A Two-Dimensional Statistical Model of an Uneven Road Surface*

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This paper describes a mathematical model of an uneven road surface where the statistical characteristics of the roughness are isotropic. A spectral density matrix is obtained in consideration of a correlation of the roughness between the right and the left tracks. By the spectral factorization of the rational spectral density matrix, which results from the approximation of the cross-spectral density function, the transfer function matrix is formulated, and the minimal realization of the shaping filter is obtained analytically. The results of the numerical simulation using the shaping filter are in good agreement with the analytical ones.

Key Words: Gaussian Random Process, Correlation, Transfer Function, Uneven Road Surface, Power Spectrum, Shaping Filter

1. Introduction

In order to analyze and evaluate theoretically the dynamic characteristics and the structural strength of vehicles which run over the ground, such as automobiles and locomotive robots, it is very important to describe precisely the road profiles which are input to the vehicles. To discuss the three-dimensional motions of the vehicles including not only the vertical and pitching motions, but also the rolling motions, we need to consider not only the statistical characteristics of the longitudinal road roughness along a single track but also the correlation of the roughness between the right and left tracks. This is provided in the ISO(International Organization for Standardization) draft proposal⁽¹⁾ associated with a one-dimensional power spectral density function of the road

irregularities along a single track. However, a common method of dealing with the two-dimensional road irregularities has been not established. Little research has been carried out on the subject, except for that on a model⁽²⁾ of guideway irregularities based on the plane surface estimate of the two tracks for magnetic levitation vehicles and a model⁽³⁾ of the uneven road surface by means of a two-dimensional filtered Poisson process.

This paper describes a model of a two-dimensional uneven road surface, where the statistical characteristics of the roughness are isotropic. A spectral density matrix is obtained in consideration of the correlation of the roughness between the right and left tracks. Also, by the spectral factorization of the rational spectral density matrix, which results from the approximation of the cross-spectral density function, the transfer function matrix is formulated, and the minimal realization of the shaping filter is obtained analytically. Subsequently, the statistical characteristics of the shaping filter are discussed by the numerical simulations.

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2. Spectral Density Matrix of an Uneven Road Surface

In this paper, the irregularities of the road surface are assumed to be a stationary Gaussian random process. There are many measured results of the power spectral density function of the longitudinal road roughness along a single track; in ISO/DP 8608⁽¹⁾, it is formulated as a function of the spatial frequency n (the reciprocal of the wave number)

$$P(n) = A/n^2, \quad (1)$$

where A is the constant which denotes the road roughness. But in this expression, the power spectrum increases limitlessly as n approaches zero. Hence, we assume that the power spectral density function of an uneven road surface along a single track is

$$P_{11}(n) = P_{22}(n) = A/(n^2 + n_0^2), \quad (2)$$

where $P_{11}(n)$ and $P_{12}(n)$ are the power spectral density function of the right and left tracks, respectively, assuming their statistical characteristics are identical. n_0 is a constant (the cutoff frequency), and when n is sufficiently larger than n_0 , Eqs. (1) and (2) almost coincide. By the inverse Fourier transform of Eq. (2), the autocorrelation function of the road roughness is obtained as a function of the distance l ,

$$\begin{aligned} r_{11}(l) &= r_{22}(l) = E[z_1(x)z_1(x+l)] \\ &= E[z_2(x)z_2(x+l)] \\ &= \int_{-\infty}^{\infty} P_{11}(n) \exp(i2\pi nl) dn \\ &= \pi A \exp(-2\pi n_0 l) / n_0, \end{aligned} \quad (3)$$

where z_1 and z_2 are the road heights of the right and left tracks, respectively, as shown in Fig. 1. $r_{11}(l)$ and $r_{22}(l)$ are the autocorrelation function for z_1 and z_2 , respectively, and $E[\]$ denotes a mathematical expectation.

Next, we consider the correlation of the roughness between the right and left tracks. The statistical characteristics of the uneven ground are generally assumed to be isotropic, so that they are independent of the direction. In this case, the correlation of the roughness between two arbitrary points on the uneven road surface is a function of only the distance l , independent of the direction of the track. Hence, the cross-correlation function between the right and left

Fig. 1 Geometry of uneven road surface

tracks can be expressed as

$$\begin{aligned} r_{12}(l) &= r_{21}(l) = E[z_1(x)z_2(x+l)] \\ &= E[z_1(x)z_1(x+\sqrt{l^2+w^2})] \\ &= r_{11}(\sqrt{l^2+w^2}) \\ &= \pi A \exp(-2\pi n_0 \sqrt{l^2+w^2}) / n_0, \end{aligned} \quad (4)$$

where, as shown in Fig. 1, w is the tread, i.e., the distance between the right and left tracks, which are parallel. By the Fourier transform of Eq. (4), the cross-spectral density function between the right and left tracks is obtained⁽⁴⁾;

$$\begin{aligned} P_{12}(n) &= P_{21}(n) \\ &= \int_{-\infty}^{\infty} r_{12}(l) \exp(-i2\pi nl) dl \\ &= 2\pi A w K_1(2\pi w \sqrt{n^2+n_0^2}) / \sqrt{n^2+n_0^2}, \end{aligned} \quad (5)$$

where $K_1(\)$ denotes the modified Bessel function. By rearranging Eq. (5),

$$\begin{aligned} P_{12}(n) &= P_{21}(n) \\ &= (2\pi w \sqrt{n^2+n_0^2}) K_1(2\pi w \sqrt{n^2+n_0^2}) P_{11}(n) \end{aligned} \quad (6)$$

is obtained. From Eq. (36) in the appendix,

$$\lim_{w \rightarrow 0} P_{12}(n) = \lim_{w \rightarrow 0} P_{21}(n) = P_{11}(n). \quad (7)$$

As a matter of course, in the case where the tread $w = 0$, the cross-spectral density function between the right and left tracks is coincident with the power spectral density function along a single track. From Eqs. (2) and (6), the coherence function between the right and left tracks is obtained:

$$\begin{aligned} \gamma^2(n) &= |P_{12}(n)|^2 / [P_{11}(n)P_{22}(n)] \\ &= [(2\pi w \sqrt{n^2+n_0^2}) K_1(2\pi w \sqrt{n^2+n_0^2})]^2. \end{aligned} \quad (8)$$

Moreover, the power spectral density matrix P_{zz} can be described as

$$\begin{aligned} P_{zz}(n) &= \begin{bmatrix} P_{11}(n) & P_{12}(n) \\ P_{21}(n) & P_{22}(n) \end{bmatrix} \\ &= \frac{A}{n^2+n_0^2} \begin{bmatrix} 1 & f(n) \\ f(n) & 1 \end{bmatrix}, \end{aligned} \quad (9)$$

where $f(n) = (2\pi w \sqrt{n^2+n_0^2}) K_1(2\pi w \sqrt{n^2+n_0^2})$.

Fig. 2 Comparison of approximated $xK_1(x)$

Fig. 3 Relationship between cross-spectral density and spatial frequency

3. Transfer Function Matrix and Minimal Realization

First, we approximate the cross-spectral density function to a rational function. Namely, in letting

$$x = 2\pi w \sqrt{n^2 + n_0^2}, \quad (10)$$

$xK_1(x)$ is approximated to a rational function of x . As shown in Fig. 2, $xK_1(x)$ is the function that decreases rapidly according to increasing x . Thus, we approximate $xK_1(x)$ to

$$xK_1(x) \approx 1/(1 + ax^2), \quad (11)$$

where a is a constant which is determined so that Eq. (11) corresponds with $xK_1(x)$. In Fig. 2, Eq. (11) is in good agreement with $xK_1(x)$ when $a=0.7$. Then the cross-spectral density is approximated to a rational function

Fig. 4 Relationship between coherence and spatial frequency

$$\begin{aligned} P_{12}(n) &= P_{21}(n) \\ &\approx P_{11}(n)/[1 + a(2\pi w)^2(n^2 + n_0^2)] \\ &= A/[(n^2 + n_0^2)[1 + a(2\pi w)^2(n^2 + n_0^2)]]. \end{aligned} \quad (12)$$

Simultaneously, the coherence is approximated as

$$\gamma^2(n) \approx 1/[1 + a(2\pi w)^2(n^2 + n_0^2)]^2. \quad (13)$$

Figure 3 shows the theoretical Eq. (6) and the approximated Eq. (12) of the cross-spectral density $P_{12}(n)$, as the solid line and the dotted line, respectively, where we specify the tread $w=1$ m, the road roughness $A=1.0 \times 10^{-5}$ c·m, $n_0=5.0 \times 10^{-3}$ c/m, and $a=0.7$. Figure 4 shows the theoretical Eq. (8) and the approximated Eq. (13) of the coherence. From Figs. 2 to 4, if we select $a=0.7$ in Eq. (11), the approximated values are in good agreement with the theoretical ones.

In the following, the spectrum is expressed as a function of the wave number $k (=2\pi n)$ instead of the spatial frequency n . Thus the spectral density matrix S_{zz} is expressed as a rational function of the wave number k as follows:

$$S_{zz}(k) = \begin{bmatrix} S_{11}(k) & S_{12}(k) \\ S_{21}(k) & S_{22}(k) \end{bmatrix} = (1/2\pi) P_{zz}(k/2\pi) \approx \frac{2\pi A}{k^2 + k_0^2} \begin{bmatrix} 1 & \frac{1}{1 + aw^2(k^2 + k_0^2)} \\ \frac{1}{1 + aw^2(k^2 + k_0^2)} & 1 \end{bmatrix}, \quad (14)$$

where $k_0 = 2\pi n_0$.

Next, we consider a minimal realization of a shaping filter, i.e., a linear dynamical system which generates the spectral density matrix $S_{zz}(k)$. Here we consider road surfaces that change spatially, so the shaping filter is expressed as a function of the distance x

$$\left. \begin{aligned} \frac{dy(x)}{dx} &= A_c y(x) + B_c w(x) \\ z(x) &= C_c y(x) \end{aligned} \right\}, \quad (15)$$

where

$$z(x) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad w(x) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Here, A_c , B_c , and C_c are the $m \times m$, $m \times 2$, and $2 \times m$ matrices,

respectively, and $\mathbf{y}(x)$ is an $m \times 1$ vector, in which m is determined later as the dimension of the minimal realization. Also, $\mathbf{w}(x)$ is a Gaussian white noise vector; its elements w_1 and w_2 are independent of each other, their mean values are both zero and their variances are both σ^2 , so that

$$\left. \begin{aligned} E[\mathbf{w}(x)] &= \mathbf{0} \\ E[\mathbf{w}(x_1)\mathbf{w}^T(x_2)] &= \sigma^2 \mathbf{I} \delta(x_1 - x_2) \end{aligned} \right\}, \quad (16)$$

where $\mathbf{I}$ denotes a 2×2 unit matrix and $\delta(\cdot)$ denotes the Dirac delta function.

The Laplace transform of Eq.(15) with the space transform variable p is

$$\mathbf{Z}(p) = \mathbf{C}_c(p\mathbf{I} - \mathbf{A}_c)^{-1} \mathbf{B}_c \mathbf{W}(p), \quad (17)$$

where $\mathbf{Z}(p)$ and $\mathbf{W}(p)$ denote Laplace transforms of $\mathbf{z}(x)$ and $\mathbf{w}(x)$.

Here, in letting the transfer function matrix be

$$\mathbf{H}(p) = \mathbf{C}_c(p\mathbf{I} - \mathbf{A}_c)^{-1} \mathbf{B}_c, \quad (18)$$

we obtain the spectral representations

$$\mathbf{S}_{zz}(k) = |\mathbf{H}(ik)|^2 \mathbf{S}_{ww}(k), \quad (19)$$

where $\mathbf{S}_{ww}(k)$ denotes the spectral density matrix of the white noise vector $\mathbf{w}(x)$, which is expressed as follows:

$$\mathbf{S}_{ww}(k) = (\sigma^2/2\pi) \mathbf{I}. \quad (20)$$

In letting the variance be

$$\sigma^2 = (2\pi)^2 A \quad (21)$$

from Eqs.(14), (19) and (20), we obtain

$$|\mathbf{H}(ik)|^2 = \frac{1}{k^2 + k_0^2} \begin{bmatrix} \frac{1}{1 + aw^2(k^2 + k_0^2)} & \frac{1}{1} \\ \frac{1}{1 + aw^2(k^2 + k_0^2)} & \frac{1}{1} \end{bmatrix}. \quad (22)$$

By the spectral factorization⁽⁵⁾ of Eq.(22), we obtain the transfer function matrix

$$\mathbf{H}(p) = \frac{1}{2(p - p_1)(p - p_2)} \begin{bmatrix} p_1 + p_3 - 2p & -p_1 + p_3 \\ -p_1 + p_3 & p_1 + p_3 - 2p \end{bmatrix}, \quad (23)$$

where

$$\left. \begin{aligned} p_1 &= -k_0 \\ p_2 &= -\sqrt{k_0^2 + 1/aw^2} \\ p_3 &= \pm \sqrt{2p_2^2 - p_1^2} \end{aligned} \right\}. \quad (24)$$

Moreover, the minimal realization of the transfer function matrix is formulated as follows⁽⁵⁾:

$$\mathbf{A}_c = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_2 \end{bmatrix}, \quad (25)$$

$$\mathbf{B}_c = \frac{1}{p_1 + p_3} \begin{bmatrix} -(p_1 + p_2) & -(p_1 + p_2) \\ p_2 - p_3 & p_1 + p_2 \\ p_1 + p_2 & p_2 - p_3 \end{bmatrix}, \quad (26)$$

$$\mathbf{C}_c = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \quad (27)$$

Consequently, the dimension of the minimal realization is determined as $m=3$. Also, p_1 and p_2 , which are the poles of the transfer function matrix, are both negative and the shaping filter is stable. Moreover, if we select the negative sign of p_3 in Eq.(24) so that the zeros of the transfer function matrix are also negative, the shaping filter is a minimum phase system.

In numerical calculation using a digital computer, discrete systems are more convenient than continuous systems. The shaping filter is expressed as discrete systems as follows:

$$\left. \begin{aligned} \mathbf{y}(i+1) &= \mathbf{A}\mathbf{y}(i) + \mathbf{B}\mathbf{w}(i) \\ \mathbf{z}(i) &= \mathbf{C}\mathbf{y}(i) \end{aligned} \right\}, \quad (28)$$

where, with the sampling interval τ ,

$$\mathbf{A} = \exp(\mathbf{A}_c \tau) = \begin{bmatrix} \exp(p_1 \tau) & 0 & 0 \\ 0 & \exp(p_2 \tau) & 0 \\ 0 & 0 & \exp(p_2 \tau) \end{bmatrix}, \quad (29)$$

$$B = \left[\int_0^\tau \exp(A_c \zeta) d\zeta \right] B_c = \frac{1}{p_1 + p_3} \begin{bmatrix} -(\exp(p_1 \tau) - 1)(p_1 + p_2)/p_1 & -(\exp(p_1 \tau) - 1)(p_1 + p_2)/p_1 \\ (\exp(p_2 \tau) - 1)(p_2 - p_3)/p_2 & (\exp(p_2 \tau) - 1)(p_1 + p_2)/p_2 \\ (\exp(p_2 \tau) - 1)(p_1 + p_2)/p_2 & (\exp(p_2 \tau) - 1)(p_2 - p_3)/p_2 \end{bmatrix}, \quad (30)$$

$$C = C_c = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad (31)$$

$$\begin{aligned} E[w(i)] &= 0 \\ E[w(i)w^t(j)] &= (\sigma^2/\tau)I\delta_{ij} = (4\pi^2 A/\tau)I\delta_{ij} \end{aligned} \quad (32)$$

and $\delta_{ij} = 0 (i \neq j)$, $\delta_{ij} = 1 (i = j)$.

4. Numerical Results

Here, the parameters used in the numerical simulation are the tread $w = 1$ m, the road roughness $A = 1.0 \times 10^{-5}$ c/m, $n_0 = 5.0 \times 10^{-3}$ c/m, the sampling interval $\tau = 5$ cm, and $a = 0.7$. The positive sign of p_3 in Eq. (24) is used. In the statistical calculations, such as the power spectrum and so on, the FFT method is applied⁽⁶⁾; here one block is composed of a series of 2 048 data and the statistical values are smoothed using the average of 100 blocks.

Figure 5 shows an example of the road profiles of

Fig. 5 An example of the roughness of parallel tracks generated by the shaping filter

Fig. 6 Result of power spectrum of a single track

the right and left tracks generated by the shaping filter with the white noise vector.

Figure 6 shows the power spectral density for the right track. Because the diagonal elements of the spectral density matrix that are the power spectral density along a single track are, by nature, the rational functions independent of the approximation of Eq.(11), the theoretical power spectral density along a single track generated by the shaping filter is

Fig. 7 Result of cross-spectral density between parallel tracks

Fig. 8 Result of coherence between parallel tracks

just as in Eq.(2). In Fig. 6 the simulated result is in good agreement with the theoretical ones. The results for the left track are as same as those for the right.

Figures 7 and 8 show the cross-spectral density and the coherence, respectively, between the right and left tracks, in which the solid lines denote the simulated results and the dotted lines denote the approximated theoretical ones. In Fig. 7, there is some unevenness of the simulated result and some differences between the simulated and theoretical results in the high-frequency domain. This is considered to be due to the lack of smoothing, and the levels of the differences are relatively small. In Figs. 7 and 8, the simulated results associated with the correlation between the right and left tracks are in good agreement with the theoretical ones.

Similar simulated results are obtained when the negative sign of p_3 is used in Eq.(24).

5. Conclusions

In this paper, a two-dimensional statistical model of an uneven road surface, where the characteristics of the roughness are isotropic, was proposed. First, a spectral density matrix was stated, considering the correlation between the right and left tracks. Second, a cross-spectral density was approximated to a rational function, and by the spectral factorization of the spectral density matrix, the transfer function matrix was derived. Thus, the minimal realization of the shaping filter of the uneven road surface was formulated. By numerical simulation, it was confirmed that an uneven road surface with the desired statistical characteristics was realized using the shaping filter.

Appendix

The modified Bessel function $K_1(x)$ is expressed as follows :

$$K_1(x) = \frac{1}{x} + \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{k!(k+1)!} [\log(x/2) - \psi(k+1)/2 - \psi(k+2)/2], \quad (33)$$

where $\psi(\)$ is the polygamma function in which γ is Euler's constant expressed as

$$\psi(k+1) = \sum_{m=1}^k (1/m) - \gamma. \quad (34)$$

From Eq.(33), $xK_1(x)$ is

$$xK_1(x) = 1 + \sum_{k=0}^{\infty} \frac{(x/2)^{2k+2}}{k!(k+1)!} [2 \log(x/2) - \psi(k+1) - \psi(k+2)]. \quad (35)$$

For $xK_1(x)$,

$$\lim_{x \rightarrow 0} xK_1(x) = 1. \quad (36)$$

When $|x|$ is large enough, the asymptotic expansion is

$$xK_1(x) \sim \sqrt{\pi x/2} e^{-x} \sum_{n=0}^{\infty} (1, n)/(2x)^n, \quad (37)$$

where $(1, n)$ is expressed by the gamma function $\Gamma(\)$

$$(1, n) = \Gamma(n+3/2)/[n! \Gamma(-n+3/2)]. \quad (38)$$

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