The Natural Mode of Vibration of a Circular Plate with a Solid Shaft*

(Correction of Natural Frequency under Consideration of Elastic Deformation on the Boundary between a Circular Plate and a Solid Shaft, and Bending Deformation of the Shaft)

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The authors used Mindlin's equation, and calculated the natural mode of a circular plate with a solid shaft under the boundary condition of a fixed inner diameter. As the result, the calculated natural frequency of the low mode of the thick circular plate differed greatly from the one measured by modal analysis. The causes were analyzed on the (1,0) mode of vibration, which had the biggest error, of 78 percent. It was revealed that the elastic deformation on the boundary between the circular plate and the solid shaft, and the bending deformation of the shaft had a strong influence on the natural frequency. By taking both influences into account, the natural frequency of the (1,0) mode was corrected to within a 5 percent error.

Key Words : Machine Element, Gear, Mode of Vibration, Natural Frequency, Modal Analysis

1. Introduction

The authors have fundamentally researched an estimation theory of sound power radiated from gears using a circular plate with a solid shaft. The final research object is to establish a useful method to estimate the sound noise radiated from rotating gears if the gear design parameters are given.

In the research, it was necessary to determine accurately the natural vibration modes of a thick circular plate. The equations of vibration for such a thick elastic plate were derived in Ref. (1) taking the rotatory inertia and shear of the plate into account. Reference (2) applied the equations to the calculation of the flexural vibration of a gear body. Hence, the authors also used the equations to calculate the natural vibration modes of the thick circular plate with a

solid shaft. However, the calculated natural frequency differed greatly from the one measured with the experimental modal analysis, particularly in some low modes. The calculation was based on the boundary condition that the circular plate was fixed on the shaft diameter. The difference between the calculated and experimental natural frequencies of the (1,0) mode, which showed the largest error among all the modes, was analyzed. Then the causes were the elastic deformation on the boundary between the circular plate and the solid shaft, and the bending deformation of the shaft. Finally, by taking these effects into account, the calculated natural frequencies were corrected with an almost sufficient accuracy.

2. Measurement of Natural Mode

The natural vibration modes of six circular plates with a solid shaft or a center hole were measured with the experimental modal analysis instruments shown in Table 1. The tested plates were made of iron, and were 5 mm, 15 mm, and 25 mm thick. The diameters were all 200 mm, and the shaft or center hole diameters were all 70 mm. Figure 1 schematically shows the circular plate with a solid shaft supported by the 80

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mm-wide, 80 mm-deep, and 250 mm-high iron base. The circular plates with a center hole were suspended with a rubber cord to obtain the completely free supporting condition.

Table 2 shows the measured and calculated natural frequencies of the circular plates with a center hole. The calculation used the equations of vibration written in Ref. (1) and was based on the boundary condition that the circular plate was free on both the inner and outer diameters.

As is clear in Table 2, the calculated frequencies

Table 1 Instruments for experimental modal analysis

Frontend	Multi-channel FFT Analyzer S C A D A S
Accelerometer	B & K 4 3 7 5 ($\times 2$) (0.1 Hz ~ 16.5 KHz)
Impact hammer	B & K 8 2 0 2
Modal analysis software	L M S C A D A X - 5 2 0 0 Multi-input ver. X - 5 2 1 0

Table 2 Natural mode frequency of a circular plate with a center hole

Mode of vibration		(2,0)	(3,0)	(4,0)	(5,0)
5	Meas.	580Hz	1471Hz	2617Hz	4007Hz
	Cal.	582Hz	1479Hz	2633Hz	4033Hz
	Error	0.23%	0.51%	0.61%	0.64%
25	Meas.	2734Hz	6625Hz	11090Hz	15894Hz
	Cal.	2690Hz	6498Hz	10849Hz	15520Hz
	Error	-1.63%	-1.91%	-2.17%	-2.35%
Mode of vibration		(0,1)	(1,1)	(2,1)	(3,1)
5	Meas.	1021Hz	2091Hz	3840Hz	5932Hz
	Cal.	1035Hz	2101Hz	3877Hz	5986Hz
	Error	1.37%	0.44%	0.94%	0.91%
25	Meas.	4800Hz	8497Hz	14874Hz	----
	Cal.	4742Hz	8346Hz	14571Hz	----
	Error	-1.20%	-1.78%	-2.04%	----

Fig. 1 Circular plate with a solid shaft supported on a base

coincided well with the measured one, within 3% error. This result showed that the equations in Ref. (1) gave the accurate natural frequencies of a thick circular plate if the correct boundary conditions were given.

Table 3 shows the comparison of the calculated and measured natural frequencies on the circular plates with a solid shaft. The calculation used the boundary condition that the circular plate was fixed on the inner diameter and was free on the outer diameter. From the table, the following was found: (1) All the calculated natural frequencies of the 5 mm-thick circular plate and the high mode natural frequencies of the 15 mm- and 25 mm-thick circular plates coincided well with the measured ones, within 5% error, except for the double-frame part in the table; (2) In the low modes of the 15 mm- and 25 mm-thick circular plates, shown with the double frame, the calculated natural frequencies differed greatly from the measured ones, with the largest error of 78%.

These results showed that the used boundary condition, that is, the circular plate was fixed on the inner diameter, was correct for all the modes of the 5 mm-thick circular plate, and for the high modes of the 15 mm- and 25 mm-thick circular plates. But it was not correct for the low modes of the 15 mm- and 25 mm-thick circular plates.

Table 3 Natural mode frequency of a circular plate with a solid shaft

Mode of Vibration	Natural Frequency			
	Thickness	5 mm	15 mm	25 mm
(1,0)	Measured	907Hz	2043Hz	2240Hz
	Calculated	939Hz	2640Hz	3976Hz
	Error	3.5%	29.2%	77.4%
(2,0)	Measured	1067Hz	2668Hz	3921Hz
	Calculated	1097Hz	3021Hz	4536Hz
	Error	2.8%	13.2%	15.7%
(3,0)	Measured	1696Hz	4540Hz	7001Hz
	Calculated	1689Hz	4700Hz	7123Hz
	Error	-0.4%	3.5%	1.7%
(5,0)	Measured	4129Hz	10897Hz	15906Hz
	Calculated	4058Hz	10876Hz	15573Hz
	Error	-1.7%	-0.2%	-2.1%
(10,0)	Measured	14728Hz	---	---
	Calculated	14427Hz	32623Hz	40807Hz
	Error	-2.0%	---	---

3. Natural Frequency of the (1,0) Mode

The causes as to why the fixed boundary condition was not correct for the low modes of the 15mm- and 25 mm-thick circular plates were concretely analyzed for the (1,0) mode, which had the largest calculation error (78%), and also had the simplest mode shape to analyze. By using the analysis result, the calculated (1,0) mode natural frequency was corrected.

3.1 Decrease of natural frequency due to deformation on the boundary between the circular plate and shaft

In the (1,0) mode, the circular plate with a solid shaft was assumed to be a lumped mass-spring vibration system: the circular plate was rigid and was connected to the rigid shaft with a torsional spring S_1 (the spring coefficient: k_1), as shown in Fig. 2. The shaft was completely fixed on a base B. The natural frequency f_1 was given by the following equation:

$$f_1 = (1/2\pi) \sqrt{k_1/J}, \quad (1)$$

where J : the moment of inertia around a node diameter of the circular plate.

The natural frequency f_1 calculated by Eq. (1) should be equal to the calculated frequency shown in Table 3. Hence, the equivalent torsional spring coefficient k_1 was determined by using the known moment of inertia J of the circular plate.

Here it was assumed that small deformation occurred on the boundary between the circular plate and the shaft. The base B then rotated slightly around the node diameter. This state is shown in Fig. 3. The slight rotation of the base B was approximated with the equivalent torsional spring S_2 (the spring coefficient: k_2) against the real fixed base C (the spring S_1 is not drawn in Fig. 3). The torsional spring S_2 came into series with S_1 in the torsional vibration

Fig. 2 Approximation of the (1,0) vibration mode of a circular plate with a solid shaft (The boundary between the circular plate and the shaft was fixed.)

system. The equivalent torsional spring coefficient k_{eq} was given by the following equation:

$$1/k_{eq} = 1/k_1 + 1/k_2, \quad (2)$$

Hence the natural frequency f_{eq} was given by the following equation:

$$f_{eq} = (1/2\pi) \sqrt{k_{eq}/J}. \quad (3)$$

Substituting Eqs. (1) and (2) into Eq. (3), we obtain

$$f_{eq} = [\sqrt{1 + k_1/k_2}]^{-1} f_1. \quad (4)$$

From Eq. (4), it was found that the natural frequency f_{eq} became smaller than f_1 , which was calculated under the fixed inner diameter boundary condition.

3.2 Equivalent torsional spring due to deformation on the boundary between the circular plate and shaft

The elastic deformation on the boundary between the circular plate and the shaft, and the bending deformation of the shaft itself were analyzed. And the equivalent torsional spring S_2 due to those deformations was derived.

3.2.1 Equivalent torsional spring due to elastic deformation on the boundary between the circular plate and shaft

It was assumed that the elastic deformation, as shown in Fig. 4, occurred on the boundary between the circular plate and the shaft and caused the relative rotational displacement $\psi_{r,e}$ between the circular plate and the shaft. As it was very difficult to solve the elastic deformation analytically, the relative displacement was approximated by using the analytical result for the case in which the linear stress due to a moment M was applied on a semi-infinite plate.

According to Ref. (3), the displacement w of the plate surface along the z direction was given by the following equation:

$$w = \frac{-\rho_0}{\pi E a} \left[(y^2 - a^2) \ln \frac{y+a}{y-a} - 2ay \right]. \quad (5)$$

Fig. 3 Approximation of the (1,0) vibration mode of a circular plate with a solid shaft (including the elastic deformation on the boundary between the circular plate and the shaft)

As the inclination angle $\phi_{r,s}$ is $\phi_{r,s} = \tan \phi_{r,s} = \partial w / \partial y$
 $\phi_{r,s} = C_a p_0$, (6)

where $C_a = 4/\pi E$ when approximated by the center point inclination (the broken line m_1), and $C_a = 2/\pi E$ when approximated by the average inclination which was determined from the displacement on the maximum stress points (the broken line m_2), E : Young Modulus. Let M be the moment working on the b deep part of the semi-infinite plate in the direction perpendicular to this paper. The moment M was given by the following equation:

$$M = \int_{-a}^a (p_0 b/a) y^2 dy \\ = (2/3) p_0 a^2 b. \quad (7)$$

Hence, the equivalent torsional spring k_s was given by M/ϕ_s

$$k_s = C_s a^2 b, \quad (8)$$

where $C_s = \pi E/6$ when approximated by the center point inclination, and $C_s = \pi E/3$ when approximated by the maximum stress point inclination.

Here the circular plate was assumed to incline by $\phi_{r,e}$ due to the moment M , as shown in Fig. 6. Let M be the moment working on a small dx wide part; ΔM

Fig. 4 Angular displacement of the circular plate due to elastic deformation on the boundary between the circular plate and shaft

Fig. 5 Surface deformation of a semi-infinite plate due to moment

is given by $k_s \phi_{r,e}$. By substituting $a^2 = r_s^2 - x^2$, $b = dx$ in Eq. (8),

$$\Delta M = C_s (r_s^2 - x^2) dx \phi_{r,e}, \quad (9)$$

Then,

$$M = \int_{-r_s}^{r_s} \Delta M = C_s r_s^3 \phi_{r,e}, \quad (10)$$

where, $C_e = 2\pi E/9$ when approximated by the center point inclination, and $C_e = 4\pi E/9$ when approximated by the maximum stress point inclination. The equivalent torsional spring k_e on the boundary between the circular plate and the shaft is given by $M/\phi_{r,e}$. Thus,

$$k_e = C_s r_s^3. \quad (11)$$

3.2.2 Bending deformation of shaft By using the relation between the bending moment M and the inclination angle ψ of a beam as shown in Fig. 7, the equivalent torsional spring k_b due to the bending deformation of the shaft was given by

$$k_b = \begin{cases} [C_b(1/3 - \xi + \xi^2)]^{-1} & \text{(simply supported)} \\ [C_b(\xi - 4\xi^2 + 6\xi^3 - 3\xi^4)]^{-1} & \text{(fixed supported)} \end{cases}$$

where $C_b = l_0/EI$, $\xi = l/l_0$

$$I = \pi r_s^4/4$$

moment of inertia of area. (12)

Then, the equivalent torsional spring coefficient k_2 was given by the following equation:

$$k_2 = (1/k_e + 1/k_b)^{-1}. \quad (13)$$

3.3 Correction of natural frequency

Table 5 shows the (1,0) mode natural frequencies corrected by using the equations derived in the previ-

Fig. 6 Angular displacement of a circular plate due to bending moment

Fig. 7 Deformation of a beam due to bending moment

Table 4 Values for the correction of natural frequency

Thickness of plate	5 mm	15 mm	25 mm
Moment of inertia of circular plate J ($\times 10^{-2} \text{ Nms}^2$)	0.306	0.919	1.532
Equivalent torsional stiff. of circular plate k_1 ($\times 10^6 \text{ Nm/rad}$)	0.107	2.528	9.561
Equivalent torsional stiff. of elastic deform. on boundary k_e ($\times 10^6 \text{ Nm/rad}$)	6.160 (center point incl.) 12.320 (maximum stress point incl.)		
Equivalent torsional stiff. of bending deformation of shaft k_b ($\times 10^6 \text{ Nm/rad}$)	18.190 (simply supported shaft) 27.280 (fixed shaft)		

Table 5 Correction of the (1,0) mode natural frequency of a circular plate with a solid shaft

Thickness of plate	5 mm	15 mm	25 mm
Torsional stiff. determined by center incl.			
Simply supported shaft	928Hz (+2.4%)	2121Hz (+3.8%)	2267Hz (+1.2%)
Fixed shaft	929Hz (+2.4%)	2153Hz (+5.4%)	2333Hz (+4.2%)
Torsional stiff. by maximum stress point incl.			
Simply supported shaft	932Hz (+2.8%)	2277Hz (+11.5%)	2502Hz (+11.7%)
Fixed shaft	933Hz (+2.9%)	2317Hz (+13.4%)	2603Hz (+16.2%)

ous section, and also by using the values given in Table 4.

The following was found from the table: (1) The equivalent torsional spring approximated by the center point inclination gave the natural frequencies better than the one approximated by the maximum stress point inclination did; (2) The simply supported shaft gave a better result than the fixed shaft; (3) The combination of the center point inclination and the simply supported shaft gave the best result, and the corrected natural frequencies coincided well with the measured ones, within a 5% error in all the plates; (4) The largest 78% error of the 25 mm-thick circular plate was decreased to only 2%, and the 29% error of the 15 mm-thick circular plate was also decreased to 4%. However, the natural frequency improvement of the 5 mm-thick plate was very small because the error was small from the beginning. Namely, the 4% error was only decreased to 3%.

As a result, the following was revealed. The elastic deformation on the boundary between the circular plate and the shaft, and the bending deformation of the shaft had a great influence on the (1,0) natural frequency. The influence was great on the

Table 6 Natural frequency corrected by using only boundary elastic deformation or by using shaft bending deformation

Thickness of plate	5 mm	15 mm	25 mm
Correction of Elastic deformation on boundary			
Frequency (error)	930Hz (+2.5%)	2176Hz (+6.5%)	2399Hz (+7.1%)
Correction of Bending deformation of shaft			
Frequency (error)	936Hz (+3.2%)	2470Hz (+20.9%)	3220Hz (+43.7%)

thick circular plates in particular. By taking those influences into account, the accurate natural frequency was obtained; it was close to the measured one.

The boundary elastic deformation was compared with the shaft bending deformation in terms of the influence on the (1,0) mode natural frequency of the circular plate with a solid shaft. The compared case was a combination of the center point inclination and the simply supported shaft. Table 6 shows the corrected natural frequencies by using only the boundary elastic deformation effect or by using only the shaft bending deformation effect. From the table, it was found that the natural frequency's error of the 25 mm-thick plate decreased from 78% to 11% if the boundary elastic deformation was corrected. On the other hand, the error decreased from 78% to only 44% if the shaft bending deformation was corrected. The same tendency of the natural frequency was found in the 15 mm-thick circular plate. Thus, it was revealed that the boundary elastic deformation between the circular plate and the shaft had a greater effect on the natural frequency's accuracy than the shaft bending deformation.

4. Conclusions

The natural mode frequencies of the circular plates with a solid shaft were calculated by using Mindlin's equation and also were determined by the experimental modal analysis. By analyzing the difference between the calculated and measured (1,0) mode natural frequency, the following was revealed:

(1) The elastic deformation on the boundary between the circular plate and the shaft, and the bending deformation of the shaft had a great influence on the natural frequency of the circular plate. The accurate natural frequencies were obtained by correcting these effects.

(2) The boundary elastic deformation had a greater influence on the natural frequency than the shaft bending deformation.

(3) It was effective to approximate the boundary

elastic deformation effect by the equivalent torsional spring determined with the center point inclination of a semi-infinite plate.

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