Chaotic Behaviour of a Nonlinear Vibrating System with a Retarded Argument*

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This paper investigates the behaviour of a nonlinear vibrating system, i. e., van der Pol and Duffing's system, with a retarded argument under a harmonic stimulating force. An approximate analytical method, i. e., the averaging scheme, is used to analyse subharmonic oscillations of the order $1/2$. The algorithm used to study periodic solutions and their stabilities with higher-order approximations, is presented. A computer simulation is used to obtain Poincaré mapping and invariant manifolds. Through the application of the approximate analytical procedure presented here and numerical simulation, both symmetrical and unsymmetrical subharmonic solutions are observed in addition to period-doubling bifurcations and chaotic behaviour. In addition, one of the Lyapunov exponents, which is positive, has verified that the vibrating system possesses chaotic phenomena.

Key Words: Vibration, Forced Vibration, Nonlinear Vibration, Subharmonic Oscillation, Chaos, Poincaré Mapping, Retarded Argument, Invariant Manifold, Period-Doubling Bifurcation

1. Introduction

Over the past few decades, studies on nonlinear vibrating phenomena arising from physical systems, which are governed by rather simple deterministic equations, have focused on the analyses of chaotic phenomena and consequently, many investigative reports on the subject are available^{(1)~(3)}. As a result, chaotic motion has been observed to occur in a variety of nonlinear dynamical systems.

On the other hand, certain systems governed by ordinary differential equations are dependent upon the past state of the process. Such systems may be recognized in many fields of engineering and technology such as rolling mills, chemical reactors, grinding processes and control systems^{(4)~(6)}. The equation governing such systems is called a retarded

differential difference equation, and the space of this equation is defined by infinite dimensions. Therefore it may be very difficult to analyse its behaviour^{(7),(8)} and it seems many problems yet to be solved.

This study investigates the forced vibrating system which shows a nonlinear characteristic as well as retardation. Approximate analytical solutions of the main resonance and subharmonics of the order $1/2$ are evaluated using the averaging method. This approximate analytical scheme illustrates the effects of nonlinear properties in dynamic behaviour. A numerically analytical algorithm is presented here. This procedure has generated solutions more accurate than those obtained using the averaging scheme.

Examination of the unstable frequency region in both of the above responses using digital simulation has clarified the occurrence of irregular motion, i. e., chaotic behaviour. In computing Lyapunov exponents corresponding to irregular motion, one positive Lyapunov exponent indicates that the motion is chaotic. Furthermore, this consideration has demonstrated the existence of both symmetrical and unsymmetrical solutions associated with subharmonic solutions of both $1/2$ and $1/3$ orders.

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2. Theoretical Analysis

The system equation under consideration is expressed as

$$x(t)'' - \{h - \gamma \cdot x^2(t)\} \cdot x(t)' + \omega_0^2 \cdot x(t) + q \cdot x^3(t) + b \cdot x(t - \tau) = f \cdot \cos \omega t + G, \quad (1)$$

where $x(t)$ is a nondimensional displacement and the other symbols are constant system parameters and, in particular, τ is a positive small retarded argument and $(\cdot)'$ represents the differentiation with respect to $t: d(\cdot)/dt$. Although this system has been previously discussed by other researchers⁽⁹⁾⁻⁽¹³⁾, there are still many problems to be solved.

In the following numerical analysis, the norm $\|x(t)\|$ of the solution $x(t)$ of Eq. (1) has been utilized. The definition of norm $\|x(t)\|$ is expressed as follows.

As $x(t)$ is a response waveform, it may be expanded into the Fourier series:

$$x(t) = a_0 + \sum a_n \cdot \cos(2n\pi t/T - \phi_n), \quad (2 \cdot a)$$

where T denotes the period of $x(t)$. Then, norm $\|x(t)\|$ is defined as

$$\|x(t)\| = \sqrt{2a_0^2 + \sum a_n^2}. \quad (2 \cdot b)$$

2.1 Main resonant responses

The approximate solution in the main resonant frequency region is assumed to take the form

$$x = u \cdot \cos \omega t - v \cdot \sin \omega t = \alpha \cdot \cos(\omega t + \theta), \quad (3)$$

where the amplitudes u and v are slowly varying functions of time t .

Substituting Eq. (3) into Eq. (1) and applying the averaging scheme yields the following averaging equations:

$$\begin{aligned} u' &= -\{(z' + \omega^2 x)/\omega\} \cdot \sin \omega t \\ v' &= -\{(z' + \omega^2 x)/\omega\} \cdot \cos \omega t, \end{aligned} \quad (4)$$

where $z = x'$.

In steady-state solutions of oscillations, the conditions $u' = 0$ and $v' = 0$ hold. Applying the Routh-Hurwitz criterion to Eq. (4) generates stable conditions for the oscillations⁽⁹⁾⁻⁽¹¹⁾.

2.2 Subharmonic responses of order one-half

The approximate expression of the phenomena is assumed as follows:

$$x = R + u \cdot \cos(\omega t/2) + v \cdot \sin(\omega t/2) + A \cdot \cos \omega t, \quad (5)$$

where the amplitudes R , u and v are slowly varying functions of time t and the fundamental amplitude A is assumed to approximately have the expression $f/(1 - \omega^2)$ and $c = \sqrt{u^2 + v^2}$ is the amplitude of the subharmonics of the order $1/2$.

Similar to the case in the previous section, the averaging procedure provides a set of simultaneous averaging equations:

$$\begin{aligned} R' &= [-g + \omega(u - v)\{p - w\{R^2 + (u^2 + v^2)/2\}\} \\ &\quad + \{k + s\{R^2 + 3(u^2 + v^2)/2\}\}R]/p, \end{aligned}$$

$$\begin{aligned} u' &= [(k - \omega^2)v + 3sv\{(u^2 + v^2)/4 + R^2\} \\ &\quad + w\omega u(u^2 + v^2)]/(2\omega), \\ v' &= [-a + (k - \omega^2)u + 3su\{(u^2 + v^2)/4 + R^2\} \\ &\quad - w\omega v(u^2 + v^2)]/(2\omega), \end{aligned} \quad (6)$$

where $p = 2(h - b\tau)/(2 - b\tau^2)$, $w = 2a/(2 - b\tau^2)$, $k = 2(\omega_0^2 - b)/(2 - b\tau^2)$, $s = 2q/(2 - b\tau^2)$, $a = 2f/(2 - b\tau^2)$ and $g = 2G/(2 - b\tau^2)$.

Equation (6) may be utilized in order to consider steady-state solutions of vibrations and their stability conditions⁽⁹⁾⁻⁽¹¹⁾.

The existence of subharmonic vibrations of the order $1/2$ suggests the possibility of period-doubling bifurcations from the $2T$ -periodic solution in Eq. (6) to the $4T$ -periodic solution. Namely, we define $x_0(t)$ as the steady-state solution of subharmonic oscillations of the order $1/2$ and $\delta x(t)$ as a small disturbance. Then, by adding the small disturbance $\delta x(t)$ to the steady-state solution $x_0(t)$ we define the following equation:

$$x^*(t) = x_0(t) + \delta x(t).$$

Inserting it into the system equation (1), we construct the variational equation, i.e., the variational Hill's equation. Due to the appearance of the harmonic terms of frequency $\omega/2$ and $3\omega/2$, which are involved as parametric excitation components in Hill's equation, we may expect the possible cascade of period-doubling bifurcations from the Floquet theory^{(14),(15)}. Therefore, we readily conclude that such a cascade may lead to chaotic motion^{(11),(14),(15)}. Further considerations for these phenomena will be discussed in section 4.1.

3. Numerical Analytical Scheme with Method of Direct Numerical Integration

Since it is quite difficult to analytically obtain approximate periodical steady-state solutions with high accuracy for nonlinear oscillating systems, the search for more accurate solutions must rely on direct numerical integration methods. Moreover, there are some difficulties in the analytical procedures when chaotic phenomena are considered.

Accordingly, we construct an algorithm which enables us to seek periodic steady-state solutions in a precise and efficient manner. This procedure will clarify the details of the dynamic behaviour of this system and compare its results to those obtained using the averaging method. The outline of this numerical procedure is given as follows.

(a) The system under consideration is assumed to be,

$$dy/dt = f(y, t), \quad (7)$$

where y and f are n -dimensional vectors; that is,

$$\begin{aligned} y &= (y_1, \dots, y_n), \\ f &= (f_1, \dots, f_n), \end{aligned}$$

and $'(\cdots)$ means the transposition of the vector $(\cdots)$.

(b) A variational equation corresponding to Eq. (7) is

$$d\eta/dt = A(y, t) \cdot \eta, \quad (8)$$

where $\eta = '(\eta_1, \cdots, \eta_n)$ is a variation of the variable y and $A = \partial f / \partial y$ represents the Jacobian matrix. Moreover, functions f and A are assumed to be periodic functions of t with period T .

(c) The initial value of Eq. (7) is given as

$$y^0 = y(0), \quad (9)$$

and the value at $t = T$ is expressed as follows.

$$y^1 = y(T). \quad (10)$$

If $y^1 = y^0$, this y^0 corresponds to the initial value which accurately provides a (steady-state) periodic solution, and the search for these periodical solutions is equivalent to the investigation of initial values such as y^0 given above.

Therefore, the problem of solving periodic solutions is, in effect, the problem of evaluating the following equation:

$$y^1 - y^0 = 0, \quad (11)$$

where y^1 is a function of y^0 , i. e.,

$$y^1 = \text{fun.}(y^0). \quad (12)$$

(d) B denotes the fundamental matrix solution of Eq. (8),

$$B = '(\eta^1, \cdots, \eta^n). \quad (13)$$

In applying the Newton-Raphson scheme to Eq. (11), the algorithm for finding periodic solutions thus takes the following form:

$$(B - E_n) \cdot \delta y^0 = y^0 - y^1, \quad (14)$$

where E_n is the unit matrix of order n and δy^0 is the correction of the initial value y^0 .

We utilize Eq. (14) repeatedly until any given initial value y^0 converges.

(e) When Eq. (14) indicates a periodic solution, we must investigate the stability of the solution. That is, if all the absolute values of eigenvalues of matrix

B are less than unity, the corresponding periodic solution is thought to be stable.

All the numerical analytical solutions presented in this study have been determined using this algorithm.

4. Numerical Results and Discussion

An example of the response curves for the systems of Eqs. (4) and (6) is illustrated in Fig. 1, where the abscissa is the frequency, ω , of the stimulating force. In this figure, solid and broken lines correspond to the stable and unstable states, respectively, and the parameters are as follows: $h = \alpha = 0.1$, $\omega_0 = 1.0$, $q = 3.0$, $b = -2.0$, $\tau = 0.1$ and $f = 5.0$. In the following, however, the condition $G = 0$ is retained in Eq. (1).

From Fig. 1, the subharmonic oscillations of the order 1/2 are recognized in the vicinity of the primary resonant frequency region. Although the occurrence of two kinds of subharmonic oscillations is expected, these two subharmonic vibrations are heterogeneous. That is, the subharmonic of the order 1/2, which occurs in the frequency region $\omega \leq 1.75$, corresponds to the response which has been excited in this region. On the other hand, in the frequency region $\omega \geq 1.75$ the symmetrical solution becomes unstable and simultaneously, the unsymmetrical solution appears. Thus, the subharmonic component of the order 1/2, which exists at the frequency region $\omega \geq 1.75$ in Fig. 1, corresponds to the subharmonic components of the order 1/2 accompanied by the unsymmetrical solutions, as mentioned above. These symmetric and unsymmetric solutions will be discussed later.

Although, as shown in Fig. 1, it may be possible for the averaging procedure to clarify considerably the characteristics of the main resonance and subharmonic oscillation of the order 1/2, a scheme with this degree of approximation may be disadvantageous when explaining the difference between these subharmonic vibrations.

Simulation results are illustrated in Fig. 2. This figure shows the norm $\|x(t)\|$ of the displacement $x(t)$ versus the frequency ω , applying the scheme mentioned previously in section 3. The results of Fig. 2 qualitatively agree with those of Fig. 1. For example, both figures show typical characteristics of hard spring. However, some significant differences exist between them, as follows.

(a) In the frequency region higher than the main resonance, the symmetric solution becomes unstable at point A and as ω increases, bifurcates into the unstable symmetric solution, which corresponds to the branch AB, and the stable unsymmetric one, which corresponds to the branch AC, as shown in the figure.

Fig. 1 Response curves by averaging method

Fig. 2 Response curves by numerical simulation

(a) Characteristics in frequency region lower than main resonance

(b) Characteristics in frequency region higher than main resonance

Fig. 3 Details of response curves of Fig. 2

(b) In the frequency region lower than the main resonant region, the characteristic curves of the system are very complex and furthermore unsymmetrical motions appear, bifurcating from the symmetrical ones as ω decreases, as shown in Fig. 2. The details of this situation are illustrated in Fig. 3(a), where only the fundamental component is drawn.

From Fig. 3(a), as ω decreases, the stable symmetric solution becomes unstable at point A and the stable unsymmetric motion which corresponds to branch AB appears. After the unsymmetric solution creates an orbit similar to the loop BCD, this solution again meets the symmetric one at point F. It is clear from the figure that intervals BC and DF are unstable and CD is stable. On the other hand, the symmetric solution also repeats the bifurcation at point E. This system repeatedly generates these bifurcations and combinations of the response curves. Furthermore, the stable and unstable states become entangled on the response curves, and chaotic phenomena may be expected to occur in their unstable states, which will be considered later.

(c) On the nonresonant branch, characteristic curves, like isolated islands, appear. One of them corresponds to subharmonic vibrations of the order 1/2 and the other to subharmonic oscillations of the order 1/3, where subharmonic components of the orders 1/2 and 1/3, respectively, become predominant (see Fig. 2 and Fig. 3(b)).

These islandlike response curves bifurcate into symmetric and unsymmetric solutions and possess stable states which are quite limited. Thus, chaotic motion is expected to arise in the unstable regions of the islandlike curves.

Namely, as shown in Fig. 3(b), the stable, symmetric solution of subharmonic oscillations of the order 1/3, AB, bifurcates into both an unstable symmetric solution BD and unsymmetric solution BC, which possess a narrow stable region when ω decreases. As ω decreases further, the unstable symmetric solution BD again meets the unstable unsymmetric BC at point E. The symmetric solution near point E is stable. On the other hand, when ω decreases, the stable solution FG, which corresponds to the subharmonic vibrations of the order 1/2, bifurcates into the unstable solutions GH and GJ at point G. When ω decreases further, these unstable solutions again meet at point K, where the solution near point K is stable. Note that the solution of subharmonics of the order 1/2 is unsymmetric.

Figure 4 represents the influence of the retarded argument on the behaviour of subharmonic oscillations of the order 1/2. As this argument increases, the island become smaller. It is remarkable that this

argument can control the behaviour of subharmonic oscillations of the order $1/2$.

Figure 5 shows the influence of nonlinearity on the response. As nonlinearity decreases, the skeleton curve of the resonance becomes steeper and the unsymmetric solutions, appearing in the frequency region higher than the primary resonance, have a

Fig. 4 Influence of retardation on subharmonics of the order $1/2$

Fig. 5 Influence of nonlinearity on the response

Fig. 6 Bifurcation diagram for the response of Fig. 3(b)

tendency to become larger.

4.1 Chaotic behaviour

We have demonstrated the existence of a progression to chaos via a cascade of period-doubling bifurcations in section 2.2. In order to verify the theoretical approximate prediction, the characteristics of the system over the unstable regions, as mentioned above, have been discussed in detail, using a computer simulation.

A bifurcation diagram obtained through digital simulation is shown in Fig. 6, which corresponds to the response in Fig. 3(b). From the figure, we can easily recognize the occurrence of chaotic motion in the frequency region. The chaotic behaviour associated with Fig. 6 is illustrated in Figs. 7 and 8.

Figure 7 shows the chaotic attractor which comes into existence in the unstable frequency region, $\omega = 3.987$, of subharmonics of the order $1/3$. The attractor in Fig. 7(a) is obtained using Poincaré mapping. The three small circles "O" in the figure indicate the locations of subharmonic solutions of the order $1/3$, A-1, A-2 and A-3, whose stabilities have already been lost. Figure 7(b) shows details of Fig. 7(a). An unstable invariant manifold (i.e., α -branch) and stable invariant manifold (i.e., ω -branch) are plotted in Fig. 7(c) and are associated with Fig. 7(a). The α -branches are too small to be of use in this figure. The details of Fig. 7(c) are shown in Fig. 7(d), which corresponds to the behaviour in the vicinity of point A-3 of Fig. 7(a). Figures 7(c) and (d) clearly express the existence of homoclinic orbits^{(1),(3)}. The presence of homoclinic orbits tends to promote erratic behaviour, and this underlies chaotic behaviour⁽¹⁾. Thus, we conclude that the motion in Fig. 7 is chaotic.

Similarly, in Fig. 8, the chaotic attractors, which appear in the vicinity of the frequency region of subharmonics of the order $1/2$, are plotted using Poincaré mapping, where $\omega = 4.05494$. In Fig. 8(a), the two small circles "O" express the locations of subharmonic solutions of the order $1/2$ whose stabilities have already been lost, and the symbols: A-1, A-2, and so on, represent the chaotic attractors associated with the subharmonics of the order $1/2$. Figure 8(b) shows the details of Fig. 8(a). The irregular motion of Fig. 8 represents the chaos which occurs in the frequency region where the stable unsymmetric solutions of subharmonic oscillations of the order $1/2$ on the island (see Fig. 2) have become unstable. The progression to this chaos is period-doubling bifurcation.

Figure 9 indicates the bifurcation diagram, corresponding to Fig. 3(a), which enables us to investigate the global characteristics of the system. This figure suggests the existence of chaotic phenomena, even in

(a) Poincaré map ($\omega=3.987$)

(b) Details of Fig. 7(a)

(c) Invariant manifolds (i. e., α - and ω -branches)

(d) Details of Fig. 7(c)

Fig. 7 Chaotic motion associated with subharmonics of the order $1/3$

this region.

Figure 10 has some chaotic attractors, that is, $\omega = 0.9$ in Fig. 10(a) and $\omega = 0.63$ in Fig. 10(b). All chaotic attractors in Fig. 10 appear when the corresponding unsymmetric solutions have lost their stability. Therefore, the progression to this chaos is again period-doubling bifurcation.

Now, a numerical result of the largest Lyapunov exponent λ_1 , corresponding to the motion shown in Fig. 10(a), is illustrated in Fig. 11. As shown in this figure, the Lyapunov exponent λ_1 is clearly positive.

Following the fact that the sign of the Lyapunov exponent is a criterion for chaos, this behaviour may be concluded to be chaotic^{(1),(3)}. In addition, the largest Lyapunov exponent for the motion in Fig. 7 is $\lambda_1 \sim 0.22$, in Fig. 8 is $\lambda_1 \sim 0.214$, and so on.

5. Conclusions

The averaging method and numerical analytical procedure have been used to study the dynamic behaviour of a nonlinear vibrating system with a retarded argument under a harmonic stimulating force.

(a) Poincaré map ($\omega = 4.05494$)

Fig. 9 Bifurcation diagram for the response of Fig. 3(a)

(b) Details of Fig. 8(a)

Fig. 8 Chaotic motion associated with subharmonics of the order 1/2

Furthermore, Poincaré mapping has been utilized in order to investigate the behaviour of irregular motions. A summary of the results is as follows.

Even though the restoring characteristic of the system is symmetric, the averaging method has confirmed the resonant phenomena of subharmonic oscillations of the order $1/2$ (i. e., corresponding to

unsymmetrical solutions). Although this approximate method has produced two kinds of subharmonic vibrations, they are different.

The outline of an algorithm which enabled determination of approximate higher-order, periodic steady-state solutions, has been given. This approximate higher-order computational scheme can clarify new branches of the responses which the averaging method could not generate. Namely, the symmetric solution bifurcated into the symmetric solution and the unsymmetric one in the frequency region higher than the primary resonant area. The former is unstable and the latter, stable. On the other hand, the symmetric and the unsymmetric solutions which appeared in the frequency region lower than the primary resonance became entangled, repeating bifurcations and combinations.

The computer simulation, i. e., Poincaré mapping, has confirmed the chaotic phenomena in the frequency regions where the responses are unstable and their progressions to chaos are considered to be period-doubling bifurcations.

References

- (1) Guckenheimer, J. and Holmes, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York (1983).
- (2) Szemplinska-Stupnicka, W., Plaut, R. H. and Hsieh, J.-C., *Period Doubling and Chaos in Unsymmetric Structures under Parametric Excitation*, *Trans. ASME, J. Appl. Mech.*, Vol. 56, No. 4 (1989), p. 947.
- (3) Moon, F. C., *Chaotic Vibrations*, John Wiley & Sons, New York, (1987).
- (4) Kolmanovskii, V. B. and Nosov, V. R., *Stability of Functional Differential Equation*, Academic Press, Inc. (1986).
- (5) Minorsky, N., *Self-Excited Mechanical Oscillations*, *J. Appl. Phys.*, Vol. 19, No. 4 (1948), p. 332.
- (6) Inoue, Y. and Ueda, Y., *Analysis of a Self-Oscillatory System Containing Dead Time by Averaging Method*, *Trans. of Institute of Elec. and Comm. Eng. Japan (in Japanese)*, Vol. J64-No. 3 (1981), p. 243.
- (7) Schlesinger, M. F., et al., *Perspectives in Nonlinear Dynamics*, World Scientific (1986), p. 281.
- (8) Hale, J. K., *Theory of Functional Differential Equations*, Springer-Verlag, New York (1977).
- (9) Tsuda, Y., Sueoka, A., Yoshitake, Y. and Tamura, H., *Vibrations of a Forced Self-Excited System with a Retarded Argument*, *Bull. JSME*, Vol. 29, No. 256 (1986), p. 3476.
- (10) Tsuda, Y., Tamura, H., Sueoka, A. and Hinaga, T., *Vibrations of a Forced Self-Excited System with a Retarded Argument (Influence of Initial Conditions)*, *JSME Int. J. Ser. III*, Vol. 31, No. 3

- (1988), p. 494.
- (11) Tsuda, Y., Tamura, H. and Sueoka, A., On Dynamics of a Forced Self-Excited Nonlinear Vibrating System with a Retardation, Proceedings the 1989 ASME Pressure Vessels and Piping Conference, ASME, PVP-Vol. 179 (1989), p. 89.
 - (12) Plaut, R. H. and Hsieh, J. -C., Chaos in a Mechanism with Time Delays under Parametric and External Excitation, Journal of Sound and Vibrations, Vol. 114, No. 1 (1987), p. 73.
 - (13) Awrejcewicz, J., A Route to Chaos in a Nonlinear Oscillator with Delay, Acta Mechanica, Springer-Verlag, Vol. 77 (1989), p. 111.
 - (14) Szemplinska-Stupnicka, W. and Bajkowski, J., The $1/2$ Subharmonic Resonance and its Transition to Chaotic Motion in a Nonlinear Oscillator, Int. J. of Nonlinear Mech., Vol. 21, No. 5 (1986), p. 401.
 - (15) Szemplinska-Stupnicka, W., Secondary Resonance and Approximate Models of Routes to Chaotic Motion in Non-Linear Oscillators, J. Sound Vid., Vol. 113, No. 1 (1987), p. 155.
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