A Novel Control Algorithm for Rapid and Precise Positioning*

Yaw-Shih SHIEH** and An-Chen LEE**

This paper investigates the positioning technology of a control system using an optimal control method. In this method, the control algorithm is based on minimization of a quadratic performance index, which consists of the position error and the weighted derivative of the error during a specific time interval. Minimization of this index allows us to obtain the optimal step control input command. It is shown that the control algorithm is efficient for positioning and simple for on-line implementation. The stability of the step control system is analyzed by the root locus method, and a means of selecting both the sampling period and the weighting matrix according to the specifications of the system is presented. Comparisons of the proposed control algorithm with bang-bang control and the optimum regulator problem are made. Also, a track-seeking servo control system of an optical disk driver is used to illustrate application of the proposed control method.

Key Words: Optimal Control, Position Control, Step Control

1. Introduction

In classical control systems, the controller design usually involves a trial-and-error process in which various methods of analysis are used iteratively to determine the design parameters of an "acceptable" system. The performance of a classical controlled system is limited only by the use of output feedback for the lack of information about the system. Consequently, modern control systems using state feedback were developed, and in particular, the optimal control technique was introduced to obtain optimum performance. The determination of the weighting matrix in optimal control designs is still generally a trial-and-error process, however, and the control laws for optimal control are usually either too complex to be implemented or too sensitive to changes in the system parameters. For example, Ananthanarayanan⁽¹⁾ derived bang-bang control for fast response, but the

switching time was extremely sensitive. Kirk⁽²⁾, Kailath⁽³⁾ and Anderson and Moore⁽⁴⁾ have all solved the regulator problem, yet this led to a time-varying state feedback law which requires the solution of a set of ordinary nonlinear differential equations of the Riccati type.

In this paper, a control algorithm called step control is proposed. Step control combines the advantages of both classical and optimal control design. The basic idea behind our algorithm is to sample the states of the system at the beginning of each specific equally spaced time interval. Then, by minimizing the square error of position and weighted square error of velocity, an optimal control input with a constant value during the time interval is developed.

With the use of the classical control technique of the root locus, the closed-loop poles of a step control system can be located in the root locus plot of the z -domain and the stability of the controlled system can be studied. Furthermore, the design sampling period and the design weighting matrix can be obtained according to positioning requirements.

A second-order system is used to illustrate the usefulness of the step control algorithm. Under some

* Received 18th October, 1991

** Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsin-chu 30049, Taiwan, R. O. C.

loose assumptions that are satisfied by most systems, the relationship between the sampling period and the weighting matrix is formulated analytically in a terse equation, independent of the system parameters. Moreover, the poles of the closed-loop system can be made invariant for different system parameters so that the different systems can be controlled via the step control algorithm such that all of the closed-loop systems have similar dynamical characteristics. This is useful in synchronization of simultaneous multi-axis control. Finally, the application of the step control algorithm to a high-order system, the track-seeking servo system of an optical disk driver, is presented.

2. The Step Control Algorithm

The basic concept of the step control algorithm is that the error between actual output and the reference input can be reduced dramatically by applying a certain sequence of step control input. The control law can be built off-line by minimizing the performance index, which consists of the integral of the error over the time interval and the derivative of the error at the last time instant of the time interval. The control input is obtained by sensing the states of the system.

Consider a linear time invariant system expressed as

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$

$$y(t) = Cx(t), \quad (2)$$

where $x(t)$ is the n by 1 state vector, $y(t)$ is the q by 1 output vector, and $u(t)$ is the p by 1 control input vector. A , B and C are n by n , n by p and q by n real constant matrices, respectively. Let the control input vector $u(t) = \bar{u}$ be constant for $t_0 \leq t \leq t_s$. The solution of the output equation is

$$y(t) = K_1(t, t_0)x(t_0) + K_2(t, t_0)\bar{u}, \quad t_0 \leq t \leq t_s, \quad (3)$$

where

$$K_1(t, t_0) = Ce^{A(t-t_0)}, \quad (4a)$$

$$K_2(t, t_0) = C \int_{t_0}^t e^{A(t-\tau)} d\tau B, \quad (4b)$$

and $x(t_0)$ is the initial condition of the state vector at $t = t_0$. The object is to find the optimum $\bar{u}$ at time t_0 in order to make the output $y(t)$ track/follow the reference input as closely as possible during $t_0 \leq t \leq t_s$.

The time derivative of the output vector is as follows:

$$\dot{y}(t) = K_3(t, t_0)x(t_0) + K_4(t, t_0)\bar{u}, \quad \text{for } t_0 \leq t \leq t_s, \quad (5)$$

where

$$K_3(t, t_0) = CAe^{A(t-t_0)}, \quad (6a)$$

$$K_4(t, t_0) = C \left[I + A \int_{t_0}^t e^{A(t-\tau)} d\tau \right] B. \quad (6b)$$

Let $r(t)$ be the reference input vector, and the error vectors be defined as

$$e(t) = r(t) - y(t), \quad (7a)$$

$$\dot{e}(t) = \dot{r}(t) - \dot{y}(t). \quad (7b)$$

The performance index is then formulated as

$$\begin{aligned} J &= \int_{t_0}^{t_s} e^T(t) Q_1 e(t) dt + \dot{e}^T(t_s) Q_2 \dot{e}(t_s) \\ &= \int_{t_0}^{t_0+T_s} e^T(t) Q_1 e(t) dt + \dot{e}^T(t_0+T_s) Q_2 \dot{e}(t_0+T_s), \end{aligned} \quad (8)$$

where $t_s = t_0 + T_s$, and Q_1 and Q_2 are q by q , non-negative definite weighting matrices of the error vectors $e(t)$ and $\dot{e}(t)$, respectively. The objective of minimization of the performance index is to obtain a minimum error output while limiting the rate of change by $\dot{e}^T(t_0+T_s) Q_2 \dot{e}(t_0+T_s)$ during the time interval. The overshooting of the output can be suppressed by choosing an adequate Q_2 . A similar objective can be achieved in the LQR problem by minimizing control input in the performance index.

The step control algorithm is, therefore, designed to obtain the optimal control input vector $\bar{u}^*$, constant for $t_0 \leq t \leq t_0 + T_s$, such that the performance index J is minimum over the time interval. Let the desired output vector $r(t) = \bar{r}$ be a constant vector for $t_0 \leq t \leq t_0 + T_s$. With a change in the variable in Eq. (8), i.e., $\tau = t - t_0$, together with Eqs. (3)-(7), Eq. (8) can be rewritten as

$$\begin{aligned} J &= \int_0^{T_s} [\bar{r} - K_1(\tau, 0)x(t_0) - K_2(\tau, 0)\bar{u}]^T Q_1 \\ &\quad * [\bar{r} - K_1(\tau, 0)x(t_0) - K_2(\tau, 0)\bar{u}] d\tau \\ &\quad + [K_3(T_s, 0)x(t_0) + K_4(T_s, 0)\bar{u}]^T Q_2 \\ &\quad * [K_3(T_s, 0)x(t_0) + K_4(T_s, 0)\bar{u}]. \end{aligned} \quad (9)$$

To obtain the optimal $\bar{u}^*$, the necessary condition is that the first derivative of J with respect to $\bar{u}$ be set to zero, or

$$\begin{aligned} \frac{\partial J}{\partial \bar{u}} \Big|_{\bar{u}=\bar{u}^*} &= \int_0^{T_s} [-K_2^T(\tau, 0)(Q_1^T + Q_1)\bar{r} \\ &\quad + K_2^T(\tau, 0)(Q_1^T + Q_1)K_1(\tau, 0)x(t_0) \\ &\quad + K_2^T(\tau, 0)(Q_1^T + Q_1)K_2(\tau, 0)\bar{u}^*] d\tau \\ &\quad + K_4^T(T_s, 0)(Q_2^T + Q_2)K_3(T_s, 0)x(t_0) \\ &\quad + K_4^T(T_s, 0)(Q_2^T + Q_2)K_4(T_s, 0)\bar{u}^*] = 0, \end{aligned} \quad (10)$$

where Q_1 and Q_2 are symmetric matrices, which are usually diagonal, and $\bar{r}$, $x(t_0)$, and $\bar{u}^*$ are constant vectors. After minor manipulations, Eq. (10) becomes

$$L_1 \bar{u}^* - L_2 \bar{r} + L_3 x(t_0) = 0, \quad (11)$$

where

$$\begin{aligned} L_1 &= \int_0^{T_s} K_2^T(\tau, 0) Q_1 K_2(\tau, 0) d\tau \\ &\quad + K_4^T(T_s, 0) Q_2 K_4(T_s, 0), \end{aligned} \quad (12a)$$

$$L_2 = \int_0^{T_s} K_2^T(\tau, 0) Q_1 d\tau, \quad \text{and} \quad (12b)$$

$$\begin{aligned} L_3 &= \int_0^{T_s} K_2^T(\tau, 0) Q_1 K_1(\tau, 0) d\tau \\ &\quad + K_4^T(T_s, 0) Q_2 K_3(T_s, 0). \end{aligned} \quad (12c)$$

The sufficient condition for a minimum of Eq. (9) is that the weighting matrices Q_1 and Q_2 must be

at least non-negative definite⁽⁶⁾. For a quadratic performance index with non-negative weighting matrices, with the added constraint of constant control input during the time interval, Eq. (11) gives unique optimal control. It is important to note that in Eqs. (12), when the system parameters A , B and C are given, L_1 , L_2 and L_3 are functions of only T_s and the weighting matrices Q_1 and Q_2 .

If the inverse of A exists, Eqs. (12) can be further simplified into algebraic form for easy application calculation. We obtain

$$L_1 = B^t(A^{-1})^t \int_0^{T_s} (e^{A^t\tau} Q_3 e^{A\tau} - e^{A^t\tau} Q_3 - Q_3 e^{A\tau} + Q_3) d\tau A^{-1} B + B^t e^{A^t T_s} Q_4 e^{A T_s} B, \quad (13)$$

where

$$Q_3 = C^t Q_1 C, \quad (14 \cdot a)$$

$$Q_4 = C^t Q_2 C. \quad (14 \cdot b)$$

We define

$$\Gamma(T_s) = \int_0^{T_s} e^{A^t\tau} Q_3 e^{A\tau} d\tau, \quad (15)$$

then

$$\begin{aligned} \Gamma(T_s)A &= \int_0^{T_s} e^{A^t\tau} Q_3 e^{A\tau} A d\tau \\ &= \int_0^{T_s} e^{A^t\tau} Q_3 \frac{d}{d\tau} (e^{A\tau}) d\tau \\ &= e^{A^t T_s} Q_3 e^{A T_s} \Big|_0^{T_s} - \int_0^{T_s} \left(\frac{d}{d\tau} e^{A^t\tau} \right) Q_3 e^{A\tau} d\tau \\ &= (e^{A^t T_s} Q_3 e^{A T_s}) - Q_3 - A^t \Gamma(T_s). \end{aligned} \quad (16)$$

Equation (16) is the Lyapunov matrix equation⁽⁶⁾, and $\Gamma(T_s)$ expressed in Eq. (15) is its solution. For a given A , T_s and Q_3 , $\Gamma(T_s)$ can be calculated by comparing the coefficients of the Lyapunov matrix equation. Then Eq. (13) becomes

$$\begin{aligned} L_1 &= B^t(A^{-1})^t [\Gamma(T_s) + T_s Q_3 - (e^{A^t T_s} - I)(A^t)^{-1} \\ &\quad \times Q_3 - Q_3(e^{A^t T_s} - I)A^{-1}] A^{-1} B + B^t e^{A^t T_s} Q_4 \\ &\quad \times e^{A T_s} B. \end{aligned} \quad (17)$$

Similarly,

$$L_2 = B^t(A^{-1})^t [(e^{A^t T_s} - I)(A^t)^{-1} - T_s I] C^t Q_1, \quad (18)$$

$$\begin{aligned} L_3 &= B^t(A^{-1})^t [\Gamma(T_s) - Q_3(e^{A^t T_s} - I)A^{-1}] \\ &\quad + B^t e^{A^t T_s} Q_4 A e^{A T_s}. \end{aligned} \quad (19)$$

Once T_s , Q_1 and Q_2 are determined, L_1 , L_2 and L_3 can be calculated from Eqs. (17)-(19) off-line preceding the execution of the step control algorithm. For practical applications, the on-line computation task consists only of several multiplications and additions in Eq. (11) at every sampling instant. If A^{-1} does not exist, L_1 , L_2 and L_3 cannot be simplified as Eqs. (17)-(19); thus, calculation of the integrals in Eqs. (12) is

necessary.

For a bounded control input, a common practical control system situation, the control algorithm can be modified by taking the bounded constraint of control input $\bar{u}^*$ in the derivation of Eq. (11). For simplicity of description, consider the single input case where $\bar{u}$ is a scalar and $-u_b < \bar{u} < u_b$. The optimal step control input obtained from Eq. (11) then has to be modified as

$$\bar{u}^* = \begin{cases} \bar{u}^*, & \text{for } |\bar{u}^*| \leq u_b \\ u_b, & \text{for } \bar{u}^* > u_b \\ -u_b, & \text{for } \bar{u}^* < -u_b \end{cases} \quad (20)$$

3. Stability Analysis of the Step Control System and Determination of the Design Time Interval and Weighting Matrix

The step control algorithm can be represented exactly by the block diagram shown in Fig. 1. The stability of the step control system can be analyzed using the sampled-data control technique, where the time interval is the sampling period.

The equivalent discrete-time state equation for Eqs. (1), (2) is

$$x(k+1) = Gx(k) + Hu(k), \quad (21)$$

$$y(k) = Cx(k), \quad (22)$$

where

$$G = e^{AT_s}, \quad (23)$$

$$H = \int_0^{T_s} e^{A(T_s-\tau)} d\tau B. \quad (24)$$

It is important to point out that unlike most digital control systems, Eqs. (21)-(24) involve no approximation, and the control input vector u is constant between any two consecutive sampling instants. The discrete-time state equation given in Eqs. (21), (22) is the zero-order hold equivalent⁽⁷⁾ of the continuous time state equation of Eqs. (1), (2). The discrete-time block diagram of the step control system is shown in Fig. 2; thus, combining Eq. (11) and Eqs. (21), (22), the state equation of the closed-loop system is

$$x(k+1) = [G - HL^{-1}L_3]x(k) + HL^{-1}L_2r(k), \quad (25)$$

$$y(k) = Cx(k). \quad (26)$$

The characteristic equation of the closed-loop system is

$$p(z) = \det[zI - G + HL^{-1}L_3] = 0, \quad (27)$$

where I is the n by n identity matrix. From the locations of the roots in Eq. (27), the stability of the

Fig. 1 The block diagram of the step control system

closed-loop system can be determined.

Since the characteristic equation is a function of T_s , Q_1 , Q_2 and system parameters, so are the roots. For a given system, the absolute and relative stability are affected by T_s , Q_1 and Q_2 . In fact, from the definition of the performance J in Eq. (8), we can set $Q_1=1$ without losing the generality of the control algorithm. An attempt can then be made to choose T_s and Q_2 for the best performance, which depends on the specifications of the closed-loop system.

By the use of root locus analysis in the z -domain, for each given Q_2 , the root locus of the closed-loop system can be plotted in the z -plane as varying with T_s or vice versa. Thus, the change of roots due to the influences of Q_2 and T_s can be shown clearly by using the two-parameter root locus in the z -plane, and the equivalent damping ratio and the damped natural frequency of the closed-loop system can be obtained by the locations of the roots.

In practice, T_s depends on the computation capabilities of the microprocessor used. As a result, the design T_s is the minimum time consumption of the microprocessor for calculating the control law in Eq. (30). The design Q_2 is thus determined once the design T_s is chosen.

4. Applying the Step Control Algorithm to a Second-Order System

In this section, an example of a second-order system is presented to demonstrate the usefulness of the step control algorithm. We assume that the system requires fast response with no oscillation, which amounts to specification of the critical damping response. The system is expressed by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u, \quad (28)$$

$$y = [1 \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (29)$$

where ξ and ω_n are the damping ratio and natural frequency, respectively. For the system considered, the weighting matrices are scalars, i.e., $Q_1 = q_1$, $Q_2 = q_2$. Now, L_1 , L_2 and L_3 in Eqs. (12·a-c) are

$$L_1 = \gamma_1 + q_1 T_s + q_1 (\alpha_1 - 1) \frac{4\xi}{\omega_n} + \frac{2q_1 \alpha_3}{\omega_n^2} + q_2 \omega_n^4 \alpha_2^2, \quad (30)$$

$$L_2 = q_1 \left[\frac{2\xi}{\omega_n} (\alpha_1 - 1) + \frac{\alpha_3}{\omega_n^2} + T_s \right], \quad (31)$$

$$L_3 = \begin{bmatrix} -\gamma_1 - q_1 \frac{2\xi}{\omega_n} (\alpha_1 - 1) - \frac{q_1 \alpha_3}{\omega_n^2} + q_2 \omega_n^2 \alpha_2 \alpha_3, \\ -\gamma_2 - q_1 \frac{2\xi}{\omega_n} \alpha_2 - q_1 \frac{\alpha_4 - 1}{\omega_n^2} + q_2 \omega_n^2 \alpha_2 \alpha_4 \end{bmatrix}, \quad (32)$$

where L_1 and L_2 are scalars, and L_3 is a 2 by 1 row vector. The notations γ_1 , γ_2 , α_1 , α_2 , α_3 and α_4 are defined in Eqs. (A 4) - (A 9) of Appendix A. Detailed derivations of L_1 , L_2 and L_3 are also given in Appendix A. With the given reference input r , the control law is obtained and expressed as follows,

$$u(k) = \frac{L_2}{L_1} r(k) - \frac{L_{31}}{L_1} x_1(k) - \frac{L_{32}}{L_1} x_2(k), \quad (33)$$

where L_{31} and L_{32} are the first and second elements of L_3 , respectively. The system parameters of the discrete-time state equation in Eq. (21) are

$$G = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}, \quad (34)$$

$$H = (e^{AT_s} - I)A^{-1}B = \begin{bmatrix} 1 - \alpha_1 \\ -\alpha_3 \end{bmatrix}. \quad (35)$$

The parameters of the discrete-time state equation, G and H in Eqs. (34), (35), respectively, and the step control feedback law in Eq. (33) are nonlinear functions of T_s , q_1 , q_2 , ξ and ω_n , and since γ_1 , γ_2 , α_1 , α_2 , α_3 and α_4 are transcendental functions, so is the characteristic equation. The closed-loop poles, i.e., the roots of the characteristic equation, are tedious to solve analytically. Two methods of obtaining the location of the closed-loop poles will be presented here. First, with the two-parameter root locus method, the control law can be determined. Next, under some nonrestricted assumptions, an approximate result is obtained. In the process, some important properties of the step control algorithm can be observed.

4.1 Two-parameter root locus approach

Consider the case where $\omega_n = 10$ rad/sec, $\xi = 0.5$, and $q_2 = 5.353 \times 10^{-9}$. The root locus of the closed-loop system is shown in Fig. 3 while T_s varies from 0 to 0.01 sec. There are two breakaway points, at $T_s = 0.003$ sec and $T_s = 0.0048$ sec. If T_s is greater than 0.006 sec, one of the roots is located outside the unit circle and the system is unstable. If T_s is 0.003 sec, the closed-loop system shows critical damping behavior. In the case of $T_s > 0.003$ sec, as T_s is increasing, the

Fig. 2 The equivalent discrete-time block diagram of the step control system

magnitude of oscillation increases, and the damping ratio of the closed-loop system decreases. When $T_s < 0.003$ sec, all roots are positive real values, in which case the response is in overdamped conditions. The responses for different T_s are shown in Fig. 4. Under the critical damping condition, the suitable T_s is 0.003 sec. Similarly, for different q_2 , other closed-loop root loci can be plotted, and other suitable values of T_s for the critical damping response can then be chosen.

4.2 Approximate approach

We assume $\omega_d T_s$ is small enough, taking the Taylor series expansion of α_1 , α_2 , α_3 and α_4 in Eqs. (A 4)-(A 7) as follows:

Fig. 3 The root locus plot of the closed-loop system with $q_2 = 5.353 \times 10^{-9}$

(a) position output

(b) control input

Fig. 4 The responses of various sampling periods with $q_2 = 5.353 \times 10^{-9}$

$$\alpha_1 = 1 - \frac{1}{2} \omega_n^2 T_s^2 + \frac{1}{3} \sigma \omega_n^2 T_s^3, \quad (36)$$

$$\alpha_2 = T_s - \sigma T_s^2 + \left(\frac{1}{2} \sigma^2 - \frac{1}{6} \omega_d^2 \right) T_s^3, \quad (37)$$

$$\alpha_3 = -\omega_n^2 T_s + \sigma \omega_n^2 T_s^2 - \left(\frac{1}{2} \sigma^2 - \frac{1}{6} \omega_d^2 \right) \omega_n^2 T_s^3, \quad (38)$$

$$\alpha_4 = 1 - 2\sigma T_s + \left(\frac{3}{2} \sigma^2 - \frac{1}{2} \omega_d^2 \right) T_s^2 + \left(\frac{2}{3} \sigma \omega_d^2 - \frac{2}{3} \sigma^3 \right) T_s^3. \quad (39)$$

With the substitution of Eqs. (36)-(39) into Eqs. (30)-(32) and neglect of the higher-order terms of T_s , L_1 , L_2 and L_3 can be expressed approximately as

$$L_1 = q_2 \omega_n^4 T_s^2, \quad (40)$$

$$L_2 = \frac{1}{6} q_1 \omega_n^2 T_s^3, \quad (41)$$

$$L_3 = \left[\frac{1}{6} q_1 \omega_n^2 T_s^3 \quad q_2 \omega_n^2 T_s^3 \right]. \quad (42)$$

Similarly, both G and H in Eqs. (41), (42), respectively, are presented as polynomials of T_s by substituting α_1 , α_2 , α_3 and α_4 in Eqs. (43)-(46). Thus, the characteristic equation is

$$p(z) = \det \begin{bmatrix} z - 1 + \frac{q_1}{12q_2} T_s^3, & -\frac{1}{2} T_s + \frac{2}{3} \sigma T_s^2 \\ -\left(\frac{3}{2} \sigma - \frac{3}{2} \sigma^2 + \frac{1}{2} \omega_d^2 \right) T_s^3 & \\ \frac{q_1}{6q_2} T_s^2 - \frac{q_1}{6q_2} \sigma T_s^3, & z + \sigma T_s \\ -\left(3\sigma - 3\sigma^2 + \frac{\omega_d^2}{3} \right) T_s^2 + \left(3\sigma^2 - \frac{10}{3} \sigma^3 \right) T_s^3 & \end{bmatrix} = 0. \quad (43)$$

From the above equation, the ratio of $\frac{q_1}{q_2}$ dominates the characteristics of the closed-loop system rather than the separate weighting. With the manipulation of Eq. (43) and omission of the higher-order terms of T_s , the following equation holds:

$$p(z) = z^2 + \left(-1 + \sigma T_s + \frac{q_1}{12q_2} T_s^3 \right) z + \left(\sigma T_s + \frac{q_1}{12q_2} T_s^3 \right) = 0. \quad (44)$$

To obtain the double roots of z under consideration of the critical damping response, the discriminator of Eq. (44) has to be zero, and the assumption $\omega_d T_s \approx 0$ implies $T_s \sigma \ll 1$ automatically in the case of a light damping system. Thus,

$$\frac{1}{144} \left(\frac{q_1}{q_2} T_s^3 \right)^2 - \frac{1}{2} \left(\frac{q_1}{q_2} T_s^3 \right) + 1 = 0 \quad (45)$$

and $\frac{q_1}{q_2} T_s^3 = 2.06$ and 70, the first and second break-away points, respectively. As a result, for the critical damping response, the condition $\frac{q_1}{q_2} T_s^3 = 2.06$ should be satisfied, and the corresponding double roots are $z_{1,2} = 0.414$. The invariant locations of the double roots of the closed-loop system and the relationship between

q_2 and T_s have now been established.

In other words, if the assumptions $T_s \omega_n \approx 0$ and $T_s \sigma \ll 1$ are met, this property must hold in spite of variations in ω_n and ξ . Table 1 shows the results of applying $\frac{q_1}{q_2} T_s^3 = 2.06$ for several values of both ξ and ω_n , and all of the roots are close to 0.414. Consequently, once T_s is chosen, all of the systems with different ξ and ω_n in Table 1 have similar dynamic characteristics controlled by the step control algorithm. Output responses with $T_s = 0.003$ sec are shown in Fig. 5. All systems show similar responses except for $\omega_n = 30$ and $\xi = 0.9$, because $T_s \sigma$ is not small enough in Eq. (44) in this case. However, by increasing the sampling rate, the deviation in Fig. 5 can be further reduced. This property of the step control algorithm is useful for the synchronization of simultaneous multi-axis control systems in which the ξ and ω_n of each axis have distinct values.

4.3 Summary

Although the locations of the double roots in the z -plane are the same, smaller suitable T_s causes a faster response because a high sampling rate results in the large bandwidth of the closed-loop system as shown in Fig. 6. Since $\frac{L_2}{L_1}$, $\frac{L_{31}}{L_1}$ and $\frac{L_{32}}{L_1}$ can be computed off-line, only three multiplications and two additions are needed to implement the step control law in

Table 1 The locations of the closed-loop poles in the z -domain with different system parameters

System parameters	Locations of poles
$\omega_n = 30 \text{ rad/sec}$ $\xi = 0.1$	$0.4083 + j0.05$ $0.4083 - j0.05$
$\omega_n = 30 \text{ rad/sec}$ $\xi = 0.9$	0.4387 0.2456
$\omega_n = 10 \text{ rad/sec}$ $\xi = 0.5$	$0.4020 + j0.039$ $0.4020 - j0.039$
$\omega_n = 0.1 \text{ rad/sec}$ $\xi = 0.1$	$0.4142 + j0.002$ $0.4142 - j0.002$
$\omega_n = 0.1 \text{ rad/sec}$ $\xi = 0.9$	$0.4140 + j0.006$ $0.4140 - j0.006$

Fig. 5 The system responses with different natural frequencies and damping ratios controlled by the step control algorithm ($T_s = 0.003$ sec)

the on-line application.

Next, comparisons on the responses obtained by the step control algorithm, bang-bang control and the linear quadratic regulator (LQR) problem⁽³⁾ are made. If we consider a case with a bounded input, e.g., $u_b = 85$, the regulator problem can be expressed as the minimization of the performance index

$$J_2 = \int_0^\infty (x^T R_{xx} x + u^T R_{uu} u) dt, \quad (46)$$

where

$$R_{xx} = \begin{bmatrix} 1.5 \times 10^6 & 0 \\ 0 & 180 \end{bmatrix}, R_{uu} = 200 \quad (47)$$

are representative of the weighting matrices such that the output is the critical damping response. Basically, the determination of the weighting matrices is a trial-and-error process. In this example, for fixed R_{xx} , in the limiting case of $R_{uu} = 0$, the closed-loop system's poles are two distinct negative real values in the s -plane. If R_{uu} is infinite, the elements of the state feedback matrix are all zero, which means that the feedback paths are open and the closed-loop system's poles are the same as those of plant to be controlled. The damping ratio and the bandwidth of the closed-loop system decrease as R_{uu} increases. On the other hand, if R_{uu} is fixed, when the weighting factor of the velocity term is decreased, both the velocity feedback gain and the damping ratio of the closed-loop system decrease.

The weighting matrices of J_2 are set to be

$$R_{xx} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, R_{uu} = \rho \geq 0, \quad (48)$$

when the derivative of the output is neglected. The meaning of J_2 is similar to that of J , the performance index of the step control algorithm, except that the response is depressed using $u^T R_{uu} u$ in the integral of J_2 . This is achieved in the step control algorithm by limiting the rate of change of the output at the end of the time interval. The poles of the closed-loop system are a complex conjugate pair in the s -plane, regardless of the value of ρ . This means that the critical

Fig. 6 The comparison of responses obtained at $T_s = 0.003$ and 0.006 sec

damping response never takes place.

A comparison of the output responses is shown in Fig. 7. There is little difference between the responses obtained by the step control algorithm ($T_s=0.003$ second) and bang-bang control; what difference does exist can be further reduced by decreasing both the sampling period T_s and the weighting matrix q_2 . This example demonstrates the effectiveness of the proposed control scheme.

5. Applying the Step Control Algorithm to a Higher-Order System

An application of the step control algorithm to an optical disk track-seeking servo system is given by Lee and Shieh⁽⁸⁾. The system configuration for the two-stage servo system is shown in Fig. 8. This system has a coarse motor which moves the optical head in a radial direction, and a fine motor on the optical head, which moves the objective lens in the tracking direction. There is a relative displacement sensor to detect position errors between the optical head (or coarse motor) and the objective lens (or fine motor). The input voltages for the fine and coarse motors are the outputs of the system controller and the relative displacement sensor, respectively. A block diagram for the system is shown in Fig. 9.

The problem is how to design the system controller so as to obtain rapid and precise positioning of x_2

Fig. 7 The comparison of responses of the step control algorithm, bang-bang control and the LQR

Fig. 8 The model of mechanism of the track-seeking system

M: mass driven by coarse motor
 m: mass driven by fine motor
 ks1: relative displacement sensor gain
 k: stiffness coefficient of suspending element
 d: damping coefficient of suspending element
 kd1: coarse driver gain
 ktach: coarse motor tach feedback gain
 kt1, kt2: torque constants of coarse and fine motors, respectively
 ke1, ke2: voltage constants of coarse and fine motors, respectively
 L1, L2: inductances of coarse and fine motors, respectively
 R1, R2: resistances of coarse and fine motors, respectively
 x1: absolute displacement of optical head (or coarse)
 x2: absolute displacement of objective lens (or fine)
 xe: relative displacement, $x_e = x_2 - x_1$
 F1: output torque of the coarse motor
 F2: output torque of the fine motor
 v2: input voltage of the fine motor, which is output of system controller

Fig. 9 The block diagram of the track-seeking servo system of the optical disk driver

Fig. 10 The responses of the track-seeking servo system obtained by the step control algorithm and the common control scheme

without oscillation, both for reducing access time and for increasing the efficiency of the optical disk. For this system, then, x_2 is specified to show a critical damping response.

With application of the system parameters, the transfer function of the system is

$$\frac{x_2(s)}{v_2(s)} = \frac{30s^3 + 48750s^2 + 2.4 \cdot 10^7 s + 2.362 \cdot 10^{10}}{s^5 + 1696s^4 + 9.354 \cdot 10^5 s^3 + 8.759 \cdot 10^8 s^2 + 1.6 \cdot 10^{10} s} \quad (49)$$

For the sampling period $T_s = 0.0013$ sec, the position output $x_2(t)$ and the control input voltage $v_2(t)$ are shown in Fig. 10. The reference input is 0.01 m, one-third of the full stroke of the optical disk.

A common control scheme, studied by Ananthanarayanan⁽¹⁾, Lee and Shieh⁽⁸⁾, and Browns and Ma⁽⁹⁾, is first to obtain a reduced model of the original plant. Then, bang-bang control is used to generate the control input profile with respect to the position error. Finally, the system controller uses a modified profile which is chosen via testing, based on the obtained profile, because a difference exists between the reduced model and the original system. Though commonly employed, this procedure leads to difficulties in analysis and modification of the profile.

Figure 10 shows a comparison of the responses obtained by the step control and the common control schemes. The access time is reduced efficiently by the

step control scheme. Consequently, whereas bang-bang control for higher-order systems is still an unsolved problem, step control provides a systematic way of designing control algorithms for higher-order systems with satisfactory results.

6. Conclusions

The proposed control algorithm called step control provides accurate results with simple computation, which can be easily implemented by a microcomputer in on-line applications. As we have shown, the performance of this method is close to that of bang-bang control while the limitations and disadvantages of the bang-bang control method are eliminated.

Acknowledgements

This study was partially supported by the National Science Council, Republic of China, under contract number NSC 80-0401-E-009-04.

Appendix A

The Calculation of L_1 , L_2 and L_3 for a Second-Order System. The second-order system is expressed in Eqs. (28), (29) and the weighting matrices are scalar q_1 and q_2 . We define $\sigma = \xi\omega_n$ and $\omega_d = \omega_n\sqrt{1-\xi^2}$, then

$$e^{At} = \begin{bmatrix} e^{-\sigma t} \left(\cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t \right) & e^{-\sigma t} \frac{1}{\omega_d} \sin \omega_d t \\ -e^{-\sigma t} \frac{\omega_n^2}{\omega_d} \sin \omega_d t & e^{-\sigma t} \left(\cos \omega_d t - \frac{\sigma}{\omega_d} \sin \omega_d t \right) \end{bmatrix} \quad (A 1)$$

$$Q_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} q_1 \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} q_1 & 0 \\ 0 & 0 \end{bmatrix} \quad (A 2)$$

$$Q_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} q_2 \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} q_2 & 0 \\ 0 & 0 \end{bmatrix} \quad (A 3)$$

Also, we define $\Gamma(T_s) = \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{bmatrix}$ and $e^{A\tau_s} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}$, where α_1 , α_2 , α_3 and α_4 can be calculated by taking $t = T_s$ in Eq. (A 1),

$$\alpha_1 = e^{-\sigma T_s} \left(\cos \omega_d T_s + \frac{\sigma}{\omega_d} \sin \omega_d T_s \right), \quad (A 4)$$

$$\alpha_2 = e^{-\sigma T_s} \frac{1}{\omega_d} \sin \omega_d T_s, \quad (A 5)$$

$$\alpha_3 = -e^{-\sigma T_s} \frac{\omega_n^2}{\omega_d} \sin \omega_d T_s, \quad (A 6)$$

$$\alpha_4 = e^{-\sigma T_s} \left(\cos \omega_d T_s - \frac{\sigma}{\omega_d} \sin \omega_d T_s \right). \quad (A 7)$$

Furthermore, $\Gamma(T_s)$ in Eq. (16) can be calculated by comparison of coefficients, as follows:

$$\gamma_1 = -\frac{4\xi^2+1}{4\xi\omega_n} q_1 (\alpha_1^2 - 1) - \frac{\omega_n}{4\xi} q_1 \alpha_2^2 + q_1 \alpha_1 \alpha_2, \quad (A 8)$$

$$\gamma_2 = -\frac{q_1}{2\omega_n^2} (\alpha_1^2 - 1), \quad (A 9)$$

$$\gamma_3 = -\frac{q_1}{4\xi\omega_n} \left[\frac{1}{\omega_n^2} (\alpha_1^2 - 1) - \alpha_2^2 \right]. \quad (\text{A } 10)$$

Then

$$Q_3(e^{A^T s} - I)A^{-1} = [(e^{A^T s} - I)(A^T)^{-1}Q_3]^T \\ \begin{bmatrix} -q_1 \frac{2\xi}{\omega_n} (\alpha_1 - 1) - \frac{q_1}{\omega_n^2} \alpha_3 & 0 \\ -q_1 \frac{2\xi}{\omega_n} \alpha_2 - \frac{q_1}{\omega_n^2} (\alpha_4 - 1) & 0 \end{bmatrix}^T, \quad (\text{A } 11)$$

$$B^T e^{A^T s} Q_4 e^{A^T s} B \\ = [0 \ \omega_n^2] \begin{bmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{bmatrix} \begin{bmatrix} q_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_n \end{bmatrix} \\ = q_2 \omega_n^4 \alpha_2^2, \quad (\text{A } 12)$$

$$B^T (A^{-1})^T = (A^{-1}B)^T = [0 \ \omega_n^2] \begin{bmatrix} -\frac{2\xi}{\omega_n} & 1 \\ -\frac{1}{\omega_n^2} & 0 \end{bmatrix} \\ = [-1 \ 0]. \quad (\text{A } 13)$$

Combining Eqs. (A 4)-(A 13), and referring back to Eq. (17), we obtain L_1 as

$$L_1 = \gamma_1 + q_1 T_s + q_1 \frac{4\xi}{\omega_n} (\alpha_1 - 1) + \frac{2q_1}{\omega_n^2} \alpha_3 + q_2 \omega_n^4 \alpha_2^2. \quad (\text{A } 14)$$

From Eq. (18), L_2 becomes

$$L_2 = [0 \ \omega_n^2] \begin{bmatrix} -\frac{2\xi}{\omega_n} & 1 \\ -\frac{1}{\omega_n^2} & 0 \end{bmatrix} \left(\begin{bmatrix} \alpha_1 - 1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) q_1 \\ \begin{bmatrix} -\frac{2\xi}{\omega_n} & 1 \\ -\frac{1}{\omega_n^2} & 0 \end{bmatrix} - \begin{bmatrix} T_s & 0 \\ 0 & T_s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} q_1 \\ = q_1 \left[\frac{2\xi}{\omega_n} (\alpha_1 - 1) + \frac{1}{\omega_n^2} \alpha_3 + T_s \right]. \quad (\text{A } 15)$$

With Eq. (A 11), then,

$$B^T (A^{-1})^T [\Gamma(T_s) - Q_3(e^{A^T s} - I)A^{-1}] \\ = [1 \ 0] \begin{bmatrix} \gamma_1 + q_1 \frac{2\xi}{\omega_n} (\alpha_1 - 1) + \frac{q_1}{\omega_n^2} \alpha_3 & \gamma_2 \\ \gamma_2 + q_1 \frac{2\xi}{\omega_n} \alpha_2 + \frac{q_1}{\omega_n^2} (\alpha_4 - 1) & \gamma_3 \end{bmatrix}^T$$

$$= \begin{bmatrix} -\gamma_1 - q_1 \frac{2\xi}{\omega_n} (\alpha_1 - 1) - \frac{q_1}{\omega_n^2} \alpha_3, \\ \gamma_2 - q_1 \frac{2\xi}{\omega_n} \alpha_2 - \frac{q_1}{\omega_n^2} (\alpha_4 - 1) \end{bmatrix}, \quad (\text{A } 16)$$

$$B^T e^{A^T s} Q_4 A e^{A^T s} \\ = [0 \ \omega_n^2] \begin{bmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{bmatrix} \begin{bmatrix} q_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\omega_n^2 - 2\xi\omega_n \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \\ = q_2 \omega_n^2 [\alpha_2 \alpha_3 \ \alpha_2 \alpha_4]. \quad (\text{A } 17)$$

Adding Eqs. (A 16) and (A 17), we obtain

$$L_3 = \begin{bmatrix} -\gamma_1 - q_1 \frac{2\xi}{\omega_n} (\alpha_1 - 1) - \frac{q_1}{\omega_n^2} \alpha_3 + q_2 \omega_n^2 \alpha_2 \alpha_3, \\ -\gamma_2 - q_1 \frac{2\xi}{\omega_n} \alpha_2 - \frac{q_1}{\omega_n^2} (\alpha_4 - 1) + q_2 \omega_n^2 \alpha_2 \alpha_4 \end{bmatrix}. \quad (\text{A } 18)$$

References

- (1) Ananthanarayanan, K. S., IEEE Trans. on Magnetic, Vol. MAG-18, No. 3 (1982), p. 888.
- (2) Kirk, D. E., Optimal Control Theory, Prentice-Hall, (1970).
- (3) Kailath, T., Linear Systems, Prentice-Hall, (1980).
- (4) Anderson, B. D. O. and Moore, J. B., Optimal Control, Linear Quadratic Method, Prentice-Hall, (1989).
- (5) Sage, A. P. and White, C. C., Optimal Systems Control, Prentice-Hall, (1977).
- (6) Chen, C. T., Linear System Theory and Design, Holt, Rinehart and Winston, (1984).
- (7) Ogata, K., Discrete-Time Control System, Prentice-Hall, (1987).
- (8) Lee, A. C. and Shieh, Y. S., The Controller Design of the Track-Seeking Servo System of the Optical Disk Driver, Tech. Report J1-79002, The National Chiao Tung University, (1990).
- (9) Browns, C. J., and Ma, J. T., IBM J Res. Develop, (1968), p. 372.