The Meniscus and Sloshing of a Liquid in an Axisymmetric Container at Low-Gravity*

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The equilibrium and dynamic behavior of a liquid in an axisymmetric container at low gravity are investigated. Partial spherical coordinate systems are introduced 1) to express the liquid surface as a single-valued function and 2) to satisfy the compatibility of the liquid surface displacement at the container wall. The former 1) is convenient for the mathematical formulation while the latter 2) is essential to the precise evaluation of the gravity and surface tension potential. The spherical coordinates, moreover, present an analytical method for solving the sloshing problem in various axisymmetric containers, for which it has been customary to resort to numerical methods such as FEM and BEM. The basic equations are obtained from variational principles, by which the surface tension potential can be calculated more efficiently than by using Laplace's law. Numerical results are shown for the equilibrium liquid surface configurations, slosh frequencies and slosh forces.

Key Words: Vibration of Continuous System, Fluid Vibration, Meniscus, Liquid Sloshing, Low-Gravity, Surface Tension, Bond Number, Variational Principle, Galerkin's Method, Frobenius' Method

1. Introduction

Study of the low-gravity sloshing in a propellant container is essential for the detailed analysis of spacecraft dynamics and control[10]. The theoretical determination of the meniscus shape, i.e., the liquid surface configuration under the equilibrium between surface tension and gravity[10], is also a subject of fundamental importance as a preliminary step to the sloshing analysis. Studies of this kind have been conducted by a number of investigators[11-19] who have used cylindrical coordinates in the mathematical formulation for various axisymmetric containers. However, the cylindrical coordinates are not appropriate for the spherical or ellipsoidal containers, which are frequently used for space vehicles. In low gravity, the liquid surface markedly curves due to the significant role of the surface tension, so that the function expressing the liquid surface becomes two-valued. This makes the formulation extremely complicated.

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Another problem with using the cylindrical coordinates is that one cannot reasonably treat the kinematic compatibility condition of the liquid surface motion, which requires that the liquid surface displacement at the container wall is tangent to the container wall. In conventional methods, the liquid surface displacement is directed only in the axial direction of the container, so that the liquid surface penetrates and detaches from the container wall unless the wall is vertical. It should be noted that this incompatibility causes a serious error on the liquid surface area and liquid surface elevation, which must be calculated carefully for the accurate evaluation of the surface tension potential and gravity potential, respectively. In order to overcome such problems, partial spherical coordinate systems are introduced in this paper.

Two spherical coordinate systems are used. One of them is used for the determination of the meniscus shape and the other is for the dynamic sloshing analysis. In the previous paper[10], the author showed the latter spherical coordinate system enables us to solve the sloshing problem analytically for an arbitrary axisymmetric container with nonvertical wall. In this
paper, the detailed computational procedures for calculating the meniscus shape, which were omitted in the previous paper due to the page limit, are presented. The reason for the possibility of applying the analytical method to an arbitrary generatrix shape of the container is also explained together with the mathematical manipulation. Discussion of the numerical results, which was not included in the previous report, is presented in this paper.

The basic equations governing the equilibrium behavior and the sloshing of a liquid in this paper are derived from the variational principles under the assumption that the liquid is inviscid, incompressible and irrotational. By using the principle of virtual work, the differential equation for the meniscus can be obtained more briefly than through the direct use of Laplace's law. This is because the virtual work done by the surface tension can be related to the normal vector of the meniscus, and its calculation is much easier than that of the principal radii used in Laplace's law. The differential equation governing the meniscus shape is nonlinear and contains unknown parameters to be determined by the computational iterative procedures. The variational principle for the sloshing can be transformed into the eigenvalue problem corresponding to the frequency equation. This process can be done analytically for various axisymmetric containers by using the spherical coordinates introduced, so that the computation time for the present analysis is much shorter than that for the conventional numerical method, i.e., FEM or BEM.

2. Spherical Coordinate Systems

Figure 1 shows the cylindrical coordinates used in the conventional method. The axisymmetric meniscus $M$ and the oscillating liquid surface $F$ are expressed by

$$M: z = z_{M}(r)$$

where $\eta$ is the liquid surface elevation. One can find that the functions $z_{M}, z_{r}$ and $\eta$ are two-valued with respect to $r$ and $\varphi$. This is due to the fact that the liquid surface curves due to the surface tension and that the contact angle between the propellant and the container wall is nearly 0°. One can also see from Fig.1 that the liquid surface displacement $\eta$ at the container wall is not tangent to the wall surface. Because of this incompatibility, the surface tension potential related to the area of the oscillating liquid surface cannot be calculated with reasonable accuracy even with the assumption of small amplitude sloshing.

These problems can be resolved by using spherical coordinate systems $O-R\theta \varphi$ and $O'-R' \theta' \varphi$, shown in Figs. 2 and 3, respectively; the former, $O-R\theta \varphi$, is used for expressing the meniscus $M$:

$$R = R_{M}(\theta),$$

while the latter, $O'-R' \theta' \varphi$, is used for describing the oscillating liquid surface $F$:

$$R' = R'(\theta', \varphi, t) = R_{M}(\theta') + \xi(\theta', \varphi, t).$$

The origin $O$ is set at the top of the container for the case of spherical and ellipsoidal containers and also the other axisymmetric containers with roof. The position of the origin $O$ need not be determined uniquely, however, it must be higher than the height of flat liquid surface free from surface tension, which can be determined by the prescribed liquid volume.

![Fig. 1 Cylindrical coordinates in the conventional method (Liquid domain is indicated by $V$).](image1)

![Fig. 2 Spherical coordinate systems $O-R\theta \varphi$ for expressing menisci $M$.](image2)

![Fig. 3 Spherical coordinate systems $O'-R' \theta' \varphi$ for sloshing analysis.](image3)
The origin $O'$ is the top of the cone tangent to the container at the contact line of the meniscus $M$ and the container wall $W$. One-to-one correspondence is found between $R$ and $\theta$ on the meniscus $M$ and between $R'$ and $(\theta', \phi)$ on the disturbed liquid surface $F$, so that the functions $R_W, R_B, R_F$ and $\xi$ in Eqs. (3) and (4) are single-valued with respect to their arguments. One can also find that the liquid surface displacement $\xi$ satisfies the compatibility at the container wall.

3. Variational Principle

The basic equations for the meniscus and the sloshing can be derived from the variational principle:

$$\delta \int_0^T \left[ \int_F (p_t - p) dV - \int_F \alpha dF \right] dt = 0, \quad (5)$$

where $p_t$ is the liquid pressure, $p$ the gas pressure and $\alpha$ the surface tension, i.e., the surface-free energy per unit area. In Eq. (5), the Lagrangian shown by $\int \int_F \alpha dV$ can be determined as follows: First the Lagrangian corresponding to the liquid motion without surface tension is given by $\int \int_F \rho dV^{(12)}$ from the equality of the Lagrangian density with the liquid pressure $p_t$ under the assumption of inviscid flow. Taking account of the potential energy due to the gas pressure $p_t$ and of the surface tension $\int_F \alpha dF$, one obtains Eq. (5), where the gas pressure $p_t$ is assumed to be constant since the gas density is much smaller than the liquid density. The liquid pressure $p_t$ can be expressed in terms of the velocity potential $\Phi$ from the pressure equation of irrotational flow:

$$p_t = \rho_0 - \rho_0 \left[ \frac{\partial \Phi}{\partial t} + \rho_0 \Phi (R_W(0) - R' \cos \theta) \right] + \frac{1}{2} \rho_0 (\Phi'^2 + G(t)), \quad (6)$$

where $p_0$ is the static liquid pressure at the bottom of the meniscus ($R = R_W(0), \theta' = 0$) and $G(t)$ the arbitrary time-dependent function. The liquid mass density $\rho_0$ is set to be a constant under the assumption of incompressible flow. The parameter $\epsilon$ is 1 or $-1$, respectively, when the origin $O'$ is above or below the container (see Fig. 3).

4. Determination of Meniscus Configuration

The purpose of this section is to determine the function $R_W(\theta)$ in Eq. (3). At the equilibrium stage, the variational principle (5) can be reduced to the principle of virtual work by replacing the oscillating liquid surface $F$ by the meniscus $M$ and considering the liquid pressure $p_t$ as the hydrostatic pressure: $\delta W_p + \delta W_\alpha = 0$, \quad (7)

where $\delta W_p$ and $\delta W_\alpha$ are the virtual work performed by the pressure and the surface tension, respectively. This work is given by

$$\delta W_p = \int \int_F (p_t - p) \delta (dV), \quad (8)$$

$$\delta W_\alpha = \int \alpha \delta (dM), \quad (9)$$

The virtual increment of the static liquid volume element $\delta (dV)$ in Eq. (8) can be expressed by the virtual displacement of the meniscus $\delta R_W$ as $\delta (dV) = d(V) - \delta R_W \cos \gamma dM$, \quad (10)

where $\gamma$ is the angle between the $R'$-direction and the inner normal of the meniscus $M$, and $dM$ is the surface element of the meniscus $M$. The virtual increment of the surface element of the meniscus $\delta (dM)$ in Eq. (9) can also be related to $\delta R_W$ by

$$\delta (dM) = \text{div} (N \cdot \delta R_W \cos \gamma dM), \quad (11)$$

where $N$ is the inner normal unit vector of the meniscus $M$. The derivation of Eq. (11) is shown in Appendix A. Substituting Eqs. (10) and (11) into Eqs. (8) and (9) and considering Eq. (7), one obtains $p_t - p_0 = \sigma \text{div } N$, \quad (12)

which governs the equilibrium to be satisfied at the liquid-gas interface. Equation (12) must be expressed in terms of the spherical coordinates $O-R\theta\phi$ shown by Fig. 2 in order to obtain the differential equation with respect to the meniscus configuration $R_W(\theta)$. Expressing Eq. (3) in the form of $f(R, \theta, \phi, \theta') \equiv R - R_W(\theta') = 0$, one obtains the unit normal vector of the meniscus $N_R$ as follows:

$$N_R = \text{grad } f / |\text{grad } f| = (f_\theta, -f_\phi, (R \sin \theta)^{-1} f_\theta)$$

$$= \left[ (f_\theta + R^2 f_\phi + (R \sin \theta)^{-1} f_\theta)^{1/2} \right]^{-1} R_W(0) - R_W(\theta) \cos \theta), \quad (13)$$

where subscripts denote differentiation with respect to the subscripted variable, e.g., $f_\phi = \partial f / \partial \theta, R_W = \partial R_W / \partial \theta$. The static liquid pressure $p_t$ is

\begin{align*}
p_t &= p_0 - \rho_0 g (R_W(0) - R_W \cos \theta), \quad (14)
\end{align*}

Substitution of Eqs. (13) and (14) into Eq. (12) leads to the differential equation with respect to $R_W(\theta)$ of the implicit form $g(R_W, R_B, R_m) = 0$, which must be written in the explicit form $R_m = g(R_W, R_B)$ as

\begin{align*}
R_m &= \frac{(2R_B + 3R_m)R_W - (R_m / \tan \theta)}{2R_B + R_m} \\
&\times \left[ (R_B^2 + R_m^2) \right]^{-1/2}
\end{align*}

\begin{align*}
&= \frac{[p_0 - p_0 + \rho_0 g (R_W(0) - R_W \cos \theta)]}{(R_B^2 + R_m^2)^{1/2}} \times \left[ (R_B^2 + R_m^2) \right]^{-1/2} \quad (15)
\end{align*}

for the convenience of the numerical calculation using the Runge-Kutta-Gill method. The meniscus configuration $R_W(\theta)$ is determined by solving the nonlinear ordinary differential equation (15). The unknown parameters $R_W(0)$ and $p_0 - p_0$ in Eq. (15) are determined by the prescribed liquid volume and contact angle, respectively, by the iterative computation. Another condition needed to determine $R_W(\theta)$ is

$$R_m(0) = 0, \quad (16)$$


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which can be easily found from the axisymmetry of the meniscus. It should be noted that the factor \( R_m \tan \theta \) in Eq. (15) is indefinite at \( \theta = 0 \). This factor can be evaluated as a finite value by

\[
\lim_{\theta \to 0} R_m \tan \theta = R_m(0) \left(1 - \frac{\rho_0 - \rho_s}{2\sigma}ight) R_m(0).
\]

The derivation of Eq. (17) is given in Appendix B.

By using Eq. (11), i.e., the relation between the virtual increment of the meniscus element \( \delta(dM) \) and its normal vector \( N_m \), the differential equation (15) has been derived briefly than by the use of Laplace's law:

\[
\rho_s - \rho_l = \sigma(R_1^{-1} + R_2^{-1})
\]

since the calculation for the principal radii of curvature \( R_1 \) and \( R_2 \) is lengthier than that for the normal vector \( N_m \). The equivalence between Eqs. (12) and (18) is based on

\[
\text{div} N_m = R_1^{-1} + R_2^{-1},
\]

whose proof is given in Appendix C.

It is convenient for the numerical calculation to transform Eq. (15) into the nondimensional form:

\[
\frac{\dot{R}_m}{R_m} = \frac{2\dot{R}_1 + 3\dot{R}_2}{R_m} - \frac{(R_m \tan \theta)(\dot{R}_1 + \dot{R}_2)}{R_m}
\]

\[
- \left[ \frac{a(R_m - \rho_0)}{a} + B_0[R_m(0) - R_m \cos \theta] \right]
\times (\dot{R}_1 + \dot{R}_2)^{1/2}/R_m
\]

where

\[
\dot{R}_m = R_m/a
\]

(21)

\[
\dot{B}_0 = \rho_0 a^2/\sigma
\]

(22)

The characteristic length \( a \) is, for example, the radius of a spherical container. Equation (22) defines the Bond number \( B_0 \), which is the nondimensional parameter comparing the relative magnitude of the gravity and the surface tension. One can see from Eq. (20) that the meniscus shape \( \tilde{R}_m(\theta) \) depends only on the Bond number under the prescribed shape of container, liquid volume and contact angle between the meniscus and the container wall.

Numerical examples are shown in Fig. 4, where the spherical container is used as a typical example of the axisymmetric container and the liquid volume is expressed by the height \( h \) of the flat liquid surface free from the surface tension. The contact angle is 5°. When the Bond number \( B_0 \) is large, i.e., the gravity is predominant in comparison with the surface tension, the menisci are almost flat except in the vicinity of the container wall. With the decrease of the Bond number \( B_0 \), the menisci approach the spherical surface due to the significant role of the surface tension (see Appendix D).

If the container wall happens to be vertical at the contact point, the meniscus in a spherical container must agree with those in a cylindrical container, which have been examined experimentally. Figure 5 shows an example of such menisci, whose agreement with the photograph on p. 389 of Ref. (1) has been confirmed by the author.

5. Sloshing Analysis

The basic equations for the sloshing can be derived from the variational principle (5) by performing the variation operation with respect to the liquid pressure \( \rho_l \), the liquid domain \( V \) and the disturbed liquid surface \( F \). Calculating \( \delta \rho_l \) from Eq. (6), the variational principle can be expressed as

\[
\left[ \begin{array}{c}
\frac{1}{2} \int_V \frac{\partial \delta \rho_l}{\partial \rho_l} \mathrm{d}V + \int_V \left( \rho_s - \rho_l \right) \delta(dV) \\
- \int_V \delta(dF) \end{array} \right] \mathrm{d}t = 0
\]

(23)

Consider that the variation and the differentiation can be exchanged and use the following relations:

\[
\int_V \frac{\partial \delta \phi}{\partial t} \mathrm{d}V = \frac{\partial}{\partial t} \int_V \delta \phi \mathrm{d}V
\]

(24)

\[
\int_V \frac{\partial \delta G}{\partial t} \mathrm{d}V = \frac{\partial}{\partial t} \int_V \delta G \mathrm{d}V
\]

(25)

\[
\int \nabla \phi \cdot \nabla (\delta \phi) \mathrm{d}V = - \int \nabla \phi \cdot \nabla \delta \phi \mathrm{d}V
\]

\[
+ \int \nabla \phi \cdot N_m \delta \phi \mathrm{d}W
\]

Fig. 4 Meniscus configuration in spherical container

Fig. 5 Meniscus with contact point at vertical container wall (\( B_0 = 45 \)).
\[
\delta (dV) = \delta (d\phi) = - \delta \psi \cos \gamma dF
\]
\[
\delta (dF) = \text{div } N_r \cdot \delta \psi \cos \gamma dF,
\]
where \( N_r \) and \( N_w \) are the unit normal vectors of the disturbed liquid surface \( F \) and the container wall \( W \), and \( \gamma_r \) is the angle between \( N_r \) and the liquid surface displacement \( \psi \) in the \( R \)-direction. One should note that \( N_r \) is inner while \( N_w \) is outer relative to the liquid domain. Equations (24) and (25) can be derived from the integration formula for the case of time-varying integration domain, so that the convection term, i.e., the second term of the right-hand side of Eqs. (24) and (25), must be considered. Equation (26) can be derived by Green's theorem. The derivation of Eqs. (27) and (28) is similar to that of Eqs. (10) and (11). By using Eqs. (24) \sim (28) and noting \( \delta \phi = \delta G = 0 \) at \( t = t_1 \) and \( t_2 \), Eq. (23) can be transformed into
\[
\int_{t_1}^{t_2} \int_0^{2\pi} \int_{\Omega} \delta \phi \partial \phi dV - \int_0^{2\pi} \int_{\Omega} \nabla \phi \cdot \nabla \delta \phi dV - \int_{\Omega} \delta \phi \partial \phi dV - \int_{\Omega} \delta \phi \partial \phi dV = 0.
\]

Since the variations, \( \delta \phi \), \( \delta \psi \), and \( \delta G \) are arbitrary and independent, Eq. (29) yields
\[
\begin{align*}
V : & \quad \partial \phi = 0 \\
W : & \quad \nabla \phi \cdot N_w = 0 \\
F : & \quad \begin{cases} 
\partial \phi = \partial \phi \\
\delta \phi = \sigma \text{ div } N_r \\
\int \partial \phi \partial \phi dV = 0,
\end{cases}
\end{align*}
\]
which constitute the basic equations system governing the low-gravity liquid sloshing. The basic equations (30) through (34) can be used for the nonlinear analysis of the large amplitude sloshing since the boundary conditions (32) and (33) take account of nonlinear terms with respect to \( \phi \) and \( \psi \) and are evaluated on the oscillating liquid surface \( F \). This paper employs the linear approximation, in which the velocity potential \( \phi \) and the liquid surface displacement \( \psi \) are assumed to be of the order of \( O(\varepsilon) \) and linear terms up to \( O(\varepsilon) \) are retained in Eqs. (32) and (33) with the higher-order terms neglected. By eliminating \( \psi \) from the dynamic and kinematic boundary conditions (32) and (33), one can obtain a single equation expressed in terms of \( \phi \) only. However, such a combined boundary condition is not convenient for the subsequent analysis using Galerkin's method since the dynamic and kinematic conditions have different weighting, \( \delta \psi \) and \( \delta \phi \), respectively. Equation (34), which means that the liquid volume is constant, is due to the assumption of incompressible flow and can be derived from the other kinematic conditions, (30), (31) and (33), by using Gauss' theorem:
\[
\int_0^{2\pi} \int_{\Omega} \nabla \phi \partial \phi dV = \int_0^{2\pi} \int_{\Omega} \nabla \phi \partial \phi dV = - \int_0^{2\pi} \int_{\Omega} \partial \phi \partial \phi dV = 0.
\]
Therefore, Eqs. (30) \sim (33) are used as the governing equations system in the subsequent analysis.

Equation (29) must be expressed in terms of spherical coordinate system \( O'-R'\theta'\phi \) (introduced in section 2 (Fig. 3)) by using
\[
N_r = \varepsilon (e_x R_x \cos \theta - e_y R_y \sin \phi - e_z R_z)
\]
\[
l/[\varepsilon (R_x^2 + R_y^2 + R_z^2)^{1/2}] = \varepsilon (e_x R_x \cos \theta + R_y \sin \theta) = \varepsilon (e_x R_x \cos \theta + R_z \sin \theta)
\]
\[
dF = R_x (R_x^2 + R_y^2 + R_z^2)^{1/2} \partial \theta d\phi
\]
\[
dW = R_x (R_x^2 + R_y^2 + R_z^2)^{1/2} \partial \theta d\phi
\]
\[
\cos \gamma_r = N_r \cdot e_r = \varepsilon (R_x \cos \theta + R_z \sin \theta)
\]
\[
l/[\varepsilon (R_x^2 + R_y^2 + R_z^2)^{1/2}] = \varepsilon (R_x \cos \theta + R_z \sin \theta)
\]
where the notation \((R', \theta', \phi')\) in Fig. 3 has been altered to \((R, \theta, \phi)\) for brevity. Equations (36) and (37) are derived by expressing the liquid surface \( F \) and the container wall \( W \) of the implicit function form
\[
f_r(R, \theta, \phi, t) = R - R_r (\cos \theta + \sin \phi + \cos \theta - \sin \phi)
\]
\[
f_w(R, \theta, \phi, t) = R - R_w (\cos \theta + \sin \phi + \cos \theta - \sin \phi)
\]

and using
\[
N_r = \varepsilon \text{ grad } f_r / ||\text{grad } f_r||
\]
\[
N_w = \varepsilon \text{ grad } f_w / ||\text{grad } f_w||
\]
where \( \varepsilon \) is 1 or \(-1\) when the origin is above or below the container, respectively, (see Fig. 3). Equations (38) and (39) are derived by calculating the position vectors of the surfaces \( F \) and \( W \) as
\[
X_r(\theta, \phi, t) = R_r (\cos \theta + \sin \phi + \cos \theta - \sin \phi)
\]
\[
X_w(\theta, \phi, t) = R_w (\cos \theta + \sin \phi + \cos \theta - \sin \phi)
\]

and using
\[
dF = ||\text{grad } f_r|| d\theta d\phi
\]
\[
dW = ||\text{grad } f_w|| d\theta d\phi
\]
Equation (47) suggests that the surface element \( dF \) is equal to the area of the parallelogram formed by vectors \( \partial X_r / \partial \theta \cdot \partial \phi \cdot \partial \phi \) and \( \partial X_r / \partial \theta \cdot \partial \phi \cdot \partial \phi \), where \( \partial X_r / \partial \theta \) and \( \partial X_r / \partial \phi \) are respectively the tangential vectors along the \( \theta \) and \( \phi \) directions.

Calculating \( \text{div } N_r \) in Eq. (29) from Eq. (36), one obtains
\[
\text{div } N_r (\theta, \phi, t) = R_r^2 \left( \frac{\partial}{\partial R} (N_r \cdot e_r) + 2 (N_r \cdot e_s) + \frac{\partial}{\partial \theta} (N_r \cdot e_r) + \frac{1}{\tan \theta} \frac{\partial}{\partial \phi} (N_r \cdot e_r) \right)
\]

where \( R_r^2 = (R_x^2 + R_z^2)^{1/2} \).
\[
\begin{align*}
\times & \sin^2 \theta (2R^2 + 3R_1R_k - R_1R_{kw}) \\
& + \sin \theta (3R_1R_{kw} - R_1R_{kw}) R_{kw} + 2R_1R_k R_{kw} R_{kw} \\
& - (R_1 + R_k) R_{kw} R_{kw} \\
& - \cos \theta \sin^2 \theta (R_1^2 + R_k^2) R_{kw} \\
& - 2 \cos \theta \cdot R_{kw} R_{kw}.
\end{align*}
\]

(50)

It should be noted that since \( N_r \) on \( F \) does not contain argument \( R \), the first term in (49) vanishes. Substituting \( R_r = R_{kw} + \zeta (R_w) \) into Eq. (50) and neglecting the nonlinear terms with respect to the liquid surface displacement \( \zeta \) and its derivatives, one can obtain the linearized form of \( div \ N_r \) as

\[
\begin{align*}
div \ N_r &= \epsilon \left[ S_{1w}(\theta) + \zeta S_{1w}(\theta) + \zeta \xi S_{2w}(\theta) \\
& + \zeta^2 S_{3w}(\theta) + \zeta^2 \xi S_{4w}(\theta) \right],
\end{align*}
\]

(51)

where

\[
\begin{align*}
S_{1w}(\theta) &= R_{kw}(R_1 + R_k) - \frac{1}{2}
\times (2R^2 + 3R_1R_k - R_1R_{kw}) \\
& - \cot \theta \cdot R_{kw} (R_1 + R_k)
\end{align*}
\]

\[
\begin{align*}
S_{2w}(\theta) &= R_{kw}^2 (R_1 + R_k)^{-1/2}
\times (3R_1R_k R_{kw} - R_1R_{kw}) \\
& - \cot \theta \cdot R_{kw} (R_1 + R_k)
\end{align*}
\]

\[
\begin{align*}
S_{3w}(\theta) &= -(\sin \theta)^2 R_{kw}^2 (R_1 + R_k)^{-1/2}
\end{align*}
\]

(53)

For evaluating the other terms in Eq. (29), a similar linear approximation is made together with the Taylor expansion, as

\[
\begin{align*}
\frac{\partial \phi}{\partial t} |_{x = x_{n+1}} &= \frac{\partial \phi}{\partial t} |_{x = x_n} \\
& = \frac{\partial}{\partial t} \left[ \phi |_{x = x_n} + \frac{\partial \phi}{\partial x} |_{x = x_n} \cdot \zeta + \ldots \right] = \frac{\partial \phi}{\partial t} |_{x = x_n}.
\end{align*}
\]

(55)

By following the foregoing procedures, one obtains the spherical coordinate expression of the variational principle (29):

\[
\begin{align*}
\rho s \int_{0}^{2\pi} \int_{0}^{\pi} \left[ \frac{\partial}{\partial t} \left( \rho \phi \right) - R_{kw} \frac{\partial \phi}{\partial \theta} |_{\theta = \pi} \right] d\theta d\phi \\
& - \rho s \int_{0}^{2\pi} \int_{0}^{\pi} \left( \frac{\partial \phi}{\partial \theta} |_{\theta = \pi} \right) d\theta d\phi \\
& + \int_{0}^{2\pi} \int_{0}^{\pi} \left[ \epsilon \left( p_0 - p_\theta \right) + \rho \theta \left( R_w(0) - R_{kw} \cos \theta \right) \\
& - \sigma S_{1w}(\theta) + \epsilon \rho \frac{\partial \phi}{\partial \theta} |_{\theta = \pi} - \rho \theta \zeta \cos \theta \\
& - \sigma \left( S_{1w}(\theta) \zeta + S_{2w}(\theta) \frac{\partial \zeta}{\partial \theta} + S_{3w}(\theta) \frac{\partial^2 \zeta}{\partial \theta^2} \right) \\
& + S_{3w}(\theta) \frac{\partial^2 \zeta}{\partial \theta^2} \right] \delta \zeta R_{kw} \sin \theta d\theta d\phi = 0
\end{align*}
\]

(54)

determined by the contact point of the meniscus. Since \( \bar{\theta} = \pi/2 \), as shown in Fig. 3, \( R_w(\theta) \) and \( \theta \) are called the partial spherical coordinates. The 1st through 4th terms in Eq. (54) respectively correspond to the continuity condition in the liquid domain, the boundary condition on the container wall, and the kinematic and dynamic boundary conditions on the liquid surface.

These respective conditions can be combined by the variational principle. Such a combined form of the governing equations system given by Eq. (54) is more convenient than the separated form [Eqs. (30) ~ (34)] for the subsequent analysis using Galerkin's method.

The dynamic boundary condition on the liquid surface, i.e., the 4th term of Eq. (54), consists of static and dynamic parts. The former,

\[
\epsilon (p_0 - p_\theta) + \rho \theta \left( R_w(0) - R_{kw} \cos \theta \right) - \sigma S_{1w}(\theta) = 0,
\]

(55)

coincides with Eq. (15) governing the meniscus configuration \( R_w(\theta) \) whereas the latter,

\[
\epsilon \rho \frac{\partial \phi}{\partial t} |_{\theta = \pi} - \rho \theta \zeta \cos \theta - \sigma \left( S_{1w}(\theta) \zeta + S_{2w}(\theta) \frac{\partial \zeta}{\partial \theta} + S_{3w}(\theta) \frac{\partial^2 \zeta}{\partial \theta^2} \right) = 0,
\]

(56)

governs the oscillatory motion of the liquid surface. The variational principle (54) can be transformed into the eigenvalue problem corresponding to the frequency equation by Galerkin's method, in which the admissible functions for the velocity potential \( \phi \) and the liquid surface displacement \( \zeta \) are constructed analytically by the Laplace equation,

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0,
\]

(57)

and the contact angle condition for the case of a smooth container wall,

\[
\frac{\partial \zeta}{\partial \theta} |_{\theta = \bar{\theta}} = 0,
\]

(58)

respectively, as follows:

\[
\phi(R, \theta, \phi, t) = \omega b
\times \sum_{k=1}^{\infty} \left( a_k \left( \frac{R}{L_a} \right)^{n_k} + b_k \left( \frac{R}{L_b} \right)^{n_k} \right) \Theta_k(\theta) \cos m \varphi e^{i \omega t}
\]

(59)

\[
\zeta(\theta, \varphi, t) = \sum_{k=1}^{\infty} a_k \Theta_k(\theta) m \varphi e^{i \omega t},
\]

(60)

where

\( \omega \) : the natural frequency,

\( m \) : the circumferential wave number,

\( b \) : the characteristic length of the container,

\( a_k, b_k, and c_k \) : arbitrary real constants with dimension of length.

\( \Theta_k(\theta) \) : kth eigenfunction satisfying the Sturm-Liouville eigenvalue problem:

\[
\frac{d^2 \Theta}{d \theta^2} + \cot \theta \cdot \frac{d \Theta}{d \theta} + \left( \lambda - \frac{m^2}{\sin^2 \theta} \right) \Theta = 0
\]
\( \frac{d\theta}{d\theta} |_{\theta = 0} = 0 \) \hspace{1cm} (61)

\[ \mu_{2k} = \mu_{2k} = \frac{(-1)^{k+1} \sqrt{1 + 4\lambda_{1}}}{2} \] \hspace{1cm} (62)

\( \mu_{1k} \) and \( \mu_{2k} \): characteristic exponents related to the eigenvalue \( \lambda = \lambda_{1k} \) in Eq. (61) by \( \lambda = \mu(\mu + 1) \) \hspace{1cm} (63)

i.e.,

\[ \mu_{1,2k} = \left( -1 \pm \sqrt{1 + \frac{4\lambda_{1}}{2}} \right) \] \hspace{1cm} (64)

\( \Lambda_{0} \) and \( \Lambda_{0} \): The normalization parameter for obtaining the excellent convergence property of the series (59). With the increase of \( k \), \( \lambda_{1} \rightarrow \infty \), i.e., \( \mu_{1k} \rightarrow -\infty \) and \( \mu_{2k} \rightarrow \infty \), so that \( \Lambda_{0} \) and \( \Lambda_{0} \) are respectively the minimum and maximum of \( R \) considered: \( R_{0}(\theta) \) and \( R_{0}(\theta) \) \( 0 \leq \theta \leq \bar{\theta} \).

Associated with the liquid surface displacement (60), the boundary condition (62) can be easily found from Eq. (58). On the other hand, concerning the velocity potential (59), Eq. (62) must be obtained by taking limit \( \theta \rightarrow \bar{\theta} \) of the boundary condition on the container wall, which can be found from the second term of Eq. (54) as

\[ \left( \frac{\partial \phi}{\partial R} \right)_{R=R_{0}(\theta)}, \phi_{\theta} = 0 \hspace{1cm} (0 \leq \theta \leq \bar{\theta}) \] \hspace{1cm} (65)

Equation (65) can be rewritten as

\[ \left( \frac{1}{R_{0}(\theta)} \frac{\partial \phi}{\partial R} \right)_{R=R_{0}(\theta)} = 0 \] \hspace{1cm} (66)

When \( \theta \rightarrow \bar{\theta} \), \( |R_{0}(\theta)| \rightarrow \infty \) (see Fig. 3), so that the first term of Eq. (66) vanishes and the second term of Eq. (66) tends toward

\[ \frac{\partial \phi}{\partial \theta} \bigg|_{\theta = \bar{\theta}} = 0 \] \hspace{1cm} (67)

It should be noted that the boundary condition for the characteristic function \( \Theta(\theta) \) can be determined by the kinematic condition at the contact line \( \theta = \bar{\theta} \) only instead of that throughout the container wall. This is the reason for the possibility of obtaining the analytical procedure irrespective of the generatrix shape of the container. The Sturm–Liouville eigenvalue problem defined by Eqs. (61) and (62) is solved as follows.

By the following variable transformations,

\[ \xi = \cos \theta, \Theta(\theta) = (1 - \xi)^{m} u(\xi) \] \hspace{1cm} (68)

Equation (61) can be reduced to the differential equation with singular points \( \xi = \pm 1 \):

\[ \frac{d^{2} u}{d\xi^{2}} - \frac{2(m+1)}{1 - \xi^{2}} \frac{du}{d\xi} + \frac{\lambda - m(m+1)}{1 - \xi^{2}} u = 0 \] \hspace{1cm} (69)

whose solution in the form of power series around \( \xi = 1 \) \( (\theta = 0) \),

\[ u(\xi) = (\xi - 1)^{2} \sum a_{i}(\xi - 1)^{i} \] \hspace{1cm} (70)

can be determined by Frobenius’ method. The equation for the determination of \( \rho \) is

\[ \rho(\rho + m) = 0 \Rightarrow \rho = 0 \] \hspace{1cm} (71)

The recurrence relation for \( a_{i} \) is

\[ a_{i+1} = (\mu_{1} - i(i+1) - m(m + 1 + 2i)) \] \hspace{1cm} (72)

\( a_{0} \): nonzero arbitrary constant

By using Eq. (63), the solution (70) can be expressed in terms of Gauss’ hypergeometric series as

\[ u(\xi) = a_{0} F(m - \mu, \mu + m + 1, m + 1, (1 - \xi)/2) \]

\[ = \sum_{m=0}^{\infty} \frac{(\mu + m + 1)!(\mu + m + 2)!}{(m + 1)!(m + 2)!} \xi^{m} \] \hspace{1cm} (73)

which converges with arbitrary values of \( m \) and \( \mu \) if \(|1 - \xi|/2 < 1\), i.e., \( 0 \leq \theta < \pi \). In the present sloshing problem, \( 0 \leq \theta \leq \bar{\theta} < \pi \) (see Fig. 3), so that the convergence of the series (73) is rapid. Using the boundary condition (62), one can determine the eigenvalues \( \lambda = \lambda_{1k} \) and the associated eigenfunctions \( \Theta_{k}(\theta) \).

The important properties of \( \Theta_{k}(\theta) \) are the orthogonality,

\[ \int_{0}^{\pi} \Theta_{k}(\theta) \sin \theta d\theta = 0 \] \hspace{1cm} (74)

and the behavior at the singular point \( \theta = 0 \):

\[ \Theta_{k}(0) = 0 \] \hspace{1cm} (75)

where \( m \) is the circumferential wave number. The orthogonality (74) signifies that only few terms in series (59) and (60) are needed to obtain a sufficiently converged solution while Eq. (75) guarantees the compatibility of the liquid surface profile at the center \( \theta = 0 \).

Substituting Eqs. (59) and (60) into Eq. (54) and using Galerkin’s method, one can derive the algebraic homogeneous equations with respect to the unknown constants \( a_{1}, b_{1}, \) and \( c_{1} \). These equations are described in Ref. (10) in nondimensional form for the convenience of computation. These equations can be reduced to the equations with respect to \( c_{1} \) only. Therefore, the dimension of the eigenvalue problem to be solved is \( k \), where \( k \) is the integer at which the series (59) and (60) are truncated. The eigenvalue problem obtained can be expressed in the form

\[ - \omega^{2} M + K = 0 \] \hspace{1cm} (76)

One can see that the stiffness matrix \( K \) comes from the gravity and surface tension terms in the fourth term of Eq. (54), and the mass matrix \( M \) from the other terms of Eq. (54). It should be noted that the factor \( iw \) in Eq. (59) is essential for obtaining the frequency equation in the form of the eigenvalue problem as Eq. (76), in which the matrices \( M \) and \( K \) do not contain unknown \( \omega \).

In the present paper, the main numerical results are presented and some discussion is presented in addition to the discussion given in Ref. (10).

A spherical container is employed as a numerical
example. The container radius \( a \) is 1 m and the propellant is hydrazine \( \text{N}_{2} \text{H}_4 \) \((\rho_\ell = 1.011 \text{ kg/m}^3, \sigma = 0.0725 \text{ N/m})\). The contact angle is 5° (Ref. (7)). Figure 6 shows the natural frequencies of the asymmetric mode \( \omega_a \) and the axisymmetric mode \( \omega_n \), where subscripts denote the circumferential wave number \( m \). These modes with \( m = 1 \) and 0 are important for spacecraft dynamics and control since the former mode produces the transverse slosh force \( F_x \) and the pitching slosh moment \( M_y \), and the latter mode the axial slosh force \( F_z \). The resultant slosh forces and moment are calculated by integrating the appropriate components of the dynamic liquid pressure \( P_0 \),

\[
P_0 = -\rho_\ell \frac{\partial \phi}{\partial t},
\]

and its moment along the container wall. The slosh forces \( F_x \) and \( F_z \) and slosh moment \( M_y \) are shown in Fig. 7, where the magnitude of the liquid surface oscillation at the container wall is normalized by \( \xi = \zeta_0 = 0.2a \) (see Fig. 8). For the case of a spherical container, the relation \( F_x = -M_y/a \) is satisfied since the force \( dF_x \) and the moment \( dM_y \) acting on the surface element of the container wall \( d\sigma \) are expressed respectively by \( P_0 \sin \phi \cos \theta d\sigma \) and \( -P_0 a \sin \phi \cos \phi \cdot dW \), where \( \phi \) is the angle between the normal of \( d\sigma \) and the axis of the spherical container. In Figs. 6 and 7, the numerals of the curves indicate \( \log_{10} B_0 \), where \( B_0 \) is the Bond number defined by Eq. (22). The influences of the height of the contact line \( z_c \) on the natural frequencies and slosh forces and moment are examined for the given Bond number.

Figure 6 shows that when the Bond number is large, the natural frequencies increase with the increase of the liquid filling level \( z_e \) reflecting the property of the gravity wave. On the other hand, when the Bond number is small, i.e., the surface tension is predominant relative to the gravity, the natural frequencies are lowest at \( z_e \approx 1.5a \). This liquid filling level is found to make the meniscus widest (see Fig. 9). Namely, for the widest meniscus, the longest time is needed for the propagation of the surface tension wave throughout the liquid surface. The author has confirmed that with increase in the Bond number \( B_0 \), the presented results for \( \omega_n \) approach the theoretical and experimental results shown in p. 52 of Ref. (1), where the surface tension is neglected, i.e., \( B_0 \to \infty \) and the nondimensional frequency \( \omega_n / \sqrt{\sigma/a} \) is shown. (The nondimensional frequency corresponding to the present analysis is shown in Fig. 10)

From Fig. 7, one can find the general tendency (independent of the Bond number) that the transverse slosh force \( F_x \) and the pitching slosh moment \( M_y \) are

---

Fig. 6 Slosh frequencies \( \omega_n \) (\( m \): circumferential wave number; numerals of curves indicate \( \log_{10} B_0 \)).

Fig. 7 Slosh forces and moment \( F_x, F_z \) and \( M_y \) (Numerals of curves indicate \( \log_{10} B_0 \)).

Fig. 8 Eigenmodes of liquid surface displacement \( \xi \) and dynamic liquid pressure \( P_0 \) on container wall (Subscripts indicate circumferential wave number).

Fig. 9 Meniscus area \( M = \int dM(B_0=1) \)
maximum when \( z_c \approx 1.2a \), whereas the axial slosh force \( F_x \) is large when the liquid filling level \( z_c \) is high or low. One should note that for the case of the spherical container, the increase in the liquid volume does not necessarily result in the augmentation of the resultant slosh forces and moment. The reason for the result of \( F_x \) and \( M_x \) has been explained in detail in Ref. (10). In this paper, moreover, we discuss the result of \( F_x \). Irrespective of the Bond number, \( F_x \) is exactly 0 at the liquid filling level \( z_c \approx 1.1a \), which is a little higher than the just intermediate liquid filling level \( z_c \approx a \). Namely, the liquid pressures in the ranges of \( z < a \) and \( z > a \) contribute to \( F_x < 0 \) and \( F_x > 0 \), respectively, and cancel each other out (see Fig. 8 (b)). Furthermore, let us consider the reason why \( F_x \) exhibits maximum at \( z_c \approx 0.5a \) and \( z_c \approx 1.5a \) when the Bond number is large. Since the magnitude of the dynamic liquid pressure \( P_d \) is large near the liquid surface (see Fig. 8), \( F_x \) is greatly influenced by the liquid pressure at the contact line, which is given by

\[
P_c = P_0 \delta(x_{w0}, z_{c0}) - \rho g \frac{\xi^2}{2} \cos \theta \frac{\xi}{2} \frac{\xi}{2}.
\]  

(78)

Equation (78) is derived from Eqs. (56) and (77) under the assumption of high Bond number, i.e., \( \sigma \to 0 \). Because the load line of the pressure is normal to the container wall, the contribution ratio of the pressure \( P_c \) to the axial slosh force \( F_x \), i.e., the cosine of the angle between the pressure load line and \( z \)-axis \( \cos (\xi \cos \theta) \), equals 0 at \( z_c = a \) and increases as \( z_c \to 0 \) or \( 2a \). However, when \( z_c \to 0 \) or \( 2a \), the magnitude of \( P_c \), given by Eq. (78) decreases due to the decrease of the axial component of the liquid surface displacement at the container wall (\( \xi \cos \theta \)) since \( \delta \) decreases when \( z_c \to 0 \) or \( 2a \). Therefore, \( F_x \) decreases when \( z_c \to 0 \) or \( 2a \) (see Fig. 3). Thus, the maximum of \( |F_x| \) at \( z_c \approx 0.5a \) and \( z_c \approx 1.5a \) results from two factors, i.e., the increase of the contribution ratio of \( P_c \) to \( F_x \) and the decrease of the magnitude of \( |P_c| \). One should note that the decrease of \( (\xi \cos \theta) \) with \( z_c \to 0 \) and \( 2a \) corresponds to the reduction of the gravity potential due to the tilt of the container wall from the vertical axis. The foregoing explanation cannot be made unless the liquid surface displacement is tangent to the container wall; i.e., the compatibility of the liquid surface displacement is essential for the precise evaluation of the gravity potential. Thus, one of the most important advantages of the use of the partial spherical coordinate system has been illustrated.

Figure 8 (especially Fig. 8 (a)) shows that the partial spherical coordinates are helpful in drawing the mode of liquid surface displacement without violating its compatibility at the container wall within the assumption of the small amplitude linear sloshing. If one uses the cylindrical coordinates shown in Fig. 1, one cannot draw such a compatible mode of the liquid surface displacement and will be confronted with a complicated formulation since \( \xi \) must be defined as a two-valued function with respect to \( r \) and \( \varphi \).

The computation time and cost for the present analytical method based on the partial spherical coordinate system are much less than those for the conventional method using FEM or BEM. Long computation time is needed for FEM and BEM since the dimension of the eigenvalue problem to be solved is extraordinarily large. For the low-gravity sloshing problem, especially, the dimension becomes larger since the liquid domain shape is sharp at the contact point due to the extremely small contact angle between the liquid and the container wall. In the present analysis, the dimension of the eigenvalue problem (76) \( \bar{k} \) for obtaining the sufficiently converged result is, in most cases, only 3 to 5. It is only for a very thin meniscus, i.e., when \( z_c < 0.3a \) and \( B_0 \approx 1 \) in the presented result (Figs. 6 and 7), that the convergence is slower than in the other cases and \( \bar{k} \) must be about 10. Such an excellent convergence property is due to the orthogonality (74) of the characteristic function. The computation time is only 10 seconds in the case of \( \bar{k} = 10 \) and decreases proportional to \( \bar{k}^2 \). For the practical design, the computation must be conducted for many cases, i.e., for various values of the Bond number \( B_0 \), liquid filling level \( z_c \) and circumferential wave number \( m \), as shown in Figs. 6 and 7, so that the computation time for one case must be short for reasons of economy. The present analytical method is economical enough to be used for such practical design.

6. Conclusions

A theoretical investigation was conducted for the meniscus shape and the free oscillation of a liquid in an axisymmetric container in low gravity environments.

(1) Introducing the partial spherical coordinates has enabled us to express the liquid surface as a single-
valued function, to satisfy the compatibility of the liquid surface displacement at the container wall, and to solve analytically the sloshing problem for various axisymmetric containers. These contribute respectively to the convenience of the mathematical formulation, precise evaluation of the gravity and surface tension potential and reduction of computation time and cost.

(2) A variational principle has been given in a concise form to formulate the equilibrium and dynamic behavior of a liquid. The basic equations can be derived simply by relating the virtual work done by the surface tension to the normal vector of the liquid surface.

(3) The influences of the Bond number and the liquid filling level on the menisci configurations, slosh frequencies and slosh forces have been investigated for a spherical container. Some of the results have been compared with the past experimental results and have been confirmed to be in good agreement with them.

References


Appendix

A: Derivation of Eq. (11):

Equation (11) can be proved by expressing $\text{div} \, N_w$ in the integral form:

$$\text{div} \, N_w = \lim_{\varepsilon \to 0} \left\{ \int_D N_w \cdot n \, dA \right\},$$

where $D$ is the volume of an arbitrary space including point $P$ on which the vector $N_w$ is defined, $A$ is the closed surface bounding the space $D$ and $n$ is the outer normal unit vector of surface $A$. Let the space $D$ be the domain which the meniscus element $dM$ passes through during the virtual displacement $\delta R_w$ from $dM$ to $dM' = dM + \delta(dM)$ (see Fig. 11). Inner product $N_w \cdot n$ is $-1$ on $dM$, $1$ on $dM'$ and $0$ elsewhere $A \cdot dM - dM'$, and $D = dM \cdot \delta R_w \cos \gamma_w$, so that Eq. (79) becomes $\text{div} \, N_w = (dM' - dM) / dM \cdot \delta R_w \cos \gamma_w$, which is identical to Eq. (11).

B: Derivation of Eq. (17):

Equation (1) can be utilized to express the meniscus configuration in the neighborhood of the meniscus bottom $\theta = 0$. From Eq. (12), the differential equation with respect to $\mu = 0$ can be derived as

$$\frac{d}{d\tau} \left\{ \mu + \frac{d}{d\tau} \right\} - \frac{\rho g}{\sigma} \mu$$

$$\frac{d}{d\tau} \left\{ \mu - \frac{d}{d\tau} \right\} = \left( \frac{\rho g}{\sigma} \mu \right) / \sigma,$$

where $z_m = dz_m / d\tau$. Since $z_m \approx 0$ at $\tau \approx 0$, the non-linear term $z_m$ can be neglected; thus, we obtain the linearized form of Eq. (80) as

$$z_m + \left( \frac{d}{d\tau} \right) z_m = \left( \frac{\rho g}{\sigma} \mu \right) / \sigma.$$  

The solution of Eq. (81) can be expressed by the modified Bessel function $I_\nu$ as follows:

$$z_m(\tau) = \frac{\rho g}{\sigma} \left\{ I_\nu \left( \sqrt{\frac{\rho g}{\sigma} \mu} \right) - 1 \right\} + z(0).$$

Expanding $I_\nu(x)$ in power series and using the coordinate transformations (see Fig. 2),

Fig. 11 Virtual displacement of meniscus $\mu$
\[ r = R \sin \theta, \quad z = H - R \cos \theta, \]  \hspace{1cm} (83)

one obtains Eq. (17). In Eq. (83), \( H \) is the height of the container.

C: Derivation of Eq. (19):

According to the differential geometry, the orthogonal curvilinear coordinates \( \alpha, \alpha \) and \( \alpha \) can be defined respectively along the directions of principal curvature of an arbitrary surface and its normal direction, and the following formulas are established:

\[ \partial N/\partial \alpha = t_1/R_1, \quad \partial N/\partial \alpha = t_2/R_2, \]  \hspace{1cm} (84)

where \( t_1 \) and \( t_2 \) are unit tangential vectors in the \( \alpha \) and \( \alpha \) directions, respectively, and \( N \) is the unit normal vector of the surface. From Eq. (84),

\[ R_1^{-1} + R_2^{-1} = (\partial N/\partial \alpha) \cdot t_1 + (\partial N/\partial \alpha) \cdot t_2, \]  \hspace{1cm} (85)

Expressing the unit vectors in terms of their components with respect to the rectangular coordinates

\[ t_i = (\lambda_i, \mu_i, \nu_i) \quad (i = 1, 2), \quad N = (\lambda_0, \mu_0, \nu_0), \]  \hspace{1cm} (86)

and using relations such as

\[ \partial \lambda_0/\partial \alpha = \lambda(\partial \lambda_0/\partial x) + \mu(\partial \lambda_0/\partial y) + \nu(\partial \lambda_0/\partial z), \]  \hspace{1cm} (87)

\[ \lambda^2 + \mu^2 + \nu^2 = 1, \quad \lambda_0 \mu_0 + \lambda_0 \mu_0 + \lambda_0 \mu_0 = 0, \]  \hspace{1cm} (88)

one finally obtains

\[ R_1^{-1} + R_2^{-1} = \partial \lambda_0/\partial x + \partial \mu_0/\partial y + \partial \nu_0/\partial z \]

\[ -(1/2)(\partial \partial \alpha)(\lambda^2 + \mu^2 + \nu^2) = \text{div} \, N, \]  \hspace{1cm} (89)

which coincides with Eq. (19).

D: Meniscus Configuration at Zero Gravity:

Substitution of \( g = 0 \) into Eq. (80) yields

\[ \frac{1}{r} \frac{d}{dr}(rz_m(1 + z_m)^{-1/2}) = \mu, \]  \hspace{1cm} (90)

where

\[ \mu = (\rho - \rho_0)/\sigma, \]  \hspace{1cm} (91)

Integrating Eq. (90) under the condition of \( z_m(0) = 0 \), one obtains

\[ rz_m(1 + z_m)^{-1/2} = \frac{1}{2} \mu r^2, \]  \hspace{1cm} (92)

i.e.,

\[ z_m = \pm \mu r(4 - \mu^2 r^2)^{-1/2} \]  \hspace{1cm} (93)

Integrating Eq. (93) and expressing the integration constant in terms of \( z_m(0) \), one obtains

\[ z_m = \frac{1}{\mu}(4 - \mu^2 r^2)^{1/2} \pm \frac{2}{\mu} + z_m(0), \]  \hspace{1cm} (94)

i.e.,

\[ r^2 + (z_m(r) - z_m(0))^2 = \left( \frac{2}{\mu} \right)^2, \]  \hspace{1cm} (95)

which indicates that the surface \( z = z_m(r) \) is the sphere, whose center is \((r, z) = (0, z_m(0) \pm 2/\mu)\) and radius is \( 2/\mu \).