Constrained Lagrangian Formulation for Multibody Systems using Bond Graphs*

Yehia A. KHULIEF**

A method for deriving the constrained form of Lagrangian equations of motion for an interconnected system of rigid bodies using bond graphs is presented. Intrinsically, this approach allows the formulation of the final form of equations for holonomic systems without recourse to the Lagrangian function. Consequently, the burdens of deriving the expressions for kinetic and potential energies, and performing the necessary differentiations have been eliminated. This method calls only for constructing the Jacobian matrix of constraints, and then employing a bond graph that accounts for the generalized constraint reaction forces.

Key Words : Multibody, Lagrangian, Bond graph, Constrained

1. Introduction

The derivation of equations of motion for a constrained mechanical system is an indispensable step in dynamic analysis and control. Lagrange's equations provide a standard and convenient form of equations of motion. Moreover, the constrained Lagrangian form allows the use of a set of Lagrangian multipliers that account for constraint reaction forces. In this context, the system's configuration is described by a set of n generalized coordinates. The dimension of such a set of generalized coordinates exceeds the number of degrees of freedom by m , where m is the number of independent constraints. The term **generalized coordinates** used in conjunction with this Lagrangian analysis refers to any set of configuration coordinates which are not necessarily independent. Quite often, this procedure produces a set of equations which are simpler than those obtained by embedding

the constraints. Also, the symmetry of the problem is preserved since there are no preferred coordinates while others are eliminated, Ref. (1). In addition, of course, one obtains reaction forces in the process of the solution.

Bond-graph dynamic modeling can be very efficient in generating equations for digital simulation. The bond graph theory provides a graphical representation of power flow in physical systems, wherein power flow is expressed by the product of two generalized variables; e (effort) and f (flow). For a complete account of the bond graph theory, the reader is referred to Ref. (2). A recent update of bond graph bibliography can be found in Ref. (3). The principal emphasis of this paper, however, will be focused on the use of bond graph theory for generating the constrained Lagrangian equations of motion.

Standard bond graph techniques, when used to derive equations of motion of a constrained mechanical system, result in a system of equations with embedded constraints. That is, the system's configuration is described by a set of independent coordinates with a dimension equal to the number of degrees of freedom. In addition to the highly nonlinear nature of this

* Received 25th June, 1990

** Department of Mechanical Engineering, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia

form, extracting equations from the bond graph, in this case, is cumbersome. This is simply due to the fact that the m independent constraints will give rise to an equal number of inertia derivative causalities in the bond graph model.

In order to alleviate this difficulty in deriving the equations of motion of constrained systems using bond graphs, Karnopp (4) elaborated a scheme to derive the Lagrangian function from the bond graph. In his approach, bond graphs were utilized in setting up the kinetic energy and coenergy functions for use in Lagrange's equations. In Ref. (5), Karnopp proved that the generalized forces computed according to Lagrange's equations are identical to those computed directly from bond graphs with multiport transformer representations. Previously, Brown (6) demonstrated the possibility of constructing a bond graph consistent with a given Lagrangian. While the studies presented in Refs. (4)~(6) emphasize the Lagrangian features of bond graphs, Lagrange multipliers, however, were not considered. Therefore the convenient Lagrangian approach has been interfaced with bond graphs in the absence of an expression for constraint reaction forces.

In this paper, the constrained form of Lagrange's equations is derived directly from bond graphs. This approach does not require the formulation of the Lagrangian function. Therefore, one can avoid all of the algebraic and differentiation operations involved in setting up the Lagrangian function, and evaluating the associated derivatives, respectively. Moreover, the application of this modeling scheme results in a bond graph with a more convenient causal assignment. In the next section, we shall emphasize how the developed scheme automatically converts inertia derivative causalities into integral causalities.

Nomenclature

b	: damping coefficient
C	: constraint function
DOF	: degrees of freedom
e	: bond graph effort signal
f	: bond graph flow signal
g	: gravity constant
J	: Jacobian matrix of constraints
K	: stiffness coefficient
L	: length
MTF	: displacement modulated transformer
p	: vector of generalized momenta
q	: vector of generalized coordinates
S	: bond graph source
T	: kinetic energy
Q	: vector of generalized forces
λ	: vector of Lagrange multipliers

θ : rotational coordinate

Superscript

T : matrix transpose

2. Lagrange's Equations of Motion

A multibody system with frictionless constraints which may be holonomic or nonholonomic can be modeled by the general form of Lagrange's equations as

$$\frac{d}{dt} \left\{ \frac{\partial T}{\partial \dot{q}} \right\}^T - \left\{ \frac{\partial T}{\partial q} \right\}^T = \bar{Q} + J^T \lambda \quad (1)$$

where T is kinetic energy, q is an n -vector of generalized coordinates, and $\bar{Q}$ is an n -vector that contains all conservative and nonconservative forces acting on the system. The term $J^T \lambda$ represents the generalized constraint reaction forces, where λ is an m -vector of Lagrange multipliers. Equation (1) can be written in the form

$$\dot{p} - Q = J^T \lambda \quad (2)$$

where p is a vector of generalized momenta, and Q contains the force vector $\bar{Q}$ as well as forces resulting from gyroscopic and Coriolis terms.

To illustrate the difficulty in using standard bond graph techniques for deriving the equations of a multibody system, consider the simplified case of a system with a set of holonomic constraints in the form

$$C(q_1, q_2, \dots, q_n) = 0 \quad (3)$$

where C is an m -vector function of constraints, and $\left(\frac{\partial C}{\partial q} \right) = J$ is the $(m \times n)$ Jacobian matrix of constraints. The system described by Eqs. (2) and (3) possesses k degrees of freedom, where $k = (n - m)$. Therefore, it is possible to express the system dynamics by a set of k independent coordinates, thus obtaining a model of k differential equations of motion. To appreciate the complexity of this later model, let us eliminate Lagrange multipliers from Eq. (2). For an inertia-invariant mechanical system, one can partition Eq. (2) as

$$\begin{Bmatrix} \dot{p}_i \\ \dot{p}_d \end{Bmatrix} - \begin{Bmatrix} Q_i \\ Q_d \end{Bmatrix} = \begin{Bmatrix} J_i^T \\ J_d^T \end{Bmatrix} \lambda \quad (4)$$

where p_i is a k -vector of independent generalized momenta, and p_d is an m -vector of dependent generalized momenta. This partitioning, however, is made according to a selected set of k independent generalized coordinates. Therefore, the Jacobian submatrix J_d will have a full rank m . Conveniently, Eq. (4) can be written as two sets of equations in terms of independent and dependent coordinates, respectively. These are

$$\dot{p}_i - Q_i = J_i^T \lambda \quad (5)$$

and

$$\dot{p}_d - Q_d = J_d^T \lambda \quad (6)$$

Because J_d has a full-rank, then J_d^{-1} exists, and one

can solve Eq. (6) for λ to get

$$\lambda = (J_d^T)^{-1} \dot{p}_d - (J_d^T)^{-1} Q_d \quad (7)$$

Substituting λ into Eq. (5), one obtains

$$\dot{p}_i - Q_i = A \dot{p}_d - A Q_d \quad (8)$$

where A is a $(k \times m)$ matrix given by $J^T (J_d^T)^{-1}$. In order to eliminate $\dot{p}_d$ from Eq. (8), one can utilize the differentiated form of the constraint Eq. (3), together with the partitioned inertia matrix, and write the following expression

$$\dot{p}_d = B \dot{p}_i + E \quad (9)$$

where B is the $(m \times k)$ coefficient matrix, and E is the $(m \times 1)$ vector that contains quadratic velocity terms. Now, the final form of equations of motion with embedded constraints may be written as

$$(I - AB) \dot{p}_i - (Q_i - A Q_d) - AE = 0. \quad (10)$$

Although the model represented by Eq. (10) includes only k unknowns, it does, however, require appreciably more algebraic manipulations as manifested by A, B and E matrices. Such a model is obviously complex, highly nonlinear, and numerically inefficient.

3. Constrained Lagrangian Bond Graph

In this Lagrangian analysis, the standard procedure of the causality assignment defined in Ref. (2) as the sequential causality assignment procedure (SCAP) will be adopted. By simple examination of a bond graph model, developed according to SCAP, for a constrained mechanical system, one may introduce the following definition:

the number of degrees of freedom of a multibody mechanical system is equal to the number of inertia integral causalities in its bond graph.

For the constrained system described by Eqs. (2) and (3), standard bond graph techniques will produce a graph with k inertia integral causalities and m inertia derivative causalities. The resulting state-space form will accordingly contain k equations involving $\dot{p}_i$ momenta p_i and generalized efforts $\dot{p}_i$. Such a set of equations is independent, and when transformed to physical variables (directly measurable variables), produces a set of second-order differential equations of motion. Nevertheless, the bond graph user must deal with the model given by Eq. (10). Considering the algebraic difficulties associated with derivative causalities, one would recognize the awkwardness of using bond graphs in such an application.

In order to establish a rationale for introducing a bond graph structure equivalent to the constrained form of Eq. (2), consider a system of n independent generalized coordinates. The Lagrangian form of equations of motion for such a system can be written as

$$\dot{p} - Q = 0 \quad (11)$$

where Q includes all conservative and nonconservative forces, as well as forces associated with quadratic velocity terms. Further, let us consider another system of n generalized coordinates and m independent constraints. This latter system can be represented by the dynamic model given by Eq. (2). Comparing Eq. (11) to Eq. (2), one can draw a simple conclusion. That is, in the latter system, we have regarded the n generalized coordinates as independent provided that Eq. (11) is modified by adding equivalent forces to the right side, Ref. (7).

In order to devise an equivalent bond graph that represents Eq. (2), one would simply require that all inertia derivative causalities be converted into integral causalities. However, in order to satisfy Kirchhoff's law around the j^{th} 1-junction, one must simultaneously insert an effort source of the value $(-J_j^T \lambda)$ at an internal bond common to that 1-junction. While the insertion of this effort source serves the purpose of balancing Eq. (2), it preserves the integrity of the standard causality notations as inertia derivative causalities are replaced by integral causalities. The effort source $(-J_j^T \lambda)$ will be referred to as the **Lagrange multiplier effort source**, and designated by S_j^L . For the type of frictionless constraints considered in this paper, the constraint reaction forces do not do work. Therefore, the power flow associated with the effort source S_j^L is identically zero. This is equivalent to saying that the flow pairing up with such an effort source is simply zero. In this regard, one must emphasize that the Lagrange multiplier effort source S_j^L involves no power exchange, and may be represented by a full arrow ($\rightarrow$) that carries only an effort signal.

The procedure to establish the constrained Lagrangian bond graph is straightforward, and may be outlined as follows:

- (1) Assign causalities according to the standard causality procedure presented in Ref. (2).
- (2) Select one of the inertia derivative causalities and then convert it into an integral causality while observing that
 - * If the j^{th} derivative causality is at a 1-junction, insert a 0-junction into one of the internal bonds leading to this 1-junction structure, and then add the effort source $(-J_j^T \lambda)$ at the 0-junction. If there are several internal bonds associated with this 1-junction, then choose one of the internal bonds that does not result in causality conflict;
 - * If the j^{th} derivative causality is at a 0-junction, then add the effort source $(-J_j^T \lambda)$ at the same 0-junction.
- (3) Repeat step (2) for all inertia derivative cau-

salities; i. e. for $j=1, \dots, m$.

- (4) Derive equations using standard bond graph manipulations.

It is worthwhile to mention that the causality sign of the Lagrange multiplier effort source harmonizes with the conversion of inertia derivative causalities into integral causalities without upsetting the causal pattern of the bond graph. Furthermore, the resulting bond graph remains causal and congruent with the standard bond graph algebra. The definition of degrees of freedom introduced at the beginning of this section can now be modified, in the light of the developed Lagrangian bond graph, to read:

the number of degrees of freedom of a multibody mechanical system is equal to the number of inertia integral causalities minus the number of Lagrange multiplier effort sources in its constrained Lagrangian bond graph.

4. Methodology and Discussions

The solution procedure outlined in the previous section will be illustrated by means of a simple multibody example. In this context, the validity and robustness of the developed methodology are verified while emphasizing its simplistic features. Some useful observations will be noted and a few fundamental guidelines are drawn.

4.1 Standard bond graph solution

As a simple multibody example, consider the spring-mass-connected pendulum shown in Fig. 1. A uniform rod of mass m_r and length $2L$, is hinged to a translating mass M which is attached to a linear spring of stiffness K . One may choose to model this system by the following set of generalized coordinates, X, x, y and θ , where x and y are the Cartesian coordinates of the center of the mass of the rod. In order to set up the displacement-modulated transformer equations, let us first write expressions for the geometric constraints. Constraints between these four coordinates may be expressed as

Fig. 1 Spring-Mass Connected Pendulum

$$x = X + L \sin \theta \text{ and } y = L \cos \theta. \quad (12)$$

Now, the MTF equations are simply the differentiated forms of Eq. (12), and may be written in the form

$$\dot{x} = \dot{X} + (L \cos \theta) \dot{\theta} \text{ and } \dot{y} = -(L \sin \theta) \dot{\theta}. \quad (13)$$

The resulting bond graph is augmented as shown in Fig. 2. We have only two inertia integral causalities, which imply a two degree of freedom mechanical system, as expected. The state equations in the two state variables p_1 and p_4 are, respectively,

$$\dot{p}_1 = -\dot{p}_2 - e_s \quad (14)$$

and

$$\dot{p}_4 = m_r g L \sin \theta - (L \cos \theta) \dot{p}_2 + (L \sin \theta) \dot{p}_3. \quad (15)$$

Using standard bond graph manipulations, one can eliminate the generalized efforts $\dot{p}_2$ and $\dot{p}_3$ by writing expressions for the flows f_2 and f_3 , respectively. Such flow expressions, when utilized together with element constitutive relations, result in momentum expressions. One can differentiate the momentum expressions to obtain

$$\dot{p}_2 = \left(\frac{m_r}{M} \right) \dot{p}_1 + \left[\frac{m_r L \cos \theta}{I_G} \right] \dot{p}_4 - \left[\frac{m_r L \sin \theta}{I_G^2} \right] \dot{p}_4^2 \quad (16)$$

and

$$\dot{p}_3 = - \left[\frac{m_r L \sin \theta}{I_G} \right] \dot{p}_4 - \left[\frac{m_r L \cos \theta}{I_G^2} \right] \dot{p}_4^2. \quad (17)$$

Utilizing the expressions for $\dot{p}_2$ and $\dot{p}_3$, and recalling that $e_s = KX$, one can write Eqs. (14) and (15), in terms of physical variables, as

$$(M + m_r) \ddot{X} + KX + (m_r L \cos \theta) \ddot{\theta} - (m_r L \sin \theta) \dot{\theta}^2 = 0 \quad (18)$$

and

$$(I_G + m_r L^2) \ddot{\theta} + (m_r L \cos \theta) \ddot{X} + m_r g L \sin \theta = 0. \quad (19)$$

Equations (18) and (19) are the equations of motion with embedded constraints, and are of the same form given by Eq. (10).

The previous example can be solved using the unconstrained form of Lagrange's equations of motion. In this case, the motion may be expressed using two independent generalized coordinates, e. g., X and θ . The kinetic energy expression is then written as

Fig. 2 Standard bond-graph for a pendulum system

conflicts. Attempting to assign integral causalities to both p_3 and p_4 , in the previous example, would result in causality conflict. Therefore, (p_3, p_4) is an infeasible choice of independent momenta. In terms of generalized coordinates, one cannot use y and θ to produce X and x . In other words, The Jacobian submatrix J_d of Eq. (4) becomes singular when y and θ are chosen as independent coordinates. This deduction can be gleaned by simple examination of the Jacobian matrix. For the Jacobian matrix of Eq. (25), when y and θ are chosen as independent, the submatrix J_d is given by

$$J_d = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad (36)$$

which is singular. Consequently, one can filter out all the infeasible choices by merely examining the partitioned Jacobian matrix of constraints. In the earlier example, the other five choices are feasible; however, different forms of the constrained Lagrangian equations may result. When the developed procedure is applied to this example, for different choices of integral causalities, alternative constrained forms are obtained, in general. All such forms, nevertheless, are reduceable to the same form with embedded constraints, as given by Eqs. (18) and (19).

The form of the Lagrangian equations is not unique, simply due to the fact that the set of independent constraint equations can be structured in different equation forms. As a result, the generalized constraint reaction expression $J^T \lambda$ will assume different structures. If one wishes to obtain the same equation from either the constrained Lagrangian bond graph or the direct Lagrangian approach, the following guidelines are to be observed:

- The same Jacobian structure is to be employed in evaluating the term $J^T \lambda$ for either approach.
- In step (1) of the procedure developed in section 3, integral causalities are to be assigned to inertia elements associated with the inertial velocities which are used as inputs to the MTF structure.

Another remark is that concerning the **correctness** of different forms obtained from the Lagrangian approach, Analytically, they are all correct as long as they can be reduced to the minimal set of equations with embedded constraints using a nonsingular transformation matrix J_d . Numerically, however, caution should be exercised in selecting a set of independent coordinates. Quite often, numerical solution schemes employ coordinate partitioning to solve only for the independent set of coordinates, and then substitute to recover the dependent set. In this case, one must choose a set of independent coordinates that results in a nonsingular Jacobian submatrix at all system configurations. In the previous example, selecting $(p_1,$

$p_2)$ results in a singular transformation when $\theta = \pi$. Similarly, when (p_1, p_3) or (p_2, p_3) are chosen as independent, a singular transformation results for $\theta = 0$.

5. Application to a Simple Robotic System

In order to demonstrate the potential of using the developed approach for dynamic modeling of large-scale constrained mechanical systems, one may consider the following simplified planar two-axis mobile robot. The translating base has mass M , and the two links are modeled by two uniform rigid rods, shown in Fig. 4. The first link has mass m_1 , length $2L_1$, and moment of inertia I_1 about its center of mass, while the second link has mass m_2 , length $2L_2$, and moment of inertia I_2 about center of mass. Let $\tau_1(t)$ and $\tau_2(t)$ be the time-dependent actuating torques at revolute joints 1 and 2, where b_1 and b_2 are the corresponding linear damping coefficients, and K_1 and K_2 are the torsional stiffnesses, respectively. If the Cartesian coordinates of mass centers together with rotational coordinates of the two links are used as configuration coordinates, then the generalized coordinate vector can be written as

$$q = \{X, x_1, y_1, \theta_1, x_2, y_2, \theta_2\}^T. \quad (37)$$

The kinematic constraints between these coordinates can be expressed in the form

$$\begin{aligned} x_1 &= \dot{X} + L_1 \sin \theta_1 \\ y_1 &= L_1 \cos \theta_1 \\ x_2 &= X + 2L_1 \sin \theta_1 + L_2 \sin \theta_2 \\ y_2 &= 2L_1 \cos \theta_1 + L_2 \cos \theta_2. \end{aligned} \quad (38)$$

The MTF equations can be generated by differentiating the constraint relations given by Eq. (38). These can be written as

Fig. 4 Two-axis Mobile Robot (Planar Motion)

$$\begin{aligned}
\dot{x}_1 &= \dot{X} + (L_1 \cos \theta_1) \dot{\theta}_1 \\
\dot{y}_1 &= -(L_1 \sin \theta_1) \dot{\theta}_1 \\
\dot{x}_2 &= \dot{X} + (2L_1 \cos \theta_1) \dot{\theta}_1 + (L_2 \cos \theta_2) \dot{\theta}_2 \\
\dot{y}_2 &= -(2L_1 \sin \theta_1) \dot{\theta}_1 - (L_2 \sin \theta_2) \dot{\theta}_2.
\end{aligned} \quad (39)$$

The standard bond graph for this system is shown in Fig. 5. The augmented graph shows three inertia integral causalities, which is conformable with a 3-DOF system. The four inertia derivative causalities, however appeared as a result of the four independent geometric constraints.

The constrained Lagrangian bond graph can be obtained by adding Lagrange multiplier effort sources at the appropriate junctions. Utilizing the constraints equation, one can derive an expression for the generalized constraint reaction forces. This can be written in matrix form as

$$J^T \lambda = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ L_1 \cos \theta_1 & L_1 \sin \theta_1 & 2L_1 \cos \theta_1 & 2L_1 \sin \theta_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & L_2 \cos \theta_2 & L_2 \sin \theta_2 \end{bmatrix} \times \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{Bmatrix}. \quad (40)$$

The constrained Lagrangian bond graph is then established as shown in Fig. 6, and equations can be written as

$$\dot{p}_1 = -S_{x_1}^A - S_{x_2}^A \quad (41)$$

$$\dot{p}_2 = S_{x_1}^A \quad (42)$$

$$\dot{p}_3 = S_{y_1}^A + S_{e_1} \quad (43)$$

$$\begin{aligned}
\dot{p}_4 &= -e_8 - e_9 + e_{11} - L_1 \cos \theta_1 S_{x_1}^A + L_1 \sin \theta_1 S_{y_1}^A \\
&\quad + e_{12} - e_{13} + e_{14} - 2L_1 \cos \theta_1 S_{x_2}^A + 2L_1 \sin \theta_1 S_{y_2}^A
\end{aligned}$$

Fig. 5 Standard bond-graph for a mobile robot

$$\dot{p}_5 = S_{x_2}^A \quad (44)$$

$$\dot{p}_6 = S_{y_2}^A + S_{e_2} \quad (45)$$

$$\dot{p}_7 = -L_2 \cos \theta_2 S_{x_2}^A + L_2 \sin \theta_2 S_{y_2}^A - e_{12} + e_{13} - e_{14}. \quad (46)$$

$$(47)$$

The above set of state equations (41)~(47), when expressed in terms of physical variables, results in the required set of equations of motion in the constrained Lagrangian form as

$$M\ddot{X} - \lambda_1 - \lambda_3 = 0 \quad (48)$$

$$m_1 \ddot{x}_1 + \lambda_1 = 0 \quad (49)$$

$$m_1 \ddot{y}_1 - m_1 g - \lambda_2 = 0 \quad (50)$$

$$I_1 \ddot{\theta}_1 + b_1 \dot{\theta}_1 + K_1 \theta_1 - \tau - (L_1 \cos \theta_1) \lambda_1$$

$$- (L_1 \sin \theta_1) \lambda_2 - (2L_1 \cos \theta_1) \lambda_3 - (2L_1 \sin \theta_1) \lambda_4$$

$$- b_2 (\dot{\theta}_2 - \dot{\theta}_1) - K_2 (\theta_2 - \theta_1) + \tau_2 = 0 \quad (51)$$

$$m_2 \ddot{x}_2 + \lambda_3 = 0 \quad (52)$$

$$m_2 \ddot{y}_2 - m_2 g - \lambda_4 = 0 \quad (53)$$

$$I_2 \ddot{\theta}_2 + b_2 \dot{\theta}_2 + K_2 \theta_2 - \tau_2 - (L_2 \cos \theta_2) \lambda_3$$

$$- (L_2 \sin \theta_2) \lambda_4 - b_2 \dot{\theta}_1 - K_2 \theta_1 = 0. \quad (54)$$

One can appreciate the simplicity of using the constrained Lagrangian bond graph developed in this paper, when attempting to derive the above set of equations of motion using a direct Lagrangian approach. On the other hand, it is even more difficult to use the standard bond graph, shown in Fig. 5, because only an implicit set of equations with embedded constraints can be obtained.

6. Conclusions

A bond graph configuration equivalent to the constrained Lagrangian formulation of holonomic mechanical systems has been devised. The procedure outlined in this paper establishes a correlation between bond graph modeling elements and the generalized constraint reaction forces expressed by Lagrangian

Fig. 6 Constrained Lagrangian bond-graph for a mobile robot

ge multipliers. The principal advantage of this approach lies in the ease of obtaining the constrained Lagrangian equations without relying on the Lagrangian function. Extensions to this work may cover the application of this method to systems of several energy domains, systems with nonlinear storage elements, and systems with nonholonomic constraints.

The constrained Lagrangian form is equivalent to an explicit set of differential equations which is more efficient to solve by a simulation program. Furthermore, the constraint reaction forces are calculated along with the solution. Apparently, this approach possesses a great potential for tackling large-scale mechanical systems wherein inertia fields and geometric constraints are prevalent. Such an approach may be utilized in conjunction with modal analysis techniques presented in Ref. (8) in order to efficiently simulate constrained flexible mechanical systems, such as those presented in Ref. (9), where constraint reactions are of major interest.

Acknowledgement

The support provided by King Fahd University of Petroleum & Minerals is greatly appreciated.

References

- (1) Greenwood, R. T., Principles of Dynamics, Prentice-Hall, (1988).
- (2) Rosenberg, R. C. and Karnopp, D. C., Introduction to Physical System Dynamics, McGraw-Hill, (1983).
- (3) Bos, A. M. and Breedveld, P. C., 1985 Update of Bond Graph Bibliography, J. Franklin Institute, Vol. 319(1/2) (1985), p. 269.
- (4) Karnopp, D. C., Lagrange's Equations for Complex Bond Graph Systems, Trans. ASME, J. Dyn. Syst. Meas. Control, Vol. 99, No. 4 (1977), p. 300.
- (5) Karnopp, D. C., Dynamic Forces in Multiport Mechanics: Direct and Lagrangian formulations, J. Franklin Institute, Vol. 314, No. 5 (1982), p. 265.
- (6) Brown, F. T., Lagrangian Bond Graphs, Trans. ASME, J. Dyn. Syst. Meas. Control, Vol. 94, No. 3 (1972), p. 213.
- (7) Meirovitch, L., Methods of Analytical Dynamics, McGraw-Hill, (1970).
- (8) Margolis, D. L. and Karnopp, D. C., Bond Graphs for Flexible Multibody Systems, Trans. ASME, J. Dyn. Syst. Meas. Control, Vol. 101, No. 1 (1979), p. 50.
- (9) Khulief, Y. A., and Shabana, A. A., Dynamics of Multibody System with Variable Kinematic Structure, Trans. ASME, J. Mech. Transm. Autom. Des., Vol. 108 (1986), p. 167.