Forced Torsional Vibration of a Two-Degree-of-Freedom System Including a Clearance and a Two-Step-Hardening Spring*

Masumi KATAOKA**, Shinichi OHNO***
and Takanao SUGIMOTO**

The engine-clutch-gearbox train of an automobile is modeled by a forced torsional vibration system including two nonlinear terms, namely a clearance and a two-step-hardening spring. Analytical solutions of the equations of motion of the system are obtained in recurrent form. Based on the solutions, the effect of the torsional stiffness of the clutch disk on the idle rattle of the gearbox is investigated. The results are as follows: (1) The sum of torque impulses per cycle of the forced motion increases with the increase in the ratio of the second-step stiffness to the first-step stiffness. Therefore, there is an upper limit to the second-step stiffness for the purpose of the prevention of idle rattle. (2) Both the sum of torque impulses and the number of collisions decrease with the increase in the range of torsional displacement covered by the first-step stiffness.

Key Words: Theory of Vibration, Shock, Two-Degree-of-Freedom System, Forced Torsional Vibration, Backlash, Gear Noise, Clutch Disk

1. Introduction

Reducing the stiffness of the spring provided between the disk plate and hub of the clutch disk is said to be effective in decreasing the gear rattle generated at manual transmission during engine idling of an automobile[1]. However, the spring is subject to driving torque when the automobile is driven, generating an excessive relative torsional displacement unless the spring has a high stiffness. In order to meet this incompatible requirement, a two-step-hardening spring is used in actual cars. One of the characteristics of this spring is that its stiffness remains low when the torsional displacement is small, but increases if the displacement increases beyond the limit.

The authors showed, based on their experiments on actual transmission and simulation calculations, that this problem of gear rattle during engine idling can basically be treated as the forced torsional vibration of a two-degree-of-freedom system with a clearance called backlash, and tried to obtain the analytical solution of the equations of motion of the system[20-45]. However, all of the springs treated in the past reports had the torsional characteristics of the first step stiffness. In actual cars, however, the range where spring stiffness is low can not be made winder and the second step spring is sometimes actuated even during engine idling.

In this report, the authors used a two-step-hardening spring with symmetrical step-wise linear torsional characteristics and dealt with the case where the spring having a higher stiffness was also actuated; i.e., they studied the forced torsional vibrations of a two-degree-of-freedom system having a clearance and a two-step-hardening spring, and tried to obtain the analytical solutions for this system. Based on the results of simulation calculations, they investigated factors such as the stiffness ratio in the spring characteristics and the effects of other factors on gear teeth collisions.

* Received 3rd December, 1990. Paper No. 89-0959A
** Chiba Institute of Technology, 2-17-1 Tsudanuma, Narashino, Chiba 275, Japan
*** Institute of Industrial Science, University of Tokyo, 7-22-1 Roppongi, Minato-ku, Tokyo 106, Japan
2. Vibration System and Meaning of Symbols

The vibration system used in this research is shown in Fig. 1. This system is a model of the engine-clutch-gear box train of an automobile. The meanings of symbols used are shown below.

\( J_1 \): Moment of inertia of the mass of clutch hub and input gear

\( J_2 \): Moment of inertia of the mass of counter shaft gear

\( \omega' \): Average angular velocity of engine

\( \omega \): Circular frequency of angular velocity variations of engine \( \omega = 2\omega' \) in a 4-cycle 4-cylinder engine

\( k_1, k_2 \): Torsional spring constant of the first- and second-step springs provided at the clutch disk

\( k_3 \): Equivalent torsional spring constant of flexural rigidity of a pair of gear teeth contacting each other

\( K \): Spring constant ratio \( = k_3/k_1 \)

\( e \): 1/2 of the dead zone of the 2nd-step spring

\( \varepsilon \): 1/2 of the clearance (backlash) between a pair of gear teeth

\( c_1, c_2 \): Viscous damping coefficient acting on \( J_1, J_2 \)

\( X, Y \): Rotation angle of \( J_1, J_2 \)

\( Z \): Rotation angle of engine • flywheel

\( E \): Amplitude of \( Z \) variation

\( x, y, z \): Variation of \( X, Y, Z \) \( (z = E \sin \omega t) \)

\( t, t' \): Time

3. Equations of Motion

3.1 Equations

In the system in Fig. 1, the rotation angles of the moment of inertia, \( J_1 \) and, \( J_2 \), and of the engine flywheel, \( X, Y, \) and \( Z \), are expressed as the constant parts and varying ones, and can be expressed as \( X = \omega' t + x, Y = \omega' t + y \) and \( Z = \omega t + z \). When the equations of motion in terms of \( J_1 \) and \( J_2 \) are first shown using \( X, Y, \) and \( Z \), the constant parts are eliminated and the following equations of motion in terms of \( x, y \) expressing the varying parts only are obtained:

\[
\begin{align*}
J_1 \frac{d^2 x}{dt^2} + c_1 \frac{dx}{dt} + k_1 x &= k_1 E \sin \omega t \\
- G(x-z) + T(y-x) - c_2 \omega' &= 0 \\
J_2 \frac{d^2 y}{dt^2} + c_2 \frac{dy}{dt} &= - T(y-x) - c_2 \omega'
\end{align*}
\]

where the origin of time \( t \) is taken at the time when the varying part \( z \) of angular displacement is 0, and the origin of \( y \) is taken at the center of the clearance. The varying parts of the angular displacement or angular velocity will hereinafter be called simply “angular displacement” or “angular velocity”. \( G(x-z) \) represents the nonlinear restoring torque resulting from the 2nd-step spring, and \( T(y-x) \) represents the nonlinear torque caused by backlash, and their characteristics are shown in Figs. 2 (a) and (b), respectively. The following analysis will deal with the case where \( J_1 \) undergoes a periodic motion with an approximate period of \( 2\pi/\omega \) and the 2nd-step spring works once per halfcycle.

3.2 Fourier series expansion of \( G(x-z) \)

Of the two nonlinear terms included in the first equation of Eq. (1), \( G(x-z) \) is first expressed as a Fourier series.

As assumed above, \( G(x-z) \) becomes a periodic function with a period of \( 2\pi/\omega \). Since the relative displacement \( (x-z) \) is the function of time \( t \), if \( G(x-z) \) expressed in terms of nondimensional time \( \theta (= \omega t) \) is described as \( G' (\theta) \), \( G' (\theta) \) takes the form as shown in Fig. 3 (b). When the phase angle between the origin point of time \( O \) and the action start point \( Q \) of spring \( k_3 \) is taken to be \( \varphi \), \( G'(\theta)/(k_1 E) \) is expanded into a Fourier series as shown below:

\[
G(x-z)/k_1 E = G'(\theta)/k_1 E = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos m(\theta - \varphi) + b_m \sin m(\theta - \varphi)).
\]

Then the Fourier coefficients are determined. Since \( z \) is known here, if the angular displacement \( x \), when the spring \( k_3 \) is acting, is determined, the value \( G(x-z) \) can be derived. In this research, it is impos-

Fig. 1 Vibration system

Fig. 2 Nonlinear torque
sible to assume that the displacement $x$ of $J_1$ beforehand is determined because the periodicity and the number of gear teeth collisions caused by the clearance during each cycle of $J_1$'s periodic motion are unknown. Now the clearance $2\varepsilon$ between $J_1$ and $J_2$ is $\varepsilon \ll E$, and the equivalent torsional spring constant $k_2$ of the tooth of a pair of gears is $k_2 \gg k_1$, $k_2$; and therefore the vibration system in Fig. 1 can approximately be regarded as the forced torsional vibration of a one-degree-of-freedom system with $\varepsilon = 0$, $k_2 = \infty$, and the second-step spring acts on the moment of inertia $I = J_1 + J_2$. Then, a system where the viscous damping is neglected for simplification is called an approximate one-degree-of-freedom system. And in this approximate system, an angular displacement of $J$ is obtained in a region where spring $k_2$ is acting and it is considered as an approximate solution for angular displacement $x$ of $J_1$.

To avoid confusion of the symbols, the difference between the displacement of $J$ and $z$ in this approximate one-degree-of-freedom system is set as $\xi$, and the interval of the length of $\theta$ where the second-step spring acts during each halfcycle of $\xi (\xi \geq 0)$ is denoted as $\theta_b$. The following equation is effective in this division $\theta_b$:

$$G(x - z) = k_0(\xi - e)$$

(3)

In the halfcycle of $\xi (\xi \leq 0)$, on the other hand, $G(x - z)$ takes the reverse sign to the right side of the Eq. (3). Therefore, $a_0 = 0$, and the Fourier coefficients $a_m$ and $b_m$ can be determined from Eqs. (2) and (3) if $\xi$, $\theta$ and $\phi$ are determined.

3.3 Determining relative displacement $\xi$, interval of length and phase angle $\phi$

The values $\xi$, $\theta$ and $\phi$ can be determined by determining the solution for the case where $J$ in the approximate one-degree-of-freedom system is taking a periodic motion with the period of $2\pi/\omega_2$ by means of the inosculating method. Then $\xi$ is divided as shown in Fig. 3(c). In the first place, the division where the spring $k_2$ does not act ($-e \leq \xi \leq e$) is denoted as the division $PQ$, the corresponding division of $\theta$ as the division $PQ$, $\xi$ in this division as $\xi_0$, the nondimensional time as $t$, and the origin point of these factors is chosen at the middle point $O_2$ between $P$ and $Q$. Next the division where the spring $k_2$ acts ($\xi \geq e$) is denoted as $QR'$, $\xi$ in this division as $\xi_1$, and the origin point of $\xi_1$ is at $Q$. Further, the nondimensional time is taken as $t$ and the origin point of $\xi$ is taken at Q.

The equations of motion in terms of $J$ in the divisions $PQ'$ and $QR'$ are:

$$d^2\xi/dt^2 + Q^2\xi = -E\cos(\alpha + \phi - (\theta_b/2))$$

(4)

$$d^2\xi_1/dt^2 + Q^2(1 + K)\xi_1 = E\sin(\alpha + \phi) - \omega_1^2$$

(5)

The solutions of Eqs. (4) and (5) are:

$$\xi_1 = A_1 \cos \Omega_1 t + B_1 \sin \Omega_1 t$$

(6)

$$\xi_0 = A_0 \cos \Omega_2 t + B_0 \sin \Omega_2 t$$

$$+ \frac{E}{2Q(1 + K)} \sin(\alpha + \phi) - \frac{e}{1 + K}$$

(7)

where $\Omega = \omega_1/\omega_2$, $\omega_1 = \sqrt{Q/\kappa}$, $K = k_2/k_1$. $A_1$, $B_1$, $A_2$, and $B_2$ in Eqs. (6) and (7) are undetermined constants; and they can be determined together with the values $\theta_b$ and $\phi$ by the following inosculating conditions:

$$\xi_1|_{\theta \to \pi/2} = -\xi_0$$

(8)

$$\xi_0|_{\theta \to \pi/2} = -\xi_0$$

(9)

$$\frac{d\xi_0}{dt}|_{\theta \to \pi/2} = \frac{d\xi_1}{dt}|_{\theta \to \pi/2}$$

(10)

$$\frac{d^2\xi_0}{dt^2}|_{\theta \to \pi/2} = \frac{d^2\xi_1}{dt^2}|_{\theta \to \pi/2}$$

(11)

$$\xi_0|_{\theta = 0} = 0$$

(12)

$$\xi_1|_{\theta = 0} = 0$$

(13)

Equation (6) is substituted into Eqs. (8) and (9), and both equations are added and subtracted side by side to determine $A_1$ and $B_1$ as functions of the unknown values $\theta_b$ and $\phi$. Likewise, Eqs. (7) is substituted into Eqs. (12) and (13) to determine $A_2$ and $B_2$ as functions of $\theta_b$ and $\phi$. $A_1$, $B_1$, $A_2$, and $B_2$ and the equations resulted from differentiating Eqs. (6) and (7) with $\tau$ and $\sigma$ are substituted into Eqs. (10) and
(11) to obtain the following two equations by some calculations:

\[
\left( \frac{M_s}{M_s N_1 - M_s N_2} \right)^2 = 1
\]

\[\text{tan } \varphi = -\frac{N_1}{N_2}\]

provided that:

\[
M_s/E = \frac{2\Omega e/E}{\sqrt{1 + K}} \left[ \frac{\sqrt{1 + K}}{\tan \left( \frac{\pi}{2} - \frac{\delta_k}{2} \right)} - \frac{1 - \cos \sqrt{1 + K} \cdot \delta_k}{\sin \sqrt{1 + K} \cdot \delta_k} \right]
\]

\[
M_s/E = \frac{\Omega}{(\Omega^2 - 1) \sin \left( \frac{\pi}{2} - \frac{\delta_k}{2} \right)} - \frac{K \Omega^2 (1 - \cos \delta_k)}{(\Omega^2 - 1)(\sqrt{1 + K} - 1)} - \frac{\Omega^2 (1 + K)}{(2 + \sqrt{1 + K})} \times \frac{\sin \delta_k (1 - \cos \sqrt{1 + K} \cdot \delta_k)}{\sin \sqrt{1 + K} \cdot \delta_k}
\]

\[
M_s/E = \frac{Q^2 K \sin \delta_k}{(\Omega^2 - 1)(\sqrt{1 + K} - 1)} + \frac{\Omega}{(\Omega^2 - 1)} \times \frac{1 + \cos \delta_k}{\tan \left( \frac{\pi}{2} - \frac{\delta_k}{2} \right)} + \frac{\Omega \sqrt{1 + K}}{(2 + \sqrt{1 + K})} \times (1 + \cos \delta_k)(\cos \sqrt{1 + K} \cdot \delta_k - 1)
\]

\[
N_s/E = \frac{\Omega^2 K \sin \delta_k}{(\Omega^2 - 1)(\sqrt{1 + K} - 1)} + \frac{Q}{(\Omega^2 - 1)} \times \frac{\sin \delta_k (1 + \cos \sqrt{1 + K} \cdot \delta_k)}{\sin \sqrt{1 + K} \cdot \delta_k}
\]

\[
N_s/E = \frac{Q^2 K \sin \delta_k}{(\Omega^2 - 1)(\sqrt{1 + K} - 1)} + \frac{Q (1 + \cos \delta_k)}{(\Omega^2 - 1)} \sin \left( \frac{\pi}{2} - \frac{\delta_k}{2} \right) + \frac{Q \sqrt{1 + K}}{(2 + \sqrt{1 + K})} \times \frac{(1 - \cos \delta_k)(1 + \cos \sqrt{1 + K} \cdot \delta_k)}{\sin \sqrt{1 + K} \cdot \delta_k}
\]

Since Eq. (14) is a simple transcendental equation in terms of \(\delta_k\), the value \(\delta_k\) can be determined by solving this equation, and \(\varphi\) is determined by substituting the value \(\delta_k\) into Eq. (15). Then these values \(\delta_k\) and \(\varphi\), and \(A_2\) and \(B_2\) calculated by using them are substituted into Eq. (7) to determine \(\xi_a\).

### 3.4 Determining Fourier coefficients \(a_n\) and \(b_n\)

Considering that the right hand side \(G(x-z)\) of Eq. (2) is a function such that, for the one-cycle \(P = 2\pi\), i.e., \(2\pi\) of \(\theta, \theta = \Theta\) in the QR division, \(\Theta = \Theta\) in the ST division, and \(0\) in other divisions, Fourier coefficients \(a_n\) and \(b_n\) can be determined as follows by the general method of determining a Fourier coefficient by integrating both sides of Eq. (2) over one cycle:

\[
a_{n=1} = \frac{2K}{\pi} \left[ \frac{e/E}{1 + K} - \frac{1}{Q^2 (1 + K)^{-1}} \right] \times \left[ \frac{(Q^2 (1 + K)^{1} - (2m-1)^{2})}{(Q^2 (1 + K)^{-1})(2m-1)^{2}} \right] \times \cos \sqrt{1 + K} \cdot \delta_k \cdot \cos (2m-1)\delta_k \]
\]

\[
\left[ \frac{1}{Q^2 (1 + K)^{-1}} \right] \left[ \frac{1}{sin \sqrt{1 + K} \cdot \delta_k} \left( \cos \sqrt{1 + K} \cdot \delta_k \left( \frac{sin \varphi}{Q^2 (1 + K)^{-1}} - \frac{e/E}{1 + K} \right) \right) \right]
\]

\[
\left[ \frac{2m}{(2m-1)^{2}} \right] \times \cos \sqrt{1 + K} \cdot \delta_k \cdot \cos (2m-1)\delta_k \cdot \sin (2m-1)\delta_k - \sin (2m-1)\delta_k - \sin (2m-1)\delta_k - \sin (2m-1)\delta_k
\]

\[
b_{n=1} = \frac{2K}{\pi} \left[ \frac{e/E}{1 + K} - \frac{1}{Q^2 (1 + K)^{-1}} \right] \left[ \frac{(Q^2 (1 + K)^{1} - (2m-1)^{2})}{(Q^2 (1 + K)^{-1})(2m-1)^{2}} \right] \times \cos \sqrt{1 + K} \cdot \delta_k \cdot \cos (2m-1)\delta_k
\]

\[
\left[ \frac{1}{sin \sqrt{1 + K} \cdot \delta_k} \right] \left[ \frac{1}{(Q^2 (1 + K)^{-1})} \right] \left[ \frac{1}{sin \sqrt{1 + K} \cdot \delta_k} \left( \cos \sqrt{1 + K} \cdot \delta_k \left( \frac{sin \varphi}{Q^2 (1 + K)^{-1}} - \frac{e/E}{1 + K} \right) \right) \right]
\]

\[
\left[ \frac{2m}{(2m-1)^{2}} \right] \times \cos \sqrt{1 + K} \cdot \delta_k \cdot \cos (2m-1)\delta_k \cdot \sin (2m-1)\delta_k - \sin (2m-1)\delta_k - \sin (2m-1)\delta_k - \sin (2m-1)\delta_k
\]

\[
\left[ \frac{2m}{(2m-1)^{2}} \right] \times \cos \sqrt{1 + K} \cdot \delta_k \cdot \cos (2m-1)\delta_k \cdot \sin (2m-1)\delta_k - \sin (2m-1)\delta_k - \sin (2m-1)\delta_k - \sin (2m-1)\delta_k
\]
\[ + \sin \theta_0 \sin (2m-1) \delta_0 - (2m-1) \]
\[ + \left\{ \frac{e^E}{1 + K'} \right\} \cos (2m-1) \delta_0 - (2m-1) \]. \] (17)

4. Solution of the Equations of Motion and Torque Impulse

4.1 Solution of two one-degree-of-freedom systems after decomposition of the system

Let us return to Eq. (1). Of the two nonlinear terms, \( G(x-z) \) has been linearized. The remaining \( T(y-x) \) is an impulsive exciting torque which has a large value only at the time of collisions between \( J_1 \) and \( J_2 \) and remains as 0 at other times. Since, in the collision vibrations in this research, the times of \( J_1 \) and \( J_2 \) collisions in an excitation cycle are unknown as mentioned above, it is impossible to assume the displacement of \( J_1 \) and \( J_2 \), etc., in advance. Thus, we try to obtain the solution by regression equations utilizing Duhamel's integral method as in our previous reports\(^{(a),(b)}\).

Let us use the symbol \( t \) to express the time when the exciting torque acts and denote the exciting torque acting on \( J_1 \) and \( J_2 \) as \( F(t') \); then Fig. 1 can formally be divided into two one-degree-of-freedom systems as shown in Fig. 4 (a) and (b). If the exciting torque in terms of the nondimensional time \( \theta' = \omega t' \) is assumed to be \( f(\theta') = F(\theta'/\omega) / J_1 \), this exciting torque will cause the unit-moment of inertia to generate the momentary angular velocity change equivalent to:

\[ dx = f(\theta') \omega dt' / J_1 \] (18)

provided \( \cdot = d/d\theta \).

Now in the one-degree-of-freedom system in Fig. 4, the solutions \( x_0 \) and \( x_1 \) of the equations of motion of \( J_1 \) and \( J_2 \), when the exciting torque \( F(t') \) does not act, are (the initial displacement and initial velocity: \( x_0, y_0, x_0', y_0' \)):

\[ x_1 = e^{-\omega t} \left\{ \left( A' + \frac{\lambda x_0}{\sqrt{1 - \gamma^2}} \right) \sin \theta + D \cos \theta \right\} \]

\[ + B \sin (\theta + \delta) - E \sum_{m=1}^{\infty} \frac{1}{\psi_{2m-1}} (a_{2m-1} C \theta - b_{2m-1} S \theta) \frac{1}{\sqrt{1 - \gamma^2}} \] (19)

\[ y_1 = (1 - e^{-\omega t}) \left\{ \frac{\lambda y_0 + (1/2)}{c} \right\} + y_0 - (\theta/2) \] (20)

where

\[ \omega = \sqrt{b_1 J_1}, \lambda = \omega = \omega_0 = 1/k, \gamma = c_1 / 2 \sqrt{f_1} \]

\[ q = x_0 \sqrt{1 - \gamma^2}, c = c_1 J_1 \omega \]

\[ \delta = \tan^{-1} \left\{ -2 \gamma / (1 - \lambda^2) \right\} \]

\[ \tan P_{2m-1} = 2 \gamma (2m-1) / (1 - \lambda^2) \]

\[ \phi_{2m-1} = \sqrt{1 - \lambda^2 (2m-1)^2} + (2m-1) \lambda^2 \]

\[ A' = \gamma (x_0 - B \sin (\delta + \gamma) - AB \cos (\delta) / \sqrt{1 - \gamma^2} \]

\[ + \frac{E}{\sqrt{1 - \gamma^2}} \sum_{m=1}^{\infty} \frac{1}{\psi_{2m-1}} (a_{2m-1} C \theta - b_{2m-1} S \theta) \]

\[ - \lambda (2m-1) B \sin (\delta) - C \theta (2m-1) C \theta \]

\[ B = E / \sqrt{1 - \lambda^2} + (2m-1) \lambda^2 \]

\[ D = x_0 - B \sin (\delta + \gamma) \]

\[ + \frac{E}{\sqrt{1 - \gamma^2}} \sum_{m=1}^{\infty} (a_{2m-1} C \theta + b_{2m-1} S \theta) \]

\[ S(\theta) = \sin ((2m-1)(\theta - \phi) - P_{2m-1}) \]

\[ C(\theta) = \cos ((2m-1)(\theta - \phi) - P_{2m-1}) \]

Then, the system motion, when the exciting torque \( f(\theta') \) acts, can be determined as follows. The velocity increment of Eq. (18) is regarded as the initial velocity at the time \( \theta' \) and Eq. (19) is used to determine the increment of \( J_1 \) displacement at an arbitrary time \( \theta \) later than the time \( \theta' \) as below:

\[ dx = \lambda f(\theta') e^{-\omega t} / \sqrt{1 - \gamma^2} \sin q(\theta - \theta') \cdot d\theta' \] (21)

Likewise, the increment of \( J_2 \) displacement is

\[ dy = \frac{f(\theta')}{\omega c} (1 - e^{-\omega t}) \sin \theta \cdot d\theta' \] (22)

Therefore, the angular displacement increment caused by the exciting torque \( f(\theta') \) acting between time \( \theta' = 0 \) and \( \theta' = \theta \) can be obtained by integrating Eqs. (21) and (22) over the period from \( \theta' = 0 \) to \( \theta' = \theta' \).

4.2 Motion of \( J_1 \) and \( J_2 \) after \( S \) times of collision

The shape of the exciting torque occurring at the time of \( J_1 \) and \( J_2 \) collision is expressed by triangular waves; thus, the following nondimensional triangular waveform \( f'(\theta') \) centering at the time \( \tilde{\theta} \) is introduced:

\[ f'(\theta') = \begin{cases} 0 & \text{if } 0 < \tilde{\theta} - 1/n \\ 1/n & \text{if } 1/n \leq \tilde{\theta} < \tilde{\theta} + 1/n \\ 0 & \text{if } \tilde{\theta} + 1/n \leq \theta' \end{cases} \] (23)

where \( n \) is a constant relating to the sharpness of triangular waveform. If the first collision occurs at the time \( \tilde{\theta} \), the exciting torque \( f' \) acting between \( J_1 \) and \( J_2 \) will become as shown below, with the use of Eq. (23):

---

Fig. 4 Decomposition of the system

JSME International Journal

\[ f(\theta') = I_0 \omega f(\theta') J_f \]

where \( I_0 \) and \( \theta_0 \) are constants which are now undetermined but will be determined later. The value \( I_0 \) is called the torque impulse in this paper; it is a physical quantity corresponding to the impulse of linear motion. The displacement increment of \( J_1 \) caused by the first collision becomes as shown below by substituting Eq. (21) into Eqs. (23) and (24) and integrating them over the time from \( \theta' = 0 \) to \( \theta' = \theta' \):

\[
x = \frac{H_n}{J_f \omega \sqrt{1 - \gamma^2}} \int_{\theta' = 0}^{\theta'} e^{-\gamma \theta'} \left( n + n^2 \theta_0 \right) \sin \mu \theta' d\theta' + \int_{\theta' = 0}^{\theta'} e^{-\gamma \theta'} \left( n + n^2 \theta_0 \right) \sin \mu \theta' d\theta' \quad (25)
\]

Now, if the first collision is assumed to have occurred at the rear part of the clearance of \( J_1 \), for convenience' sake, the angular displacement of \( J_1 \) after the first collision is determined by adding Eq. (19) to Eq. (25). Here the rear part of clearance means the rear face of the tooth of the input gear relative to the gear rotation direction for an actual car. Further, the angular displacement of \( J_1 \) can likewise be obtained. Therefore, the motions of \( J_1 \) and \( J_2 \) after an \( S \) number of collisions can be obtained by adding the angular displacement increments caused by an \( S \) number of collisions. If \( \rho_i \) is defined as a constant which is \( \rho_i = 1, 2, 3, \ldots, s \) for a collision at the rear part of clearance and \( \rho_i = -1 \) for a collision at the front part of clearance, the motions of \( J_1 \) and \( J_2 \) after an \( S \) number of collisions are expressed as below:

\[
x = e^{-\gamma t} (A \sin \theta + D \cos \theta) + B \sin \theta - \gamma E \sum_{\theta = 1}^{S_{m-1}} (a_{m-1}(\theta) e_{m-1}^{(k)} + f_{m-1}^{(k)}) [F_i((\gamma x)^2 + q^2) \sin \theta - \beta_0]) + 2\gamma x q \cos \theta - \beta_0])
\]

\[
y = (e^{-\gamma t}) (y_0 + (1/2) c + y_0 - (\theta')) c - \frac{1}{c} \omega J \sum_{\theta = 1}^{S_{m-1}} (a_{m-1}(\theta) e_{m-1}^{(k)} + f_{m-1}^{(k)}) \left[ 1 - n^2 \frac{\gamma^2}{c^2} (e^{-\gamma t} + e^{\gamma t} - 2) \right]
\]

provided that:

\[
A = A' + \lambda \omega a / \sqrt{1 - \gamma^2}
\]

\[
F_i = (e^{-\gamma t} + e^{\gamma t}) \cos \theta - \beta_0)
\]

\[
F_i = (e^{-\gamma t} - e^{\gamma t}) \sin \theta - \beta_0)
\]

4.3 Determining \( \theta_s \), \( \theta_s \), and \( \theta_{s+1} \)

For determining the time \( \theta_s \) when the \( S \)th collision occurs, the equation below comes from the relation between \( J_1 \) and \( J_2 \) angular displacements immediately before the \( S \)th collision:

\[
y = \theta_{s-1} + \theta_{s-1} \cdot \theta_{s-1} = \theta_{s-1} \cdot \theta_{s-1} = \theta_{s-1} \cdot \theta_{s-1}
\]

By substituting the displacements \( x \) and \( y \) immediately before the \( S \)th collision obtained from Eqs. (26) and (27) into Eq. (28), the equation below is established:

\[
(1 - e^{-\gamma t})(y_0 + (1/2) c + y_0 - (\theta')) c - \frac{1}{c} \omega J \sum_{\theta = 1}^{S_{m-1}} (a_{m-1}(\theta) e_{m-1}^{(k)} + f_{m-1}^{(k)}) \left[ 1 - n^2 \frac{\gamma^2}{c^2} (e^{-\gamma t} + e^{\gamma t} - 2) \right] = \epsilon \rho_1
\]

\[
+ e^{-\gamma t} (A \sin \theta + D \cos \theta) + B \sin \theta - \gamma E \sum_{\theta = 1}^{S_{m-1}} (a_{m-1}(\theta) + f_{m-1}^{(k)}) [F_i((\gamma x)^2 + q^2) \sin \theta - \beta_0]) + 2\gamma x q \cos \theta - \beta_0])
\]

\[
+ \frac{H_n}{(1 - \gamma^2)^{1/2}} \sum_{\theta = 1}^{S_{m-1}} (a_{m-1}(\theta) e_{m-1}^{(k)} + f_{m-1}^{(k)}) [F_i((\gamma x)^2 + q^2) \sin \theta - \beta_0]) + 2\gamma x q \cos \theta - \beta_0])
\]

The time \( \theta_s \) when the \( S \)th collision occurs can be determined by solving Eq. (29). Next, by substituting the results obtained by differentiating once Eqs. (26) and (27) with \( \theta \) for the defining equation of rebounding coefficient:

\[
-\epsilon_0 = (\dot{y}, \dot{\theta}, \dot{\theta}) / (y, \theta, \theta)
\]

\( I_s \) can be determined as follows:

\[
I_s = (1 + \epsilon_0) \left[ (y_0 + (1/2) c + y_0 - (\theta'), c - \frac{1}{c} \omega J \sum_{\theta = 1}^{S_{m-1}} (a_{m-1}(\theta) e_{m-1}^{(k)} + f_{m-1}^{(k)}) \left[ 1 - n^2 \frac{\gamma^2}{c^2} (e^{-\gamma t} + e^{\gamma t} - 2) \right]
\]

\[\times ((qA - \gamma D) \cos \theta + \gamma (A + qD) \sin \theta) B \cos (\beta_0) - 2E \sum_{\theta = 1}^{S_{m-1}} (a_{m-1}(\theta) + f_{m-1}^{(k)}) \left[ F_i((\gamma x)^2 + q^2) \cos \theta - \beta_0]) + 2\gamma x q \cos \theta - \beta_0]
\]

\[\times (a_{m-1}(\theta) - b_{m-1}(\theta)) \left[ \frac{H_n}{(1 - \gamma^2)^{1/2}} \sum_{\theta = 1}^{S_{m-1}} (a_{m-1}(\theta) e_{m-1}^{(k)} + f_{m-1}^{(k)}) [F_i((\gamma x)^2 + q^2) \sin \theta - \beta_0]) + 2\gamma x q \cos \theta - \beta_0)
\]

\[\times \cos \theta - \beta_0])) [\rho_1((e^{-\gamma t} + e^{\gamma t} - 2) / c^3 - H_n(F_1 + F_2 \gamma x / ((\gamma x)^2 + q^2))]
\]


JSME International Journal
Next, the value $\rho_s+1$, when the $(S+1)$th collision occurs, can be obtained by comparing the displacement positions of $J_1$ and $J_2$ some time after the $5$th collision. This procedure was already mentioned in the authors' previous reports$^{34(4)}$ and explained only briefly. If the time for comparison is taken to be the time when $J_1$'s acceleration first becomes 0 after $S$ times of collisions and is represented by $\theta^*_s$, the $(S+1)$th collision will occur at:

Rear part of clearance if $y(\theta^*_s) < y(\theta^*_s)< y(\theta^*_s)$

Front part of clearance if $y(\theta^*_s) > y(\theta^*_s)$

In the following case, on the other hand:

$-\varepsilon + x(\theta^*_s) < y(\theta^*_s) < -\varepsilon + x(\theta^*_s)$  (32)

$J_2$ is positioned halfway in the clearance, so that the $J_1$ and $J_2$ displacement positions at more time after the $5$th collision must be compared. Then, a tangential line is drawn from the point $y(\theta^*_s)$ to the curve $e^*\rho_s + x(\theta)$, and the time for the contact point is regarded as the time $\theta^*_s$ for comparison. Then return to Eq. (23) and repeat the above procedure for the rest. If the inclination of the tangential line is reversed during this process, the $(S+1)$th collision will occur at the surface opposite to that where the $S$th collision occurred, and $\rho_s + 1 = -\rho_s$.

4.4 Calculation procedure

The motions of $J_1$ and $J_2$ and torque impulse $I_1$ can be determined by the regression equations (26), (27) and (31). The necessary procedure is first to obtain $6$ and $\varphi$ from Eqs. (14) and (15) and substitute these values into the Eqs. (16) and (17) to determine $\omega_{m+1}$ and $\beta_{m+1}$. Next, set up the initial conditions $c_0$, $x_0$, $\dot{x}_0$ and $\ddot{y}_0$ so that the first collision may occur at the rear surface of clearance for convenience' sake, and thus $\rho_s = 1$. Setting may be done so that the collision mentioned above may occur at the front surface. From this, the values $\theta_i$ and $I_i$ are determined from Eqs. (29) and (31). The position of the $2$nd collision is judged from Eqs. (32) and (33). This procedure is repeated from the start for analyzing the motion of subsequent collision.

5. Analysis Results

5.1 Elements of vibration system

The values commonly used in all calculations were $J_1 = 2.27 \times 10^{-4} \text{kg} \cdot \text{m}^2$, $J_2 = 1.12 \times 10^{-3} \text{kg} \cdot \text{m}^2$, $E = 0.0256 \text{rad}$, $\omega_0 = 0.91$ and $n = 70$. They are the same as those used in the previous reports$^{34(4)}$, and $\omega_0$ was a measured value. The value of $n$ was obtained by nondimensioning the average contact time which was obtained from both experiments and simulations and equating it with $2/n$. Furthermore, we assumed the number of terms $(m)$ of Fourier series as 20.

5.2 Vibration waveform

Figure 5 shows an example of the waveforms determined from the analytical solution, in which $e/E = 1.0$, $E/E = 0.03$, $\omega/\omega_1 = 1.9$, $K = 1.0$, $\omega' = 20 \pi \text{ rad/s}$, $k_1 = 16.8 \text{ N} \cdot \text{m/rad}$, and $E = 0.0256 \text{ rad}$.

Figures 5(a) and (b) show the angular velocity waveforms of $J_1$ and $J_2$, respectively, and (c) shows the angular acceleration waveform of $J_1$. Figure 5(d) shows the waveform of the nonlinear torque $G^*(\theta)$ caused by the action of spring $k_2$. As is clear from the directions of the increasing and decreasing angular acceleration of pulseline waveforms generated through collisions, $J_1$ is found to have repeated the collision many times at the rear surface of clearance for about a half of the cycle when $J_1$ acceleration waveform is increasing, and at the front surface of clearance for about a half of the cycle when it is decreasing. In other words, in the case of a two-step hardening spring, the collision conditions at the clearance are the same as those in the case of the one-step hardening spring mentioned in the previous reports$^{34(6)}$.

Figure 5(e) shows torque impulse I, where $\rho_s \cdot I_1$ is expressed as I in the vertical axis for convenience' sake. Figure 6 shows an example of the simulation calculations by the Runge-Kutta method in Eq. (1), where the main elements are the same as in Fig. 5, and (a) ~ (d) in Fig. 6 correspond to those in Fig. 5.
When Figs. 5 and 6 are compared with each other, waveforms slightly differ in (d) of both figures. In Fig. 6(d), the waveform in the range where $G'(\theta) \geq 0$ is somewhat smaller than the waveform in the range where $G'(\theta) \leq 0$, while in Fig. 5(d), they are equal in both ranges. The reason for this is that in Fig. 5, an approximate one-degree-of-freedom system which disregarded collision and viscous damping was adopted, and the nonlinear torque caused by spring $k_1$ acting in this system was used as the approximate of $G'(\theta)$. However, other waveforms by analytical solutions can be said to agree well with those by simulation calculations.

5.3 Relations between the total of torque impulse and spring constant ratio $K$, and collision frequency

Figure 7 shows the evaluation of the total of torque impulse per cycle with respect to the ratio of the spring constant $k_2$ of the second-step spring to the spring constant $k_1$ of the first-step spring ($\varepsilon/E = 1.0$, $\varepsilon/E = 0.03$). Here the total of torque impulse is $\sum |I|$ in the analytical solution and $2\pi/\omega \int_0^T |\int \ddot{y}(t)dt|^2 dt$ in the simulation solution. The two have the same physical meaning based on Eq. (24). In the calculations of this paper, it is $T = 6l/\omega$. Figure 8 shows the relations between the spring constant ratio $K$ and collision frequency per cycle. It can be seen from Figs. 7 and 8 that the collision frequency gradually increases with the increase of $K$, resulting in an increase of the total magnitude of torque impulse. As the spring in this report is a two-step-hardening spring, when $\omega/\omega_1$ is assigned to the horizontal axis and $\xi$ at the vertical axis and the resonance curve is drawn for the approximate one-degree-of-freedom system, the backbone curve will be inclined to ward the side with a larger $\omega/\omega_1$. The inclination becomes larger as the parameter $K$ increases. Then, in the range of $\omega/\omega_1 > 1$, the system comes nearer to the resonance condition as $K$ increases, even if $\omega/\omega_1$ is constant, and the $J_1$ and $J_2$ amplitude increases. As a result, it is considered that $J_1$ and $J_2$ collision frequency increases during the velocity increase and decrease process of $J_1$, as shown in Fig. 8. Also, the solid lines in Fig. 7 show the condition where the system has gone beyond the resonance point, while the broken lines show that the system has yet to reach the resonance point.
Fig. 9 Relation between the dead zone of the second-step spring and the total of torque impulse

Fig. 10 Relation between the dead zone of the second-step spring and collision frequency

5.4 Relations between $e/E$ and the total of torque impulse and collision frequency

Figures 9 and 10 show an example of the relations between $e/E$ and the total of torque impulse, and the collision frequency when $\omega/\omega_1=2.2$ and $e/E=0.03$ as obtained from the analytical solution. It can be seen from the figure that the collision frequency decreases by increasing $e/E$; and as a result the total of torque impulse also decreases. Further we should add that the sum of torque impulse/cycle and also the number of collisions/cycle takes almost the same values at $e/E=1.2$ and its vicinity for various values of $K$. Because if $e/E$ is larger than this value, the relative amplitude $x-2$ will enter the dead zone where the second-step spring does not act.

5.5 Relations between clearance size, total of torque impulse and collision frequency

Figure 11 shows the relations between the clearance size as obtained from the analytical solution when $e/E=0.1$, $\omega/\omega_1=2.2$, and the total of torque impulse. It can be seen from the figure that the total of torque impulse change little even if the clearance size changes. However, unlike the case where $e/E$ was changed, the collision frequency decreases as the clearance size increases, so that the torque impulse per collision will increase. This result agrees with the result of analyzing the characteristics of the one-step hardening spring. It can therefore be said that the difference of spring characteristics does not affect the relations between the clearance size and total of torque impulse.

6. Conclusions

With regard to the forced torsional vibrations of a two-degree-of-freedom system with clearance and a two-step hardening spring, the case where the spring had symmetrical stepwise linear characteristics was selected, and its analytical solution was obtained. As a result of comparing this solution with the simulation solution, the authors have reached the following conclusions.

(1) When the spring constant ratio $k=k_2/k_1$ is increased, the system comes near to the resonance conditions and the total of torque impulse rapidly increases. Therefore, the spring constant $k_2$ of the second-step spring should be as large as possible for the purpose of preventing any excessive relative torsional displacement in the clutch disk, but it cannot be enlarged excessively; otherwise the gears will rattle while the engine is idling.

(2) When the dead zone $e$ of the second-step spring is increased, the total of torque impulse and collision frequency decreases. In other words, the operation range of the first-step spring should be made as wide as possible, even if the second-step spring is operated, for the purpose of preventing the gear rattle at the time of engine idling.

(3) The total of torque impulse change little even if the clearance (backlash) size is changed.

References

(1) For example, Chikaya, Fujimoto and Kojima: Reduction of Transmission Idle Rattles by Clutch Disk, Society of Automotive Engineers of Japan,


