Simulation System for the Design of Link Mechanisms*
(Dynamic Analysis of Mechanisms with Controllers)

Tadashi KUROIWA**, Hiroki TAKAHARA**
and Akira MOTOE**

The authors have developed a general-purpose simulation system for the design of link mechanisms. Topological and Kinematic analyses have already been referred to in the preceding paper. In this paper, methods for the dynamic analysis of link mechanisms with controllers, such as transient response and vibration analysis, are proposed. Both the characteristics of the control system and the equations of motion of the mechanical system are automatically and simultaneously transformed into nonlinear state equations. In transient response analysis, for example, the responses of joint displacements, etc., are calculated by the numerical integration of these equations. In vibration analysis, the state equations are linearized around the desired trajectory to calculate the eigenfrequency, mode shape, and transfer functions. The proposed methods were applied to an industrial robot, and the effectiveness of the methods was ascertained.

Key Words: Mechanism, Dynamics, Linkages, Robotics, Simulation, Transient Response, Eigenfrequency, Vibration, Transfer Function, Computer-Aided Engineering

1. Introduction

In recent years, many kinds of robots and manipulators have been used for various purposes. Thus, it is required that various specifications be satisfied for their design.

It is very important in the design of mechanisms to predict the performances of kinematics, statics, and dynamics which include control systems. Computer simulation has become especially effective for such a purpose. Moreover, computer-aided engineering (CAE) systems from modeling to assessment have been developed. It is possible to reduce the repetition of design and trial manufactures and to shorten the developing period, because many parameters concerning the mechanisms and control systems can be efficiently evaluated by such computer simulation. Various systems for industrial robots which can simulate the coupling between the mechanisms and control systems have been developed[10-16].

The authors have developed a general-purpose simulation system for the design of link mechanisms. Topological and kinematic analyses have already been referred to in the preceding paper[9]. In this paper, methods for dynamic analysis, such as transient response analysis and vibration analysis of link mechanisms with controllers, are proposed. The characteristics of the control system are defined in the form of transfer functions among such nodes as the feedback value, desired value, and controlled value, etc. Both the characteristics of the controllers and the equations of motion of the mechanical system are automatically and simultaneously transformed into nonlinear state equations. In transient response analysis, the transient responses are calculated by the numerical integration of these equations. In vibration analysis, the state equations are linearized around the desired trajectory to derive the linear perturbative equations. Then, the eigenfrequency, mode shape, and transfer function are calculated.

In addition, this system enables animated representations by a surface model to help visualize the
results of the simulation.

2. Equations of motion and control law

2.1 Equations of motion

The coordinates are defined as follows:
\[ \Sigma_w = \text{work space fixed Cartesian coordinates}, \]
\[ \Sigma_a = \text{local Cartesian coordinates for each link } L_m, \]
\[ \Sigma_{a,n,j} = \text{local Cartesian coordinates at each joint } J_j \]
\[ \Sigma_{cm} = \text{local Cartesian coordinates at each center of} \]
\[ \text{gravity } G_m \text{ of link } L_m, \text{ of which axes are parallel} \]
\[ \text{to } \Sigma_a. \]

The following transformation matrices represent a homogeneous transformation describing the relative translation and rotation between two coordinate systems as follows:

\[ T_m = \text{transformation matrix from } \Sigma_a \text{ to } \Sigma_w, \]
\[ B_{a,n,j} = \text{transformation matrix from } \Sigma_{a,n,j} \text{ to } \Sigma_a, \]
\[ P_j = \text{transformation matrix at joint } J_j. \]

In the case in which joints \( J_j \) (\( j = 1, 2, \ldots \)) are connected to a link \( L_m \), the equations of motion for link \( L_m \), which are each referenced in its own local coordinate system at the center of gravity \( \Sigma_{cm} \), are formulated as follows (refer to Fig. 1):

\[
\begin{align*}
\ddot{F}_m & = \sum_{j=1}^{n} \left( P_{a,n,j} (F_{a,n,j} + F_{c}) \right), \\
\ddot{M}_m & = \sum_{j=1}^{n} \left( M_{a,n,j} (\ddot{F}_{a,n,j} + \ddot{M}_{a,n,j} \right) \\
& + \ddot{g}_{cm} \times \left( P_{a,n,j} (F_{a,n,j} + F_{c}) \right), \\
\ddot{F}_{a,n,j} & = m_a T_m^{-1} \left( \ddot{F}_{a,n,j} - \ddot{g}_w \right) - \ddot{f}_{cm}. \\
\ddot{M}_{a,n,j} & = I_{cm} \ddot{\omega}_a + \ddot{\omega}_a \times (I_{cm} \ddot{\omega}_a) - \ddot{h}_{cm},
\end{align*}
\]

where \( m_a \) denotes the mass of link \( L_m \), \( I_{cm} \) denotes the inertia matrix of link \( L_m \) about \( \Sigma_{cm} \), and \( \ddot{g}_w \) denotes gravity with respect to \( \Sigma_w \). \( \ddot{f}_{cm} \) and \( \ddot{h}_{cm} \) respectively denote the external force and moment acting on \( \Sigma_{cm} \) with respect to \( \Sigma_{cm} \) which are equivalent to the sum of those acting on arbitrary points of link \( L_m \), \( \ddot{F}_{a,n,j}, \ddot{M}_{a,n,j}, \ddot{F}_{c} \), and \( \ddot{M}_{c} \) denote the force and moment acting on the spring of joint \( J_j \), and those acting on the damper of \( J_j \), respectively. \( \ddot{g}_{cm} \) denotes the acceleration of \( G_m \) with respect to \( \Sigma_w \), \( \ddot{\omega}_a \) and \( \ddot{\omega}_m \) denote the angular velocity and acceleration of link \( L_m \) with respect to \( \Sigma_m \). In the case in which joint \( J_j \) connects links \( L_i \) and \( L_k \), matrix \( Q_{a,i,j} \) and vector \( \ddot{g}_{cm,j} \) (\( j = 1, \ldots, h \)) are defined as follows:

\[
\begin{align*}
Q_{a,i,j} & = B_{a,i,j}, \\
Q_{a,i,j} & = -B_{a,i,j} P_j^{-1}, \\
\ddot{g}_{cm,j} & = Q_{a,i,j} (0, 0, 0, 1)^T - \ddot{f}_{cm}, \\
\ddot{h}_{cm,j} & = -Q_{a,i,j} (0, 0, 0, 1)^T - \ddot{h}_{cm}.
\end{align*}
\]

Matrices \( Q_{a,i,j} \) and \( T_m \) denote \( 3 \times 3 \) submatrices which represent rotational transformation in Eqs. (1) - (4).

The driving force (if joint \( J_j \) is a translational one) or torque (if \( J_j \) is a rotational one) \( \tau_j \) is the projection of \( \ddot{F}_{a,n,j} \) and \( \ddot{M}_{a,n,j} \) onto the axis of translation or rotation of \( J_j \), respectively.

In the case of a rotational joint

\[
\tau_j = \ddot{e}_j \ddot{M}_{a,n,j}
\]

In the case of a translational joint

\[
\tau_j = -\ddot{e}_j \ddot{F}_{a,n,j}
\]

where vector \( \ddot{e}_j \) denotes the direction of the axis of translation or rotation.

\[
\ddot{e}_j = (0, 0, 1)^T
\]

The driving force and/or torque vector \( \ddot{F} \) and the displacement vector \( \ddot{q} \) are defined as follows:

\[
\begin{align*}
\ddot{F} & = (\tau_1, \tau_2, \ldots, \tau_d)^T, \\
\ddot{q} & = (q_1, q_2, \ldots, q_d)^T,
\end{align*}
\]

where \( q_j \) denotes the displacement of joint \( J_j \) (\( j = 1, \ldots, d \)), and \( d \) denotes the number of joints with a driving device.

2.2 Control law

In this system, feedback control laws in the form of transfer functions are considered in analyzing mechanisms with various control systems. Control systems to drive and control a mechanism are usually constructed independently among the joints. Therefore, the feedback control laws are defined at each joint, as shown in Fig. 2. The displacement or velocity of the joint is considered the feedback value, the desired displacement or velocity of the joint is considered the desired value, and the driving force or torque is considered the control value.

The relationships among such nodes are defined in the form of transfer functions represented by the ratio of polynomial expressions of the Laplace operator \( G_m(s) \) (\( k = 1, 2, 3 \)).

![Fig. 1 Forces and moments acting on Link Lm](image1)

![Fig. 2 Transfer functions of controller](image2)
\[ G_m(s) = \frac{Y(s)}{U(s)} = d_m + \sum_{k=1}^{m} \frac{c_{mk} s^{k-1}}{s^{m-k} + \sum_{k=1}^{m} d_{mk} s^{k-1}} \]  

where suffix \( m \) denotes the number of joints, \( U(s) \) and \( Y(s) \) denote the Laplace transforms of input \( u(t) \) and output \( y(t) \), respectively, and \( n_a \) denotes the order of the denominator of the transfer function \( G_a(s) \). In the case in which \( n_a = 0 \), the value of the term including \( \sigma \) is regarded as zero.

3. Transient response analysis

In the dynamic analysis of a mechanism with controllers, the equations of motion and the feedback control laws mentioned in section 2 can be transformed to state equations.

3.1 State variables vector

In this paper, the vector of state variables is defined as follows:

(a) Vector of joint displacement:
\[ \mathbf{z}_1 = (z_{11}, z_{21}, \ldots, z_{16})^T \]
\[ = \mathbf{d} = (\varphi_1, \varphi_2, \ldots, \varphi_6)^T, \]

(b) Vector of joint velocity:
\[ \mathbf{z}_2 = (z_{11}, z_{21}, \ldots, z_{16})^T \]
\[ = \mathbf{d} = (\dot{\varphi}_1, \dot{\varphi}_2, \ldots, \dot{\varphi}_6)^T, \]

(c) Vector of state variables of the controller at a joint:
\[ \mathbf{z}_{2, m} = (z_{2, m, 1}, z_{2, m, 2}, \ldots, z_{2, m, n_a}), \]
\[ = \mathbf{d} = (\ddot{\varphi}_1, \ddot{\varphi}_2, \ldots, \ddot{\varphi}_6)^T, \]  

where \( z_{2, m}(t) \) denote the state variables of transfer matrices \( G_a(s) \) at joint \( J_m \) for \( m = 1, \ldots, \lambda_0 \).

3.2 Construction of state equations

3.2.1 Feedback control law

Transfer functions \( G_a(s) \) at joint \( J_m \) for \( m = 1, \ldots, \lambda_0 \) are transformed to state equations.

(a) Desired value

The desired displacement \( \varphi_m \) or the desired velocity \( \dot{\varphi}_m \) of a joint is considered the desired value \( u_{m}(t) \) of the controller.
\[ u_{m}(t) = \varphi_m \quad \text{or} \quad \dot{\varphi}_m \]  

(b) Feedback value

The displacement \( \varphi_m = z_{2, m} \) or velocity \( \dot{\varphi}_m = z_{2, m} \) of a joint is considered the feedback value \( u_{m}(t) \) of the controller.
\[ u_{m}(t) = z_{2, m} \quad \text{or} \quad z_{2, m} \]  

(c) Expression for transfer functions in the form of state equations

The transfer functions of a controller of Eq. (13) are transformed to state equations as follows:
\[ \frac{d}{dt} z_{a, m}(t) = z_{a, m}(t) \]
\[ (j = 1, \ldots, n_a = 1, \quad k = 1, 2, 3), \]
\[ \frac{d}{dt} z_{a, m}(t) = -\sum_{k=1}^{n_a} c_{ak} z_{a, k}(t) + u_{m}(t) \]  

3.2.2 Driving force and/or torque of a joint

The driving force or torque of joint \( J_m \) \( m = 1, \ldots, \lambda_0 \) which denotes the driving force and/or torque of joint \( J_m \) can be defined in two ways.

(a) In the case in which no controller is defined, \( r_m(t) \) is given as a time-variant function.

(b) In the case in which its controller is defined, \( r_m(t) \) is equivalent to the driving force and/or torque.
\[ r_m(t) = y_{m}(t) \]  

3.2.3 Equations of motion of a mechanism

The equations of motion of the whole mechanism which are transformed to state equations are derived by rearranging Eqs. (1) and (2) as follows:
\[ \frac{d}{dt} \mathbf{z}_1 = (A \mathbf{z}_1)'( \mathbf{f} - \mathbf{b} ), \]
\[ \frac{d}{dt} \mathbf{z}_2 = \mathbf{z}_1 \]  

where \( A(\mathbf{z}_2) \) denotes the inertia matrix, and the bias vector \( b(\lambda_0 \times \lambda_0) \) denotes other forces and moments which are composed of the centripetal force, Coriolis force, gravity, external force and moment, and the force and moment acting on the spring and damper. The bias vector \( b \) is calculated by force analysis as a case in which the joint acceleration \( \ddot{\varphi} \) is equal to zero. Calculation of the inertia matrix \( A(\mathbf{z}_2) \) is mentioned in the next section.

3.3 Calculation of the inertia matrix

The method proposed by Walker and Orin applies to the calculation of the inertia matrix \( A(\mathbf{z}_2) \).

The \( j \)-th column of the inertia matrix \( A(\mathbf{z}_2) \) is represented by a column vector \( \mathbf{a}_j \).
\[ A(\mathbf{z}_2) = [ \mathbf{a}_1, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_{\lambda_0} ] \]  

From Eqs. (14) and (24)-(26), vector \( \mathbf{a}_j \) is found to be equivalent to \( \mathbf{f} \), the value of \( \mathbf{f} \) in case \( \mathbf{b} \) and \( \ddot{\varphi} \) are set as follows:
\[ \ddot{\varphi} = \dot{\varphi} \]
\[ \ddot{\varphi} = \mathbf{a}_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T. \]  

Therefore, the inertia matrix \( A(\mathbf{z}_2) \) is calculated after \( \lambda_0 \) iterations of the above procedure from \( j = 1 \) to \( j = \lambda_0 \).

The following shows the method for calculating vector \( \mathbf{a}_j \). Equations removing the terms concerning velocity, gravity, external force and moment, and the force and moment acting on the spring from the equations of force analysis are used for this calcula-
tion. Suffix * shows the variables which are related to the calculation of $\mathbf{J}_m$. Consideration has been given in this paper to describing the algorithm simply so that a kinematic loop has $\lambda$ pieces of joints $J_m (m=1, \cdots, \lambda)$ which connect link $L_{m-1}$ and link $L_m$. Furthermore, link $L_0$ denotes the base link fixed to the inertial space or the branching link to which more than 2 joints are connected, and link $L_1$ denotes the end link or another branching link.

(a) Analyses of position and acceleration

The transformation matrix $T_n$ from $\Sigma_n$ to $\Sigma_m$, and $\dot{T}_n$ corresponding to acceleration are calculated by the following recursive equations, respectively.

$$
T_n = T_{n-1} B_{L_{m-1}/L_m} p_n B_{L_m/L_n}^{-1},
$$

(28)

$$
T_n^* = (T_{n-1}^* B_{L_{m-1}/L_m} p_n + T_{n-1} B_{L_{m-1}/L_m} p_n^* q_n) B_{L_m/L_n}^{-1},
$$

(29)

where

$$
P_n = \partial p_n/\partial q_n.
$$

(30)

In these equations, the values of $T_2$ and $\dot{T}_2$ are already known. In the case in which link $L_0$ is the base link, $T_0$ and $\dot{T}_0$ are equal to zero matrices.

Therefore, the acceleration $\ddot{\mathbf{r}}_{cm}$ of the center of gravity $G_m$ of link $L_m$ with respect to $\Sigma_m$ is calculated as follows:

$$
\ddot{\mathbf{r}}_{cm} = \dot{T}_n^* \ddot{\mathbf{r}}_{cm},
$$

(31)

where $\mathbf{r}_{cm}$ denotes the vector of position of $G_m$ with respect to $\Sigma_m$.

(b) Analysis of angular acceleration

Angular acceleration $\dot{\beta}_m^*$ of link $L_m$ with respect to $\Sigma_m$ is calculated as follows:

(In the case of a rotational joint)

$$
\dot{\beta}_m^* = T_{n-1}^* \dot{\beta}_{m-1}^* + B_{L_m/L_n} q_n \ddot{\mathbf{r}}_{cm},
$$

(32)

(In the case of a translational joint)

$$
\dot{\beta}_m^* = T_{n-1}^* \dot{\beta}_{m-1}^* + \ddot{\mathbf{r}}_{cm}.
$$

(33)

In these equations, the value of $\dot{\beta}_m^*$ is already known. In the case in which link $L_0$ is the base link, $\dot{\beta}_m^*$ is equal to a zero vector. Matrices $T_n$, $T_{n-1}$, and $B_{L_m/L_n}$ denote $3 \times 3$ submatrices which represent the rotational transformation in Eqs. (32) and (33).

(c) Equations of motion of a mechanism

From Eqs. (1) and (3), the equation for the translational motion of link $L_m$ is as follows:

$$
\ddot{\mathbf{r}}_{cm} = \sum_{j=1}^{n} (Q_{cm,j} \ddot{\mathbf{r}}_{cm,j}),
$$

(34)

where

$$
\ddot{\mathbf{r}}_{cm} = m_0 T_n \ddot{\mathbf{r}}_{cm}^*.
$$

(35)

From Eqs. (2) and (4), the equation for the rotational motion of link $L_m$ is as follows:

$$
\dot{\mathbf{M}}_m = \sum_{j=1}^{n} \left( (Q_{cm,j} \dot{\mathbf{M}}_{cm,j}) + \dot{\mathbf{r}}_{cm,j} \times (Q_{cm,j} \dot{\mathbf{r}}_{cm,j}) \right),
$$

(36)

where

$$
\dot{\mathbf{M}}_m = I_{cm} \dot{\beta}_m^*.
$$

(37)

Matrices $T_n$ and $Q_{cm,j}$ denote $3 \times 3$ submatrices which represent the rotational transformation in Eqs. (34)–(36). The force vector $\mathbf{F}_{cm}^*$ and moment vector $\mathbf{M}_{cm}^*$ of joint $J_m$ are calculated in the same manner as the force analysis.

(d) Driving force and/or torque of a joint

From Eqs. (9-a) and (9-b), the driving force or torque $\tau_m^*$ of joint $J_m$ is calculated as follows:

(In the case of a rotational joint)

$$
\tau_m^* = -d_\alpha \mathbf{M}_{cm}^*.
$$

(38)

(In the case of a translational joint)

$$
\tau_m^* = -d_\alpha \mathbf{F}_{cm}^*.\n$$

(39)

As already mentioned, the vector for the driving force and/or torque $\tau^*$ is calculated as follows:

$$
\tau^* = (\tau_1^*, \tau_2^*, \cdots, \tau_n^*)^T.
$$

(40)

3.4 Method of analysis

The state equations (19)–(25) in time transient analysis are numerically integrated by using the Runge–Kutta–Gill method, etc.

4. Vibration Analysis

4.1 Linearization of state equations

Because equations of motion (24) have strong nonlinearity, it is difficult to derive the eigenfrequency, mode shape and transfer functions directly. In vibration analysis, the perturbation around the desired trajectory is considered to derive the linearized state equations.

The state variables are represented as the sum of the desired trajectory $\mathbf{z}_0$ and the perturbation around it $\mathbf{z}_e$ $(e=1, 2, c)$ as follows:

$$
\mathbf{z} = \mathbf{z}_0 + \mathbf{z}_e (e=1, 2, c),
$$

(41)

where $\mathbf{z}_e$ denotes all state variables concerning the control systems.

$$
\dot{\mathbf{z}} = (\dot{\mathbf{z}}_1, \cdots, \dot{\mathbf{z}}_{2n})^T
$$

(42)

The following describes the method for linearization by substituting Eq. (41) for the state equations. As the state equations (19)–(22) and (25) concerning the control systems are linear, perturbative equations become the same as the primary ones.

4.1.1 Linearization of the equations of motion

First, Eq. (24) is substituted for Eq. (41). Next, the first-order terms of Taylor's expansion are taken up. Taking into consideration the fact that Eq. (24) is held at the desired trajectory, we derive the linearized state equations as follows:

$$
\dot{\mathbf{z}}_1 = A(\mathbf{z}_0) \left[ \frac{\partial \mathbf{z}}{\partial \mathbf{z}_0} \right]_{\mathbf{z}_0} \mathbf{z}_1
$$

$$
+ \left[ \frac{\partial \mathbf{z}}{\partial \mathbf{z}_0} \right]_{\mathbf{z}_0} \mathbf{z}_2 + \frac{\partial \mathbf{z}}{\partial \mathbf{z}_0} \mathbf{z}_0, \n$$

(43)

where $\mathbf{z}_0$ denotes the value at the desired trajectory. Vector $\mathbf{z}$ for the driving force and/or torque is either equal to zero in case there is no controller, or is easily derived from the derivatives of Eq. (23) in case controllers exist. Therefore, the derivatives of the bias.
vector are needed for our purpose.

4.1.2 Derivatives of the bias vector

The bias vector is calculated by force analysis as a case in which the joint acceleration \( \ddot{q} \) is equal to zero. \( b_m \) denotes the \( m \)-th component of the bias vector \( \ddot{b} \), and is calculated as follows:

\[
b_m = \Delta \alpha_m^f [(1 - \alpha_m^f) \ddot{F}_{in} + \alpha_m \ddot{M}_{in}],
\]

(44)

\[
\alpha_m =\begin{cases} 1 : \text{rotational joint} \\ 0 : \text{translational joint}. \end{cases}
\]

(45)

The derivatives of the bias vector are represented as follows:

\[
\frac{\partial b_m}{\partial z_e} = \Delta \alpha_m^f \left[ (1 - \alpha_m^f) \frac{\partial \ddot{F}_{in}}{\partial z_e} + \alpha_m \frac{\partial \ddot{M}_{in}}{\partial z_e} \right]
\]

\[\quad (e = 1, 2, c).\]

(46)

Therefore, the derivatives of the bias vector can be derived like the calculation of the bias vector itself from the derivatives of \( \ddot{F}_{in} \) and \( \ddot{M}_{in} \).

(a) Derivatives of the force and moment acting on a joint

The force and moment acting on joint \( J_m \), which are denoted as \( \ddot{F}_{in} \) and \( \ddot{M}_{in} \), respectively, are transformed from Eqs. (1) - (4). Their derivatives by state variables \( z_e \) (\( e = 1, 2, c \)) are calculated as follows:

\[
\frac{\partial \ddot{F}_{in}}{\partial z_e} = \Delta \alpha_m^f \left[ \frac{\partial \ddot{F}_{in}}{\partial z_e} - \frac{\partial \Delta \alpha_m^f}{\partial z_e} \right] - \Sigma_{j \in \mathbb{N}} \left[ \frac{\partial Q_{lmj}^f}{\partial z_e} \left( \ddot{F}_{in} + \fbox{\text{c}} \right) \right] + \frac{\partial Q_{cmj}^f}{\partial z_e} \left( \ddot{F}_{in} + \fbox{\text{c}} \right) + \frac{\partial Q_{lmj}^f}{\partial z_e} \left( \ddot{F}_{in} + \fbox{\text{c}} \right) + \frac{\partial Q_{cmj}^f}{\partial z_e} \left( \ddot{F}_{in} + \fbox{\text{c}} \right)
\]

(47)

\[
\frac{\partial \ddot{M}_{in}}{\partial z_e} = \Delta \alpha_m^f \left[ \frac{\partial \ddot{M}_{in}}{\partial z_e} - \frac{\partial \Delta \alpha_m^f}{\partial z_e} \right] - \Sigma_{j \in \mathbb{N}} \left[ \frac{\partial Q_{lmj}^f}{\partial z_e} \left( \ddot{F}_{in} + \fbox{\text{c}} \right) \right] + \frac{\partial Q_{cmj}^f}{\partial z_e} \left( \ddot{F}_{in} + \fbox{\text{c}} \right) + \frac{\partial Q_{lmj}^f}{\partial z_e} \left( \ddot{F}_{in} + \fbox{\text{c}} \right) + \frac{\partial Q_{cmj}^f}{\partial z_e} \left( \ddot{F}_{in} + \fbox{\text{c}} \right)
\]

(48)

The derivatives of vectors \( \ddot{F}_{in} \), \( \ddot{M}_{in} \), \( \ddot{F}_{cm}, \ddot{M}_{cm}, \ddot{F}_{cm}, \ddot{M}_{cm} \), and matrix \( Q_{cmj}^f \) in Eqs. (47) and (48) are easily calculated. If the derivatives of force \( \ddot{F}_{in} \) and moment \( \ddot{M}_{in} \) of the sum of the inertial and external forces and moments of link \( L_m \) are calculated, the derivatives of the force and moment acting on a joint are recursively calculated from the end to the base of a kinematic loop. Matrix \( Q_{cmj}^f \) denotes a \( 3 \times 3 \) submatrix which represents rotational transformation in Eqs. (47) and (48).

(b) Derivatives of the forces and moments of the inertial and external forces and moments

The derivatives of Eqs. (3) and (4) by state variables \( z_e \) (\( e = 1, 2, c \)) are calculated as follows, respectively:

\[
\frac{\partial \ddot{F}_{in}}{\partial z_e} = \frac{\partial \ddot{M}_{in}}{\partial z_e} \left( \ddot{F}_{in} - \ddot{g} \right)
\]

(49)

\[
\frac{\partial \ddot{M}_{in}}{\partial z_e} = \frac{\partial \ddot{M}_{in}}{\partial z_e} \left( \ddot{M}_{in} \right)
\]

(50)

It is found that the derivatives of the acceleration \( \ddot{F}_{in} \), angular acceleration \( \ddot{\beta} \), and angular velocity \( \ddot{\omega} \) are needed from Eqs. (49) and (50). Matrix \( T_m \) denotes a \( 3 \times 3 \) submatrix which represents rotational transformation in Eqs. (49) and (50).

(c) Derivatives of acceleration

The acceleration \( \ddot{F}_{in} \) of the center of gravity \( G_m \) of link \( L_m \) is represented as follows:

\[
\ddot{F}_{in} = \frac{\partial \ddot{F}_{in}}{\partial \ddot{z}_e} \rightleftharpoons \ddot{z}_e
\]

(51)

where \( \ddot{F}_{in} \) denotes the position vector of \( G_m \) with respect to \( \Sigma_e \). Its derivatives by state variables \( z_e \) (\( e = 1, 2, c \)) are calculated as follows:

\[
\frac{\partial \ddot{F}_{in}}{\partial z_e} = \frac{\partial \ddot{F}_{in}}{\partial z_e} \left( \ddot{z}_e \right)
\]

(52)

The derivatives of \( T_m \) by state variables are needed for the derivatives of acceleration, where \( T_m \) denotes the second-order time derivatives of the transformation matrix \( T_m \).

(d) Derivatives of angular velocity and acceleration

The angular velocities \( \ddot{\omega}_m \) and \( \ddot{\beta}_m \) with respect to \( \Sigma_w \) and \( \Sigma_m \), respectively, are represented as follows:

\[
\ddot{\omega}_m = \ddot{\omega}_{m-1} + T_w \ddot{B}_{inj} \ddot{a}_m
\]

(53)

\[
\ddot{\beta}_m = T_w \ddot{\omega}_m
\]

(54)

The angular accelerations \( \ddot{\omega}_m \) and \( \ddot{\beta}_m \) with respect to \( \Sigma_w \) and \( \Sigma_m \), respectively, are represented as follows:

\[
\ddot{\omega}_m = \ddot{\omega}_{m-1} + T_w \ddot{B}_{inj} \ddot{a}_m
\]

(55)
\[ \vec{\ddot{z}}_n = T_n^{-1} \left( \vec{\ddot{w}} - T_n \vec{\ddot{w}} \right). \]  
\( (56) \)

Their derivatives by state variables \( \vec{z}_e \) \((e=1, 2, c)\) are calculated as follows:

\[ \frac{\partial \vec{\ddot{w}}_m}{\partial \vec{z}_e} = \frac{\partial \vec{\ddot{w}}_{m-1}}{\partial \vec{z}_e} + \frac{\partial T_n}{\partial \vec{z}_e} B_{l,m/m} \frac{\partial \vec{\ddot{w}}}{\partial \vec{z}_e} + T_n B_{l,m/m} \frac{\partial \vec{\ddot{w}}}{\partial \vec{z}_e} \]  
\( + T_n B_{l,m/m} \frac{\partial \dot{\ddot{w}}}{\partial \vec{z}_e} \)  
\( (57) \)

\[ \frac{\partial \vec{\ddot{w}}_m}{\partial \vec{z}_e} = \frac{\partial \vec{\ddot{w}}_{m-1}}{\partial \vec{z}_e} \frac{\partial T_n}{\partial \vec{z}_e} \frac{\partial w}{\partial \vec{z}_e} + T_n \frac{\partial \vec{\ddot{w}}}{\partial \vec{z}_e} \frac{\partial w}{\partial \vec{z}_e} \]  
\( + \frac{\partial T_n}{\partial \vec{z}_e} B_{l,m/m} \frac{\partial \vec{\ddot{w}}}{\partial \vec{z}_e} + T_n B_{l,m/m} \frac{\partial \vec{\ddot{w}}}{\partial \vec{z}_e} \frac{\partial \dot{w}}{\partial \vec{z}_e} \)  
\( + T_n B_{l,m/m} \frac{\partial \dot{\ddot{w}}}{\partial \vec{z}_e} \frac{\partial \dot{w}}{\partial \vec{z}_e} \)  
\( + T_n B_{l,m/m} \frac{\partial \ddot{w}}{\partial \vec{z}_e} \frac{\partial \dot{w}}{\partial \vec{z}_e} \)  
\( (58) \)

\[ \frac{\partial \vec{\ddot{w}}_m}{\partial \vec{z}_e} = \frac{\partial \vec{\ddot{w}}_{m-1}}{\partial \vec{z}_e} \frac{\partial T_n}{\partial \vec{z}_e} \frac{\partial w}{\partial \vec{z}_e} + T_n \frac{\partial \vec{\ddot{w}}}{\partial \vec{z}_e} \frac{\partial w}{\partial \vec{z}_e} \frac{\partial \dot{w}}{\partial \vec{z}_e} + T_n B_{l,m/m} \frac{\partial \vec{\ddot{w}}}{\partial \vec{z}_e} \frac{\partial \dot{w}}{\partial \vec{z}_e} \frac{\partial \dot{w}}{\partial \vec{z}_e} \)  
\( + T_n B_{l,m/m} \frac{\partial \dot{\ddot{w}}}{\partial \vec{z}_e} \frac{\partial \dot{w}}{\partial \vec{z}_e} \frac{\partial \dot{w}}{\partial \vec{z}_e} \)  
\( + T_n B_{l,m/m} \frac{\partial \ddot{w}}{\partial \vec{z}_e} \frac{\partial \dot{w}}{\partial \vec{z}_e} \frac{\partial \dot{w}}{\partial \vec{z}_e} \)  
\( + T_n B_{l,m/m} \frac{\partial \dot{w}}{\partial \vec{z}_e} \frac{\partial \dot{w}}{\partial \vec{z}_e} \frac{\partial \dot{w}}{\partial \vec{z}_e} \)  
\( (59) \)

where \( \vec{\ddot{w}}_m \) and \( \vec{\ddot{w}}_m \) denote the angular velocity and acceleration of joint \( J_n \), of which derivatives are easily calculated. If the derivatives of the transfer matrix \( T_n \) and its time derivatives are calculated, the derivatives of the angular velocity and acceleration are recursively calculated from the base to the end of a kinematic loop. Matrices \( T_n \), \( \dot{T}_n \), and \( B_{l,m/m} \) denote 3 × 3 submatrices which represent rotational transformation in Eqs. (52)–(54).

\((e)\) Derivatives of the transformation matrix

Derivatives of the transformation matrix \( T_n \) and its time derivatives \( \dot{T}_n \) by state variables \( \vec{z}_e \) \((e=1, 2, c)\) are calculated as follows:

\[ \frac{\partial T_n}{\partial \vec{z}_e} = \frac{\partial T_n^{-1}}{\partial \vec{z}_e} B_{l,m/m} B_{l,m/m}^{-1} \]  
\( + T_n B_{l,m/m} \frac{\partial T_n}{\partial \vec{z}_e} B_{l,m/m}^{-1} \)  
\( (60) \)

where \( P'' \) is defined by Eq. (30), and \( P'' \) is defined as follows:

\[ P'' = \frac{\partial P_n}{\partial \vec{z}_e} \]  
\( (64) \)

The derivatives of \( P'' \) and \( P'' \) are easily calculated, so the derivatives of the transformation matrix and its time derivatives by state variables can be calculated from the base to the end of a kinematic loop.

\( \frac{d}{dt} \) \( \vec{z}_n = A_{sys} \vec{z}_n + \vec{f}_{sys} \)  
\( (65) \)

where vector \( \vec{z}_n \) denotes the perturbation of the state variables,

\[ \vec{z}_n = (\vec{z}_{ni}, \vec{z}_{zi}, \vec{z}_{ci})^T \]  
\( (66) \)

\( A_{sys} \) denotes the system matrix of which components are equivalent to the coefficient of Eqs. (19)–(23) and (43). \( \vec{f}_{sys} \) denotes the perturbation vector of the external force or moment.

4.2 Eigenvalue analysis

The eigenvalue \( \lambda \) of the system matrix \( A_{sys} \) corresponds to the eigenfrequency of the coupled system of the mechanism and control system, and its eigenvector \( \vec{\eta} \), corresponds to the mode shape of vibration:

\( (\lambda I - A_{sys}) \vec{\eta} = 0 \)  
\( (67) \)

4.3 Transfer function analysis

The transfer functions between the perturbation vector of the external force or moment and the state vector \( \vec{z}_n \) composed of the displacement or velocity of a joint are calculated in the form of their Laplace transform:

\[ \vec{z}_n = (jwI - A_{sys})^{-1} \vec{f}_{sys} \]  
\( (68) \)

where \( \vec{f}_{sys} \) and \( \vec{z}_n \) denote the Laplace transforms of \( \vec{f}_{sys} \) and \( \vec{z}_n \), respectively.

5. Examples

Some examples of time transient and vibration analyses will be given by applying the above-mentioned simulation system to an industrial robot with 6 degrees of freedom. Each joint of this robot had a robot controller as shown in Fig. 3. In these examples, the stiffness of the gear at each joint was also modeled.
First, the results of time transient analysis will be shown. Figures 4 (a) and (b) show the initial and final configurations, respectively. The desired velocity for the joint was input as the desired value. The desired values from the initial configuration to the final configuration for all the joints were varied according to a trapezoid velocity pattern. Figure 5 indicates the displacement of a certain joint. In this figure, the dashed line and continuous line denote the desired value and response value, respectively. In Fig. 6, the residual vibration at the final position is overwritten by magnifying the difference between the response and final values 50 times. Figures 7 and 8 indicate the velocity and driving force ratio for the same joint.

Secondly, the results of vibration analysis will be shown. Figure 9 shows the mode shape at the final configuration for the most dominant mode turned out from the eigenvalue analysis. In this figure, the shaded image and the hidden line image show the undeformed mode and deformed mode, respectively. Figure 10 shows the Bode diagram for the transfer function between the external force acting on the hand and the joint displacement. The upper diagram shows the phase, and the lower one the gain. In addition, the most vibratory eigenfrequency in Fig. 9 agreed with the peak frequency in Fig. 10, and also with the frequency of residual vibration shown in Fig. 7.

6. Conclusions

In this paper, we proposed methods for time transient and vibration analyses of link mechanisms.
with controllers, which evaluate the displacement, velocity, acceleration, and driving torque of the joints for desired values, and calculate the eigenfrequency, mode shape and transfer functions for any configuration. This simulation system enables us to analyze dynamic properties easily without constructing complicated differential equations.

7. Acknowledgement

The authors are indebted to F. Umibe for proofreading and correcting the original English manuscript.

References


