}_2]^T$,

$$\left. \begin{array}{l} c_{j0} = a_{j0}(\mu_{j3} + a_{j0}) \\ c_{j1} = \mu_{j3} + a_{j0} \\ c_{j2} = 1 \end{array} \right\}, j=1, 2 \quad (29)$$

and

$$\left. \begin{array}{l} g_{j1} = \mu_{j0}/a_{j0}(\mu_{j3} + a_{j0}) \\ g_{j2} = \mu_{j1}/a_{j0}(\mu_{j3} + a_{j0}) \\ g_{j3} = \mu_{j2}/a_{j0}(\mu_{j3} + a_{j0}) \end{array} \right\}, j=1, 2 \quad (30)$$

The switched gains are

$$\begin{cases} \Psi_{j0} \leq -a_{j0}^2(\mu_{j3} + a_{j0}) & \text{if } x_{j0}\sigma_j > 0 \\ \Psi_{j0} \geq -a_{j0}^2(\mu_{j3} + a_{j0}) & \text{if } x_{j0}\sigma_j < 0, \\ j=1, 2 \end{cases} \quad (31)$$

$$\begin{cases} \Psi_{111} \leq -a_{10}(\mu_{13} + a_{10}) - 3 & \text{if } x_{11}\sigma_1 > 0 \\ \Psi_{111} \geq -a_{10}(\mu_{13} + a_{10}) - 3 & \text{if } x_{11}\sigma_1 < 0 \end{cases} \quad (32)$$

$$\begin{cases} \Psi_{112} \leq -(\mu_{13} + a_{10}) & \text{if } x_{12}\sigma_1 > 0 \\ \Psi_{112} \geq -(\mu_{13} + a_{10}) & \text{if } x_{12}\sigma_1 < 0 \end{cases} \quad (33)$$

and $\Psi_{121} = \Psi_{122} = \Psi_{211} = \Psi_{212} = 0$. The similar results to Eqs.(32) and (33) are obtained for Ψ_{221} and Ψ_{222} . The nonlinear terms are

$$\begin{aligned} f_1(X; t) &= x_{21}^2 + 2x_{11}x_{22} \cos x_{21} \\ f_2(x; t) &= x_{11}^2 - (\cos x_{11})x_{21} \end{aligned} \quad (34)$$

By noting that $|\cos x_{11}| \leq 1$ and $|\cos x_{21}| \leq 1$, $u_{11} = \gamma_{11}x_{21}^2$, $u_{12} = \gamma_{12}x_{11}x_{22}$, $u_{21} = \gamma_{21}x_{11}^3$, $u_{22} = \gamma_{22}x_{21}$

From Eqs.(17) and (18), γ_{jk} , $k=1,2$ and Φ_j , $j=1,2$ are given by

$$\begin{cases} \gamma_{11} \leq -1 & \text{if } \sigma_1 > 0 \\ \gamma_{11} \geq -1 & \text{if } \sigma_1 < 0 \end{cases} \quad (35)$$

$$\begin{cases} \gamma_{12} \leq -2 & \text{if } x_{11}x_{22}\sigma_1 > 0 \\ \gamma_{12} \geq 2 & \text{if } x_{11}x_{22}\sigma_1 < 0 \end{cases} \quad (36)$$

$$\begin{cases} \gamma_{21} \leq -1 & \text{if } x_{11}\sigma_2 > 0 \\ \gamma_{21} \geq -1 & \text{if } x_{11}\sigma_2 < 0 \end{cases} \quad (37)$$

$$\begin{cases} \gamma_{22} \leq -1 & \text{if } x_{21}\sigma_2 > 0 \\ \gamma_{22} \geq 1 & \text{if } x_{21}\sigma_2 < 0 \end{cases} \quad (38)$$

and

$$\Phi_j = a_{j0}(\mu_{j3} + a_{j0})v_j + h_j, \quad j=1, 2 \quad (39)$$

since there is no disturbance term in Eq.(27) and $b_{j\min} = b_{j\max} = 1$, $j=1, 2$, where h_j , $j=1, 2$ are small positive numbers and

$$v_j = \sum_{i=1}^2 g_{ji}z_i + g_{i3}(r_j - y_j), \quad j=1, 2 \quad (40)$$

6 Conclusions

A method for designing a sliding-mode servomechanism for a nonlinear multivariable system has been proposed. The servomechanism can be completely decoupled so that the input-output relations are one to one and have desirable system types.

In addition, the poles of the transfer functions can be arbitrarily assigned. Even if there are internal interactions in a system which are described by nonlinear terms in the state space equation, linear input-output relations can be obtained by the control scheme proposed in this paper, if the nonlinear terms are polynomially bounded. The design method can widely be applied to various kinds of practical control problems since the proposed control law is as simple as a conventional sliding-mode control law.

References

- (1) Hikita, H. and Kawashima, Y., Design of Type I Servomechanisms Using Sliding-Mode Control, Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 54, No. 497, C (1988), p. 183.
- (2) Utkin, V. I., Variable Structure Systems with Sliding Mode, IEEE, Vol. AC-22, No. 2 (1977), p. 212.
- (3) Itkis, U., Control Systems of Variable Structure, (1976), John Wiley.
- (4) Harashima, F. and Hashimoto, H., Sliding Mode and Its Applications I, Systems and Control, (in Japanese), Vol. 29, No. 2 (1985), p. 94
- (5) Young, K. D., Asymptotic Stability of Model Reference Systems with Variable Structure Control, IEEE, Vol. AC-22, No. 2 (1977), p. 279.
- (6) Young, K. D., Design of Variable Structure Model-Following Control Systems, IEEE, Vol. AC-23, No. 6 (1978), p. 1079.
- (7) Zinober, A. S., El-Ghezawi, O. M. E. and Billings, S. A., Multivariable Variable-Structure Adaptive Model-Following Control Systems, IEE Proc., Vol. 129- Pt.D, No. 1 (1984), p. 6.
- (8) Slotine, J. J. and Sastry, S. S., Tracking Control of Nonlinear Systems Using Sliding Surfaces, with Application to Robot Manipulations, Int. J. Control, Vol.38, No. 2 (1983), p. 465.
- (9) Drzenovic, B., The Invariance Conditions in Variable Structure Systems, Automatica, Vol. 5, No. 3 (1969), p. 287.