Steady-State Response Analysis of Elastically Point-Supported Composite Rectangular Plates*

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This paper presents an analytical method for determining the steady-state response of a laminated composite rectangular plate resting on some elastic point supports. The bending rigidities are assumed to be complex quantities considering material damping. The Ritz approach is used to yield a governing equation of motion for the plate, and the steady-state response solution to a sinusoidally varying point force is derived. The method is applied to a laminated square plate elastically supported at four points symmetrically located at the corners or on the diagonals. The mechanical impedance of the plate is calculated for three types of FRP (fiber-reinforced plastic) materials, and the effects of the point supports and the lamination properties upon the response characteristics are studied.

**Key Words**: Vibration of Continuous System, Composite Materials, Classical Lamination Theory, Mechanical Impedance

1. Introduction

With the increasing demand for composite structures in engineering systems, the vibration study of composite structural elements, such as laminated fiber composite plates, has become particularly important. Point-supported composite plates, for example, have been used extensively for space structures, electrical device foundation and so on.

Recently, the free vibrations of an elastically point-supported isotropic plate(1)(2) and also a point-supported orthotropic plate(3)(4) were studied. The response study of a viscoelastically point-supported isotropic plate was reported(5). However, few papers have been found giving results for the response of an elastically point-supported composite plate presented here.

This paper deals with the forced vibration of a laminated composite rectangular plate resting on some elastic point supports, and presents a steady-state solution to a sinusoidally varying point force. In the analysis, the bending rigidities are obtained by using classical lamination theory, and material damping is considered. The governing equation of motion for the plate is derived by the Ritz approach, where the double power series is used for assumed deflection.

By the application of the method, the mechanical impedance of the point-supported square plate is calculated for three types of FRP materials. The response characteristics of the plate are examined for the support stiffness and the location, and also for the material parameters of the fiber orientation and the number of layers. The effects of these parameters upon the response are discussed.

2. Analysis

Figure 1 shows the elastically point-supported, laminated composite rectangular plate under consideration. In this figure, the constants $\tilde{K}_i (i=1-7)$ are the translational stiffnesses of support points $P_i (x_i, y_i)$. The fiber direction (1-axis) in a layer is indicated by the angle $\theta$. The modulus of elasticity of the layer in the direction of the fiber is $E_i$, and the in-plane trans-
verse modulus is $E_2$. This laminated plate is assumed to consist of perfectly bonded layers of fiber-reinforced plastic materials, and the bonds are assumed to be infinitesimally thin as well as non-shear-deformable. The constitutive relations for a thin layer under plane stress are written as

$$
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_z
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{12} & C_{22} & 0 \\
0 & 0 & C_m
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{z1}
\end{bmatrix},
$$

(1)

The coefficients $C_{pq}(p,q=1,2,6)$ are assumed to be in complex form as

$$
\begin{align}
C_{11} &= E_1(1+\gamma_{z1})/(1-\nu_{12}\nu_{21}) \\
C_{22} &= E_2(1+\gamma_{z2})/(1-\nu_{21}\nu_{12}) \\
C_{12} &= \nu_{12}C_{22} \\
C_m &= G_{12}(1+\gamma_{zc}),
\end{align}
$$

(2)

where $\nu_{12}$ and $\nu_{21}$ are the Poisson ratios, $G_{12}$ is the shear modulus, and $\gamma_{z1}$, $\gamma_{z2}$ and $\gamma_{zc}$ are the damping factors representing the ratio of the imaginary part to the real part of the modulus at any frequency $\omega$.

For a uniform, midplane symmetrically laminated plate, there is no coupling between out-of-plane bending and in-plane stretching. Therefore, the relation of the moment resultants ($M_x$, $M_y$, $M_{xy}$) to the midplane curvature ($\kappa_x$, $\kappa_y$, $\kappa_{xy}$) is given by

$$
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_m
\end{bmatrix}
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix},
$$

(3)

where $D_{pq}(p,q=1,2,6)$ are the bending rigidities.

For transverse vibration of the plate, the strain energy $U$ is given by

$$
U = \frac{1}{2} \int_0^b \int_0^a W_{xx} \left[D_{11} D_{12} D_{16} \frac{W_{xx}}{2 W_{xy}} + D_{16} D_{26} D_m \frac{W_{yy}}{2 W_{xy}} \right] dxdy
$$

in terms of the transverse deflection $W(x, y, t)$. The subscripts to $W$, preceded by a comma, denote partial differentiation with respect to the variables following it. The first and the second terms in Eq.(4) are the strain energy stored in the plate and that in the translational spring $k_i$, respectively. The kinetic energy $T$ is expressed as

$$
T = \frac{1}{2} \rho h \int_0^a \int_0^b \left(W_{,t}^2 \right) dxdy,
$$

(5)

where $\rho$ is the mass density and $h$ is the total plate thickness.

For simplicity of analysis, the following dimensionless variables are introduced.

$$
\begin{align}
\alpha &= \frac{x}{a}, \quad \beta = \frac{y}{b}, \quad \alpha = \frac{a}{b}, \\
\kappa &= \frac{k_i}{a^2/(\rho h)}, \\
\tau &= \omega t, \quad \Omega = \omega a^2/(\rho h)^{1/2},
\end{align}
$$

(6)

where $D_0$ is a bending rigidity when the fiber directions of all layers coincide with the $x$-axis, defined by

$$
D_0 = E_i a^4/12(1-\nu_{12}\nu_{21}).
$$

(7)

It is desirable to use $D_0$ in $\Omega$ rather than $D_1$, for instance, because $D_0$ is independent of the laminate stacking sequence.

The dynamical energies of the plate can be rewritten as

$$
\begin{align}
T &= \frac{1}{2} \frac{D_0}{a^2} \int_0^\beta \int_0^\alpha \left(W_{,\alpha}^2 \right) dudv \\
U &= \frac{1}{2} \frac{D_0}{a^2} \int_0^\beta \int_0^\alpha \left(W^*\right)^2(D^*)[W^*]^2 dudv
$$

+ \frac{1}{2} \frac{D_0}{a^2} \int \sum \kappa_i (W(\alpha, \beta, t))^2,
\end{align}
$$

(9)

where

$$
\begin{align}
W^* &= \{W_{,\alpha}, W_{,\beta}, 2 W_{,\alpha \beta}\}^T, \\
(D^*) &= \{D_{11}, D_{12}, D_{16}, D_{22}, D_{26}, D_{16}, D_{26}, D_m\},
\end{align}
$$

(10)

The transverse deflection of the plate is assumed to be the form

$$
W(\alpha, \beta, \tau) = \sum_{m=0}^{N} \sum_{n=0}^{M} \sum_{\alpha=0}^{N} \sum_{\beta=0}^{M} M_{mn}(\alpha, \beta) q_m(\tau) \tilde{q}_n(\tau).
$$

(12)

By substituting Eq.(12) into Eqs.(8) and (9), the following equations are derived:

$$
\begin{align}
T &= \frac{1}{2} \frac{D_0}{a^2} \Omega^2 \sum_{m=0}^{N} \sum_{n=0}^{M} \sum_{\alpha=0}^{N} \sum_{\beta=0}^{M} M_{mn}(\alpha, \beta) q_m(\tau) \tilde{q}_n(\tau)
$$

+ \frac{1}{2} \frac{D_0}{a^2} \int \sum \kappa_i (W(\alpha, \beta, \tau))^2,
\end{align}
$$

(13)

$$
U = \frac{1}{2} \frac{D_0}{a^2} \sum_{m=0}^{N} \sum_{n=0}^{M} \sum_{\alpha=0}^{N} \sum_{\beta=0}^{M} K_m(\alpha, \beta) q_m(\tau) \tilde{q}_n(\tau),
$$

(14)

where $q_m(\tau)$ is an unknown time function and a dot denotes differentiation with respect to $\tau$. The coefficients $M_{mn}(\alpha, \beta)$ and $K_m(\alpha, \beta)$ of Eqs.(13) and (14) are given by

$$
M_{mn}(\alpha, \beta) = \int_0^\beta \int_0^\alpha u_{,\alpha}^n v_{,\beta}^m dudv
$$

(15)

$$
K_m(\alpha, \beta) = \int_0^\beta \int_0^\alpha \left(W_{,\alpha}^*\right)^T(D^*)[W_{,\beta}^*] dudv
$$

+ \sum \kappa_i (u_{,\alpha}^n)(v_{,\beta}^m)
$$

(16)

where

$$
(W_{,\alpha}^*) = \left(v_{,\beta}^m\right)^T \alpha u_{,\alpha}^n \alpha v_{,\beta}^m 2 \alpha (u_{,\alpha}^n)(v_{,\beta}^m)^T.
$$

(17)
and prime denotes differentiation with respect to \( u \) or \( v \). The equation of motion of the plate can be obtained from the following Lagrange equation:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_{\alpha}} \right) - \frac{\partial T}{\partial q_{\alpha}} (T - U) = Q_\alpha (t),
\]

where the generalized force \( Q_\alpha (t) \) corresponding to each coordinate \( q_\alpha \), is deduced from an expression for the virtual work of \( Q(u, v, \tau) \).

\[
Q_\alpha (t) = \int_0^1 \int_0^1 \delta(t) \frac{\partial Q(u, v, \tau)}{\partial q_{\alpha}} u^\alpha v^\beta du dv
\]

(19)

By substituting Eqs.(13) and (14) into Eq.(18), the equation of motion of the plate can be put in matrix form as

\[
Q^t (M^{(a)} n) (\ddot{q}_n) + (K^{(a)} n) (q_n) = \frac{\partial^2}{\partial \tau^2} \left( Q_{\alpha} \right)
\]

(20)

The steady-state response solution of the plate to a sinusoidally varying force \( \{P_{\alpha n}\} e^{i \tau} \) can be written as

\[
\{q_{\alpha n}\} = \left[ (K^{(a)} n) - Q^t (M^{(a)} n) \right]^{-1} \{P_{\alpha n}\} e^{i \tau}
\]

(21)

In the case of the plate driven transversely by \( F e^{i \tau} \) at an arbitrary point \( (u_0, v_0) \), the external point force of Eq.(19) can be expressed as

\[
Q_{\alpha} (t) = \int_0^1 \int_0^1 F \delta (u - u_0) \delta (v - v_0) u^\alpha v^\beta du dv e^{i \tau}
\]

\[
= F \{u_0\}^\alpha (v_0) e^{i \tau}
\]

(22)

by using the Dirac delta functions \( \delta (u) \) and \( \delta (v) \). Therefore, the normalized mechanical impedance of the monitoring point \( (u_0, v_0) \) is determined by

\[
|Z(u_0, v_0)/M)| = \left| \frac{Q^t \sum_{\alpha=0}^2 \sum_{\beta=0}^2 (u_0)^\alpha (v_0)^\beta r_{\alpha\beta}}{M_{\alpha\beta}} \right|
\]

(23)

where \( M_{\alpha\beta} \) is the total mass of the plate and \( r_{\alpha\beta} \) is the vector element given as

\[
\{r_{\alpha\beta}\} = \left[ (K^{(a)} n) - Q^t (M^{(a)} n) \right]^{-1} \{u_0\}^\alpha (v_0)^\beta n.
\]

(24)

The normalized driving point impedance can be easily obtained by replacing \( \{u_0, v_0\} \) in Eq.(23) by \( \{u_0, v_0\} \).

### 3. Numerical Results and Discussion

Numerical results are presented for the steady-state response of a symmetrically laminated square plate to a sinusoidally varying point force. The plate is elastically supported at four points symmetrically located at the corners or on the two diagonals. The stiffness parameters \( k_i \) are taken to have identical values \( k_i = k_i \) for all supports. The plate is made of the same filamentary composite materials. Each layer of the plate is assumed to be perfectly bonded to others, so there is no slip between the interfaces of layers. Moduli ratios of these composite materials used for the present numerical study are listed in Table 1, and are taken from reference 7.

In order to confirm the accuracy of the present numerical results, frequency parameters for a completely free and a corner point-supported isotropic square plate are compared in Table 2 with those of other authors. In the present calculation, the \( M \times N = 7 \times 7 \) terms of the series in Eq.(12) are used for each of the vibration modes. It is found that the present results agree well with the other numerical ones.

The damping factors of a laminated composite plate depend slightly on the fiber orientation and the lamination geometry, as well as on the excitation frequency. For simplicity of discussion, however, it is assumed that all the damping factors are relatively small constants \( \eta_{1} = \eta_{2} = \eta_{3} = 0.001 \).

Figure 2 shows the frequency parameters \( \Omega \) for a composite square plate elastically supported at the corners, as a function of the support stiffness parameters \( \Omega \). The laminated plate is composed of three \( E / E \) layers with the fiber orientation angle \( (30°/ - 30°/30°) \) as angle–ply stacking sequence. As the laminated plates are generally orthotropic, the mode shapes of the plate become asymmetric ones. With an increase of the stiffness parameters \( k_i \), frequency parameters monotonically increase and ultimately become the values of a simply point-supported plate \( (k_i = \infty) \). The

<table>
<thead>
<tr>
<th>Material</th>
<th>( E_1 ) (GPa)</th>
<th>( E_2 ) (GPa)</th>
<th>( G_12 ) (GPa)</th>
<th>( v_21 )</th>
<th>( \nu_{23} )</th>
<th>( E_1/E_2 )</th>
</tr>
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<tbody>
<tr>
<td>E-Glass/Epoxy</td>
<td>60.7</td>
<td>24.8</td>
<td>12.0</td>
<td>0.23</td>
<td>2.45</td>
<td></td>
</tr>
<tr>
<td>Boron/Epoxy</td>
<td>209</td>
<td>19</td>
<td>6.4</td>
<td>0.21</td>
<td>11.0</td>
<td></td>
</tr>
<tr>
<td>Graphite/Epoxy</td>
<td>138</td>
<td>8.96</td>
<td>7.1</td>
<td>0.30</td>
<td>15.4</td>
<td></td>
</tr>
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| Table 1 Lamina material properties of composite plates used in this study |

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<th>Material</th>
<th>( E_1 ) (GPa)</th>
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</tr>
</tbody>
</table>

| Table 2 Comparison of frequency parameters \( \Omega \) for free and simply point-supported isotropic square plate ; \( \alpha = 1.0, \nu = 0.30, \eta_{1} = \eta_{2} = \eta_{3} = 0 \) |

<table>
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<tr>
<th>Mode</th>
<th>( k_{\alpha} = 0 ): completely free plate</th>
<th>( k_{\alpha} = \epsilon ): simply point-supported plate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode</td>
<td>( \Omega ) (ref)</td>
<td>( \Omega ) (present)</td>
</tr>
<tr>
<td>SS-2</td>
<td>19.60</td>
<td>19.69</td>
</tr>
<tr>
<td>SS-3</td>
<td>38.46</td>
<td>38.73</td>
</tr>
<tr>
<td>SS-4</td>
<td>44.38</td>
<td>44.44</td>
</tr>
</tbody>
</table>
support stiffness has small effects on the fourth mode at \( k_s = 10 \), because nodal lines arising in the fourth mode are close to all four support points.

Figure 3 shows the normalized driving point impedance of the three-layered square plates elastically supported at the corners and driven at the center. The stacking sequence of three \( E/E \) layers is \( (30°/−30°/30°) \). The circles and the dot point on the square plate shown in the figure indicate the location of the support and the driving point, respectively. Within the frequency range of the figure, four resonant peaks appear for the stiffness parameters \( k_s = 1 \) and 10, and only two resonant peaks appear for \( k_s = 100 \). The response curves of the plate are shifted to the right-hand side of the figure with an increase of the support stiffness. The response curves of the laminated square plate at \( k_s = 100 \) are similar to those of the isotropic plate at \( k_s = 10 \), as shown by the double chain line. In addition to this results, the variation of natural frequencies of the laminated composite plates with a change of the support stiffness parameter is smaller than that of an isotropic plate. Thses facts may suggest that the response of a laminated composite plate is less sensitive to the support stiffness than that of an isotropic plate.

Figure 4 shows the normalized driving point impedance for the laminated square plates of three \( E/E \) layers having the support stiffness \( k_s = 10 \). The magnitude of the resonant peaks of \( \Omega_a \) and \( \Omega_b \) is considerably larger than those of other peaks, and has almost the same values regardless of the variation of the lamination angle \( \theta \). In addition, the response curves in the frequency range up to the first antiresonant peak are not affected much by the lamination angle.

Figure 5 shows the normalized driving point impedance of the same three-layered square plate having the support stiffness \( k_s = 100 \) as shown in Fig. 3, for the case of three different driving points. For the plate driven at different positions from the center, more resonant peaks appear than for that driven at the center. The level of the resonant peaks varies depending upon the location of the driving points. For example, in the case of the driving point on the diagonal \((u, v) = (0.75, 0.75)\), shown by the chain line, the magnitudes of the second and the fourth resonant peaks become evidently less than those of the others.

Figure 6 shows the normalized driving point impedance of \( E/E \) three-layer:

\( \theta = \theta \), laminated square plates elastically supported at the corners; \( a = 1 \), \( \eta_{e1} = \eta_{e2} = \eta_c = 0.001 \), \((u, v) = (0.5, 0.5)\), \( k_s \) : ---, 1; ---, 10; ---, 100; ---, 100 (isotropic material, \( \nu = 0.30 \))

\( \theta = 0° \), \( \theta = 10° \), \( \theta = 15° \), \( \theta = 30° \), \( \theta = 45° \)
impedance for the laminated square plates of five-layer (45°/−45°/45°/−45°/45°) consisting of E/E, B/E and G/E material layers. The stiffness parameters of corner supports are identical values $k_s=100$. It is noted that the response curve of the plate of B/E material, shown by the broken line, is similar to that of the G/E laminated plate, because the orthotropic modulus ratios $E_1/E_2$ of the two materials have nearly equal values. Also, the response curves of B/E and G/E laminated plates are shifted to the left-hand side of the figure from those of the E/E laminated plate. For $\Omega<10$, the response curves of the three different plates with the stiffness parameter $k_s=100$ agree in general with that of an isotropic plate having the stiffness parameter $k_s=10$, shown by the double chain line.

Figure 7 shows the frequency parameter $\Omega$ and corresponding mode shapes versus the location of the support points for a three-layered square plate resting on four supports symmetrically located on the diagonals. These laminated plates having the support stiffness parameter $k_s=10$ are composed of three E/E layers (30°/−30°/30°). The location of the support points $(\bar{u}, \bar{v})$ denotes the absolute value in the transformed coordinates $(\bar{u}=u-0.5, \bar{v}=v-0.5)$, and the

Fig. 5 Normalized driving point impedance of laminated composite square plates elastically supported at the corners; $E/E$ three-layer: (30°/−30°/30°), $a=1.0, \eta_{31}=\eta_{32}=\eta_c=0.001, k_s=100, (u_d, v_d)=(-0.5,0.5), (0.75,0.75), (0.75,0.75), (0.75,0.75)$

Fig. 6 Normalized driving point impedance of laminated composite square plates elastically supported at the corners; Five-layer: (45°/−45°/45°/−45°/45°), $a=1.0, \eta_{31}=\eta_{32}=\eta_c=0.001$, material: $---, E/E; -----, B/E; -----, G/E; -----, isotropic (k_s=10, \nu=0.30)$

Fig. 7 Frequency parameters of laminated composite square plates elastically supported at four points symmetrically located on the diagonals; $E/E$ three-layer: (30°/−30°/30°), $a=1.0, \eta_{31}=\eta_{32}=\eta_c=0.001, k_s=10$.

Fig. 8 Normalized transfer impedance of laminated composite square plates elastically supported at four points symmetrically located on the diagonals; $E/E$ three-layer: (30°/−30°/30°), $a=1.0, \eta_{31}=\eta_{32}=\eta_c=0.001, k_s=10, (u_d,v_d)=(0.5,0.5), (u_w,v_w)=(-1.0,1.0), (0.8,0.8), (0.6,0.6)$
circles in each plate indicate the location of the supports. The change of frequency parameters against the support location depends on the corresponding mode shapes. For example, the frequency parameters of the first and the second modes are almost identical for \( \bar{u}_1 = \bar{v}_2 < 0.37 \), and linearly increase, and those of the third and the sixth modes change in an archlike and a wavelike manner, respectively.

Figure 8 shows the normalized transfer impedance of the same three-layered square plate as shown in Fig. 7 under the action of center-force. The monitoring point \((u_M, v_M)\) is located at the support point on the upper right side of the plate. The level of the first resonant peak for the plate at \( \bar{u}_4 = \bar{v}_4 = 0.3 \) (\( u_M = v_M = 0.8 \)), shown by the broken line, has the largest value. For the plate elastically supported at the corners, shown by the solid line, the antiresonant peaks disappear in the range of the figure. It is observed that the response curves of the plates depend on the location of the support points. These results suggest that the force transmitted to the foundation through the support can be reduced by an appropriate choice for the location of the support points.

4. Conclusions

The steady-state response to a sinusoidally varying point force has been studied for a laminated composite rectangular plate resting on some elastic point supports. The bending rigidities of the laminated composite plate are assumed to be complex quantities considering material damping, and are obtained by using classical lamination theory. The governing equation for the plate is solved by the Ritz approach, where a double power series is used for assumed deflection.

The method is applied to laminated square plates which are elastically point supported at the corners or on the diagonals. The mechanical impedance of the plates driven at arbitrary points is calculated for three types of FRP materials. The effects of the point supports and the lamination properties upon the responses are discussed.

The present numerical computations were carried out on a HITAC M-682 H computer at the Hokkaido University Computing Center.

References